Rakenteiden Mekaniikka (Journal of Structural Mechanics)
Vol. 49, No 2, 2016, pp. 69 – 77
rmseura.tkk.fi/rmlehti/
©The authors 2016.
Open access under CC BY-SA 4.0 license.

VERTAISARVIOITU
KOLLEGIALT GRANSKAD
PEER-REVIEWED
www.tsv.fi/tunnus

On optimisation of structures under stability constraints - a simple example

Reijo Kouhia[1], Sami Pajunen, and Jarmo Poutala

Summary. In this paper some basic questions on optimisation of structures prone to instability behaviour are addressed by using a simple example consisting only one state variable. A multi-criteria optimisation problem, where both the material cost and imperfection sensitivity are minimized, is formulated.

Key words: optimisation, stability, structures, imperfection sensitivity

Received 30 June 2016. Accepted 7 November 2016. Published online 9 December 2016

In memory of Professor Juhani Koski

Introduction

It is well known that optimisation of structures where instability phenomena are present can result in severe imperfection sensitivity [9, 10]. Classical examples are the simple Augusti model and stiffened plates [2, 4]. In both examples, the imperfection sensitivity is due to the interaction of two buckling modes. If the critical loads corresponding to the two buckling modes are well separated, the post-buckling paths are stable, however, when these loads approach to each other, secondary bifurcations occur on the primary post-buckling paths rendering the secondary post-buckling path highly unstable.

Description of the structural model

A simple cantilever beam with a solid square cross-section and having a constant length L and loaded by a compressive dead-weight load P is modelled as a rigid bar and a non-linear rotational spring, see Fig. 1a, resulting in a discrete one degree-of-freedom (dof) model. It is also assumed that the material is fixed, but the grade of it is left free. This implies that the Young's modulus can be assumed to be a constant whereas the yield strength is apt for optimisation.

The basic optimisation problem is to design the beam such that the cost function

$$C = \sigma_{\mathrm{y}} A L, \tag{1}$$

[1]Corresponding author. `reijo.kouhia@tut.fi`

where σ_y is the yield strength and A the area of the cross-section, is minimised under the following conditions: (i) the load carrying capacity has to be larger than the design load P_d and (ii) the structure can have an imperfection of a certain prescribed amount. How to incorporate the imperfection into the model and in the optimisation problem setting will be discussed later in this paper.

The minimisation problem has thus two design variables: the yield strength of the material σ_y and the area of the cross-section, or in this case simply the side length of the cross-section b. By introducing the non-dimensional ratios

$$\eta = \sigma_\mathrm{y}/E, \quad \text{and} \quad \xi = b/L, \tag{2}$$

the cost function can be written as

$$C = \eta\xi^2 E L^3. \tag{3}$$

The elastic buckling load of the perfect structure is

$$P_\mathrm{E} = \frac{k}{L} = \frac{3EI}{L^2} = \frac{1}{4}\xi^4 E L^2, \tag{4}$$

where it has been assumed that the spring stiffness has the value $k = 3EI/L$. The plastic limit load due to normal force for the perfect structure is

$$N_\mathrm{p} = \sigma_\mathrm{y} A = \eta\xi^2 E L^2. \tag{5}$$

Simple optimisation approach

A naive optimisation strategy in which the stability limit is only taken into account by the linearized buckling eigenvalue problem, could be formulated by requiring the design load to be smaller than the plastic normal force and the elastic buckling load:

$$N_\mathrm{p} \geq P_\mathrm{d}, \quad \text{and} \quad P_\mathrm{E} \geq P_\mathrm{d}. \tag{6}$$

Giving the design load in the form $P_\mathrm{d} = \beta E L^2$, the constraints (6) result in

$$\eta\xi^2 \geq \beta, \quad \text{and} \quad \xi^4 \geq 4\beta. \tag{7}$$

If the design, buckling, and plastic capacity loads coincide, it results in the relation $\eta = \xi^2/4$, thus

$$\xi = \sqrt[4]{4\beta} \quad \text{and} \quad \eta = \tfrac{1}{2}\sqrt{\beta}. \tag{8}$$

Results for certain values of the design loads β are shown in Table 1.

Defining the non-dimensional load parameter as

$$\lambda = P/P_\mathrm{d}, \tag{9}$$

the elastic post-buckling path is simply

$$\lambda = \frac{\xi^4}{4\beta}\frac{\theta}{\sin\theta}, \tag{10}$$

which at the optimum point (8) simplifies to $\lambda = \theta/\sin\theta$, indicating that the form of the post-buckling path does not depend on the design variables. The post-buckling curves for the elastic and plastic solutions differ significantly, see Fig. 1b.

Table 1. Results from the naive optimisation.

$\beta = P_\mathrm{d}/EL^2$	$\eta^{-1} = E/\sigma_\mathrm{y}$	$\xi^{-1} = L/b$
$1.5625 \cdot 10^{-6}$	1600	20
$4.9383 \cdot 10^{-6}$	900	15
$25 \cdot 10^{-6}$	400	10

Effect of the plastic moment

For a more realistic model the effect of normal force $N = -P\cos\theta$ has to be taken into account in the fully plastic moment, which results in

$$M_\mathrm{p} = M_\mathrm{p0}\left[1 - \left(\frac{N}{N_\mathrm{p}}\right)^2\right], \tag{11}$$

where M_p0 is the fully plastic moment in pure bending $M_\mathrm{p0} = \sigma_\mathrm{y} b^3/4$. The following load-deflection relation in the post-buckling regime is then obtained

$$\lambda^2 + 4\eta\xi\beta^{-1}\cos^{-1}\theta\tan\theta\lambda - \eta^2\xi^4\beta^{-2}\cos^{-2}\theta = 0, \tag{12}$$

which at the optimum point (8) results in relation

$$\lambda = \sqrt{(2/\sqrt{\beta})(\tan\theta/\cos\theta)^2 + \cos^{-2}\theta} - (\sqrt{2}/\sqrt[4]{\beta})\tan\theta\cos^{-1}\theta. \tag{13}$$

The post-buckling paths are shown in Fig. 1b. It can be seen from the figure that the slender solutions, i.e. small design load, results in higher reduction of load carrying capacity in the post-buckling regime or in other words higher imperfection sensitivity than the solutions with stockier cross-sections.

Non-linear spring - imperfection sensitivity analysis

To imitate elastic-perfectly-plastic behaviour of the material, a non-linear elastic behaviour of the spring is adopted. The moment M and the rotation θ relationship chosen is

$$M(\theta) = M_\mathrm{p}\tanh\alpha\theta, \tag{14}$$

where M_p is the fully plastic moment and the non-dimensional coefficient is $\alpha = k/M_\mathrm{p}$, where k is the initial spring stiffness. However, it should be noted that since only loading of the spring is considered, there is no principal difference between the behaviours of a non-linear elastic and similar elasto-plastic models. For the initial spring stiffness, the value $k = 3EI/L$ has been chosen, where EI is the elastic bending stiffness of the column. It is also assumed that the unloaded position of the column is tilted by an angle θ_0, which plays role as an imperfection parameter.

The equilibrium equation can be written as

$$PL\sin\theta = M(\theta) = M_\mathrm{p}\tanh\alpha(\theta - \theta_0). \tag{15}$$

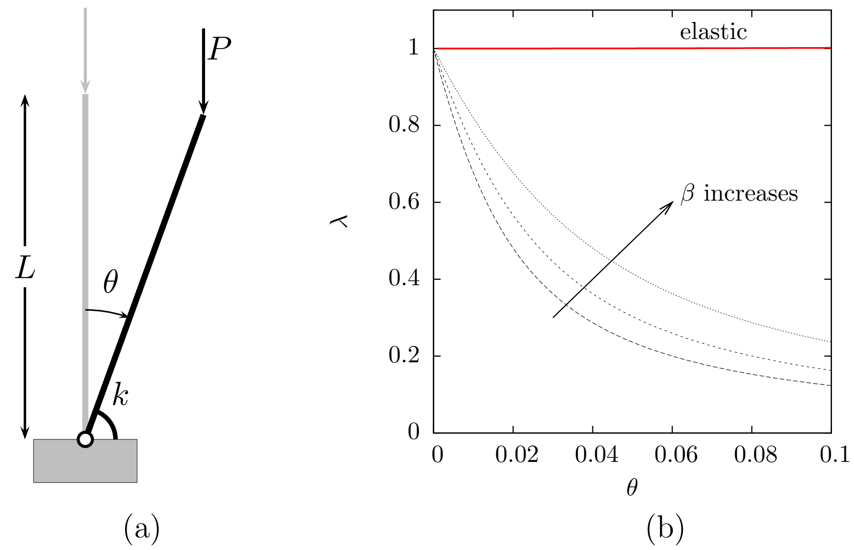


Figure 1. (a) The model and (b) post-buckling paths for purely elastic and rigid-plastic spring at the optimum point where $P = P_\mathrm{d} = P_\mathrm{E} = N_\mathrm{p}$. The design loads and design variables have the same values as in Table 1.

If the effect of normal force to the fully plastic capacity is taken into account, the fully plastic moment has the expression

$$M_\mathrm{p} = \frac{1}{4}\sigma_\mathrm{y} b^3 \left[1 - \left(\frac{-P\cos\theta}{\sigma_\mathrm{y} b^2}\right)^2\right] = \frac{1}{4}\eta\xi^3 \left(1 - \frac{\beta^2\cos^2\theta}{\eta^2\xi^4}\lambda^2\right) EL^3. \tag{16}$$

The equilibrium equation can be written in terms of the non-dimensional variables as

$$\left(\kappa\frac{\beta^2}{4\xi\eta}\tanh\alpha(\theta-\theta_0)\cos^2\theta\right)\lambda^2 + (\beta\sin\theta)\lambda - \frac{1}{4}\xi^3\eta\tanh\alpha(\theta-\theta_0) = 0, \tag{17}$$

where the load parameter λ is defined in (9) and κ indicates whether the effect of normal force to the fully plastic moment is taken into account ($\kappa = 1$) or not ($\kappa = 0$). If the effect of normal force to the fully plastic capacity is neglected, the equilibrium curve in the load-rotation space has the simple form

$$\lambda = \frac{\eta\xi^3}{4\beta}\frac{\tanh\alpha(\theta-\theta_0)}{\sin\theta}. \tag{18}$$

As it can be seen from Fig. 2a the effect of the normal force to the behaviour is significant and therefore the difference between the solutions (17) and (18) is non-negligible.

Although the numerical solution of the non-linear equilibrium equation (17) can be easily performed, an asymptotic expansion in the form

$$\lambda = \frac{\xi^4}{4\beta}\left(a_0 - a_2\theta^2 - a_{-1}\frac{\theta_0}{\theta}\right), \tag{19}$$

where higher order terms consistent with assumptions $\theta \ll 1$ and $|\theta_0/\theta| \ll 1$ are neglected, is very useful. Expanding the coefficients of the polynomial (17) in λ and the second power of λ in power series with θ gives

$$\tanh\alpha(\theta-\theta_0)\cos^2\theta \approx \alpha\theta\left[1-\frac{\theta_0}{\theta}-(1+\tfrac{1}{3}\alpha^2)\theta^2\right], \tag{20}$$

$$\lambda^2 \approx \frac{\xi^8}{16\beta^2}\left(a_0^2-2a_0a_2\theta^2-2a_0a_{-1}\frac{\theta_0}{\theta}\right), \tag{21}$$

and substituting these expressions into (17), results after some simple manipulations, the coefficients in the asymptotic expansion (19) as

$$a_0 = \frac{2}{1+\sqrt{1+\kappa\dfrac{\xi^4}{4\eta^2}}}, \tag{22}$$

$$a_2 = \frac{\dfrac{1}{3}\left(\dfrac{\xi^2}{\eta^2}-\dfrac{1}{2}a_0\right)-\dfrac{\kappa}{16}\left(1+\dfrac{\xi^2}{3\eta^2}\right)\dfrac{\xi^4}{\eta^2}a_0^2}{1+\dfrac{\kappa}{8}\dfrac{\xi^4}{\eta^2}a_0}, \tag{23}$$

$$a_{-1} = \frac{1-\dfrac{\kappa}{16}\dfrac{\xi^4}{\eta^2}a_0^2}{1+\dfrac{\kappa}{8}\dfrac{\xi^4}{\eta^2}a_0}. \tag{24}$$

The equilibrium path has a maximum limit load at

$$\theta = \sqrt[3]{\frac{a_{-1}\theta_0}{2a_2}}, \tag{25}$$

if the term a_2 is positive. Substituting this value back to the expression (19) gives the maximum load carrying capacity i.e. the load value at the limit point in terms of the imperfection magnitude. Such a relationship is also known as the imperfection sensitivity diagram

$$\lambda_{\max} = \frac{\xi^4 a_0}{4\beta}\left[1-3\frac{a_2}{a_0}\left(\frac{a_{-1}\theta_0}{2a_2}\right)^{2/3}\right] = \lambda_{\mathrm{bif}}\left(1-S\theta_0^{2/3}\right), \tag{26}$$

where the term

$$S(\xi,\eta,\kappa) = \frac{3}{2^{2/3}}a_0(\xi,\eta,\kappa)^{-1}a_2(\xi,\eta,\kappa)^{1/3}a_{-1}(\xi,\eta,\kappa)^{2/3} \tag{27}$$

is the *imperfection sensitivity coefficient*, the greater it is the more harmful are the structural imperfections to the maximum load carrying capacity. It is interesting to notice that the effect of inclusion of the normal force to the fully plastic moment to the bifurcation load of the perfect structure $\lambda_{\mathrm{bif}} = \xi^4 a_0/(4\beta)$ is solely due to the a_0-term. The asymptotic expansion of the imperfection sensitivity relation (26) is in accordance with the famous 2/3-power law of Koiter [3, 6, 5, 7, 8]; characteristic for structures exhibiting symmetric unstable bifurcational behaviour.

Taking expressions (22)–(24) into account in (27), it can be seen that the imperfection sensitivity coefficient behaves as

$$S \propto \left(\frac{\xi}{\eta}\right)^2. \tag{28}$$

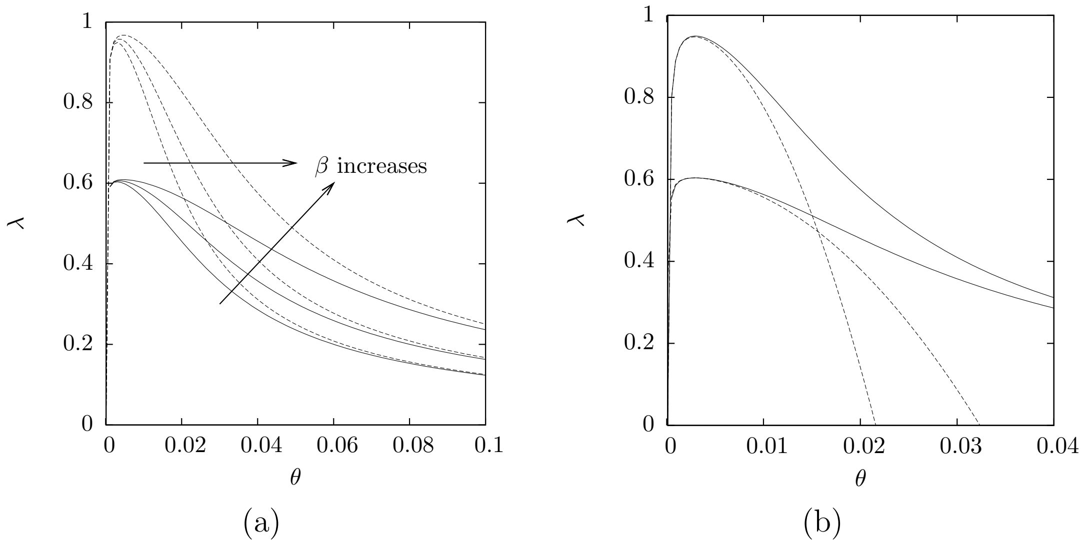


Figure 2. (a): Equilibrium paths, Eq. (17) (the effect of normal force is taken into account) is shown by solid lines, Eq. (18) shown by dashed lines. The design loads and design variables have the same values as in Table 1. (b): Comparison of the exact solution (solid lines) to the asymptotic solution (dashed lines) for the smallest design load in Table 1. Two upper curves correspond to $\kappa = 0$ and the lower ones to $\kappa = 1$. In both figures the imperfection has been $\theta_0 = 10^{-4}$.

Especially, if the effect of normal force to the fully plastic moment is neglected ($\kappa = 0$), the imperfection sensitivity coefficient has the value

$$S = 2^{-2/3}\left[\left(\frac{\xi}{\eta}\right)^2 - 1\right], \tag{29}$$

which shows that the imperfection sensitivity coefficient is positive for all practically relevant designs $\xi > \eta$, at least for metal structures. At the naive optimum point $\eta = \xi^2/4$ the imperfection sensitivity (29) has the expression

$$S = 2^{-2/3}\left(\frac{16}{\xi^2} - 1\right), \tag{30}$$

and the imperfection sensitivity is minimised by increasing the cross-section dimensions.

Optimisation problem

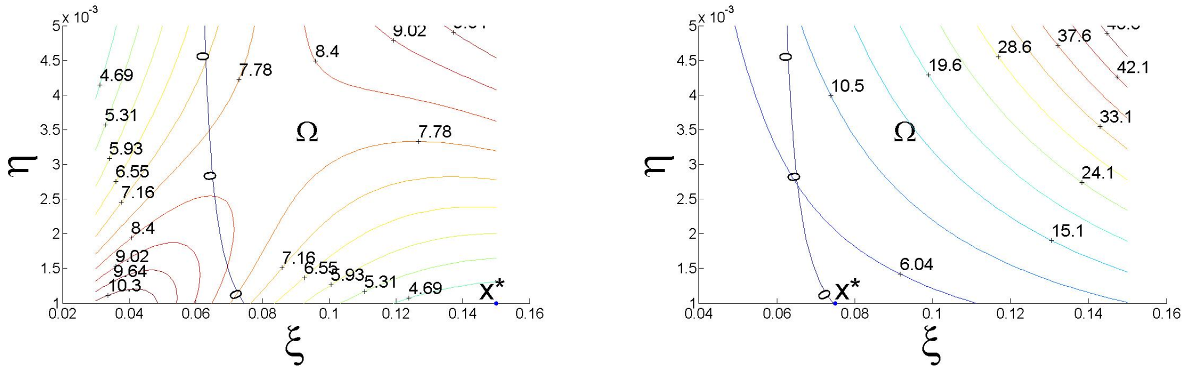
A multi-criteria optimisation problem, where both the material cost and the imperfection sensitivity are minimised, can be stated as

$$\begin{cases} \min C/EL^3 & = \min \eta\xi^2 \\ \min S(\xi,\eta,\kappa) & = \min \dfrac{3}{2^{2/3}} a_2(\xi,\eta,\kappa)^{1/3} a_{-1}(\xi,\eta,\kappa)^{2/3} a_0(\xi,\eta,\kappa)^{-1} \end{cases}, \tag{31}$$

with the constraint

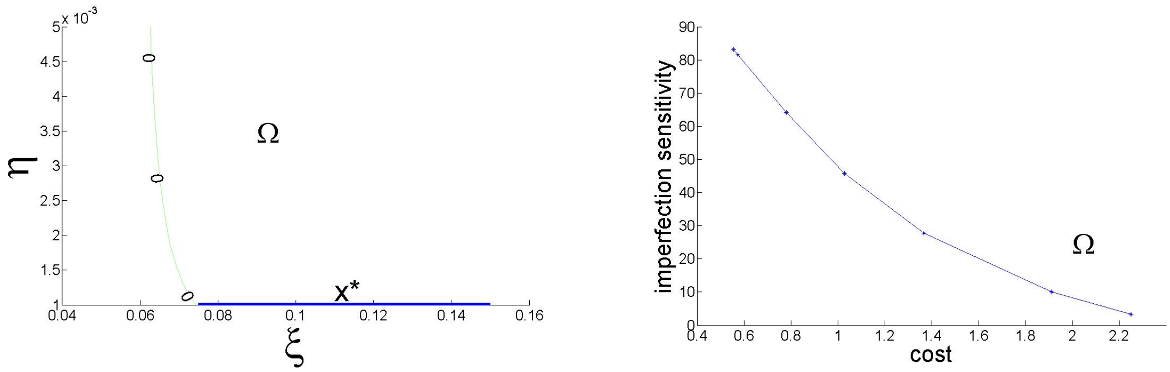
$$P_{\max} \geq P_{\mathrm{d}}, \quad \text{or} \quad \xi^4 a_0(\xi,\eta,\kappa)\left[1 - S(\xi,\eta,\kappa)\theta_0^{2/3}\right] \geq 4\beta. \tag{32}$$

Having only two design variables, the optimisation problem can be illustrated by using the graphical method [1]. Using weight factors for the objective functions, the problem



(a) (b)

Figure 3. Contour lines of the objective function and the feasible region Ω, (a) $w_1 = 0.1$ and $w_2 = 0.9$, (b) $w_1 = 0.9$ and $w_2 = 0.1$. Design load and imperfection are $\beta = 3 \cdot 10^{-6}, \theta_0 = 0.005$. Optimum is denoted with $x^\star$ and the symbol 0 refers to the constraint (32).



(a) (b)

Figure 4. Pareto-optimal solution set in design space and in objective space ($\beta = 3 \cdot 10^{-6}, \theta_0 = 0.005$).

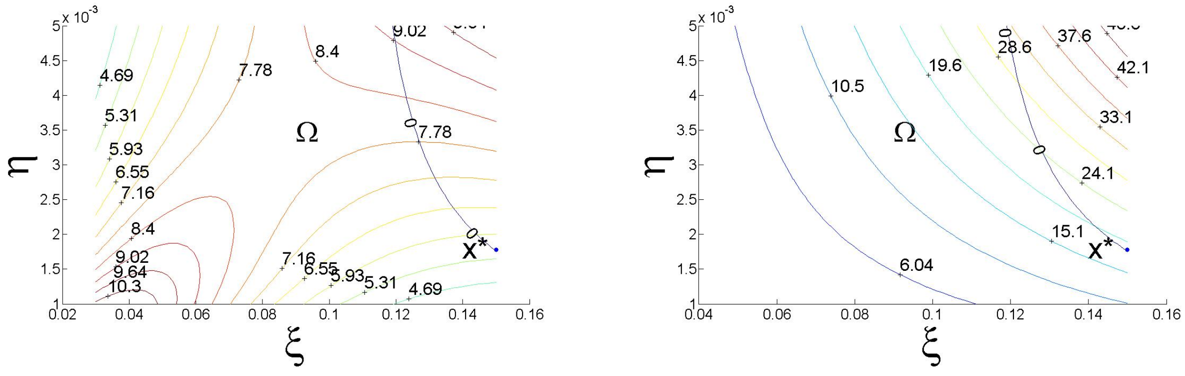
can be re-written as

$$\min \left(w_1 f \eta \xi^2 + w_2 \frac{a_2^{1/3} a_{-1}^{2/3}}{a_0} \right) \tag{33}$$

in which the sum of the weight factors w_1 and w_2 is unity, and the scaling factor f is chosen so that the values of both objective functions are of the same order in the whole region of interest ($f = 5 \cdot 10^5$).

The behaviour of the objective and constraint functions for a typical design with a small imperfection is depicted in Fig. 3 together with the different weight combinations. In Fig. 4 the Pareto-optimal set is shown.

As seen from Figs 3 and 4, in the optimal design the value of design variable η hits always its lower bound, i.e. the strength of the material is as low as possible. When the design load is increased, the Pareto front is reduced to a single optimal point as depicted in fig 5.



(a) (b)

Figure 5. Contour functions of the objective function and the feasible region Ω, (a) $w_1 = 0.1$ and $w_2 = 0.9$, (b) $w_1 = 0.9$ and $w_2 = 0.1$. Design load and imperfection are $\beta = 3 \cdot 10^{-5}, \theta_0 = 0.005$. Optimum is denoted with $x^\star$.

Concluding remarks

In the present work, a preliminary study on optimisation of a simple compressed column under stability constraints is performed. The column is modelled as a rigid bar with a non-linear elastic spring, representing elastic-plastic behaviour of the moment-curvature relation, resulting in a single degree of freedom discrete model. A multi-criteria optimisation problem is formulated where the cost and imperfection sensitivity are minimized. The material grade, i.e. the yield stress and the cross-section width are the two design variables and the cost is assumed to be linear in the yield stress. The coefficient of the imperfection sensitivity is derived by using the initial post-buckling theory of Koiter. Increasing the yield strength also increases the imperfection sensitivity. The main result of the optimisation process is to use as low grade material as possible resulting in the stockiest design.

References

[1] J.S. Arora. *Introduction to Optimum Design.* Elsevier, 3rd edition, 2012.

[2] Z.P. Bažant and L. Cedolin. *Stability of structures.* Oxford University Press, New York, Oxford, 1991.

[3] B. Budiansky. *Theory of buckling and post buckling of elastic structures*, volume 14 of *Advances in Applied Mechanics*, pages 1–65. Academic Press, London, 1974.

[4] J.G.A. Croll and A.C. Walker. *Elements of Structural Stability.* Macmillan, 1972.

[5] A.M.A. van der Heijden. *W.T. Koiter's Elastic Stability of Solids and Structures.* Cambridge University Press, 2009.

[6] J.W. Hutchinson. *Plastic buckling*, volume 14 of *Advances in Applied Mechanics*, pages 67–144. Academic Press, London, 1974.

[7] W.T. Koiter. *Over de stabiliteit van het elastisch evenwicht* (in Dutch). PhD thesis, Technische Hogeschool, Delft, 1945. English tranlations: NASA TT F10, 833 (1967) and AFFDL, TR-7025 (1970).

[8] M. Ohsaki and K. Ikeda. *Stability and Optimization of Structures - Generalized Sensitivity Analysis*. Mechanical Engineering Series. Springer, New York, 2007.

[9] J.M.T. Thompson and G.W. Hunt. *A General Theory of Elastic Stability*. Wiley, London, 1973.

[10] J.M.T. Thompson and G.W. Hunt. *Elastic Instability Phenomena*. Wiley, Chichester, 1984.

Reijo Kouhia, Jarmo Poutala
Tampere University of Technology
Department of Mechanical Engineering and Industrial Systems
P.O. Box 589, FI-33101 Tampere, Finland
`reijo.kouhia@tut.fi, jarmo.poutala@tut.fi`

Sami Pajunen
Tampere University of Technology
Department of Civil Engineering
P.O. Box 600, FI-33101 Tampere, Finland
`sami.pajunen@tut.fi`