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On some bifurcation analysis techniques for continuous systems

Nikolay Banichuk, Alexander Barsuk, Juha Jeronen, Pekka Neittaanmäki,
and Tero Tuovinen[1]

Summary. This paper is devoted to techniques in bifurcation analysis for continuous mechanical systems, concentrating on polynomial equations and implicitly given functions. These are often encountered in problems of mechanics and especially in stability analysis. Taking a classical approach, we summarize the relevant features of the cubic polynomial equation, and present some new aspects for asymptotics and parametric representation of the solutions. This is followed by a brief look into the implicit function theorem as a tool for analyzing bifurcations. As an example from mechanics, we consider bifurcations in the transverse free vibration problem of an axially compressed beam.

Key words: bifurcation analysis, continuous systems, asymptotic analysis, implicit functions, beam

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In memory of Professor Juhani Koski

Introduction

Many important problems of theoretical and applied mechanics require qualitative and quantitative analysis of solutions of cubic and higher order polynomial equations. In this context it is sufficient to refer to numerous problems related to the determination of the principal axes of stress and strain tensors.

Dynamic stability analysis for linear elastic systems, extending Euler's method, was extensively studied by Bolotin [3, 4], based on the pioneering general work by Lyapunov. Bolotin's method also leads to polynomial equations, which must then be solved to determine the local stability exponents of the system under study.

In mechanics, the method applies to both continuous systems (described by partial differential equations) and discrete systems (described by systems of ordinary differential equations). In the case of partial differential equations, the equations are first discretized in space, and Bolotin's method is applied to the resulting semi-discrete form (which is discrete in space, but continuous in time), represented as a system of ordinary differential equations.

[1]Corresponding author. `tero.tuovinen@jyu.fi`

Specifically for cubic polynomial equations, many classical methods exist for their solution, such as the historical Cardano formula, and the numerically stable trigonometric algorithm given e.g. in Press et al. [9] (which, as the authors note, dates back to the 17th century, being originally due to Viète [16]). Such specific algorithms, provided that they are numerically stable, are in the present day highly useful as routines for fast numerical solvers.

However, sometimes such an approach may meet difficulties, especially when separate continuous branches of solutions are detected, and the dependence of solutions on the problem parameters is of interest. Specifically, the solutions returned by a numerical solver may be in a random order, which in a parametric study then requires additional effort to detect which points belong to the same curve.

Some approaches to bifurcation problems and estimation of critical parameters have been presented by Nečas et al. [8] and Neittaanmäki and Ruotsalainen [7]. Modern developments of bifurcation and stability analysis are known as the theory of catastrophe (see Thompson [13]). This theory includes a variety of new problems of stability analysis and qualitative topological methods. In the books by Troger and Steindl [15] and Thomsen [14], the authors have applied bifurcation theory on many practical engineering problems. The book by Seydel [11] contains a comprehensive literature review about the topic.

As for classical approaches to bifurcation theory and its application to statical and dynamical problems, arising in mechanics and engineering, and in mathematical physics, there exists corresponding literature, such as Thompson [12] and Lacarbonara [6].

In this study, we consider some bifurcation analysis techniques. We first concentrate on the cubic equation, taking a classical approach, but presenting some new aspects as for asymptotics and parametric representation of the solutions.

This is followed by a brief look into the implicit function theorem as a tool for analyzing bifurcations, reported in more detail in our study Banichuk et al. [1]. As an example from mechanics, we will consider bifurcations in the transverse free vibration problem of an axially compressed beam.

Statement of the problem

Consider the cubic equation

$$ey^3 + ay^2 + by + c = 0\,, \tag{1}$$

where e, a, b and c are arbitrary real coefficients. Let us focus on the non-degenerate case, where $e \neq 0$. The problem is to find the solution of the equation (1) in an analytical form, and to analyze the solutions as a function of the coefficients of the considered equation.

Equation (1) is dependent on the four given parameters e, a, b and c, but generally, it can be reduced to a canonical form, which has only one parameter. To perform this reduction, we begin by dividing both sides of the equation by e, obtaining

$$y^3 + a'y^2 + b'y + c' = 0\,, \tag{2}$$

where $a' = a/e$, $b' = b/e$ and $c' = c/e$. Any one of the terms (except the highest-degree one) can be eliminated by linearly shifting the coordinates. This requires solving a polynomial of degree $3 - k$, where k is the degree of the term being eliminated.

The standard approach is to *depress* the cubic equation, i.e. eliminate the second-highest-degree term, as doing this requires solving only a linear equation. With this in mind, let us apply a shift of coordinates, defining a new auxiliary variable z:

$$y = z - h\,, \tag{3}$$

where h is a constant to be determined later. Inserting (3) into (2), we have

$$(z-h)^3 + a'(z-h)^2 + b'(z-h) + c' = 0\,. \tag{4}$$

Expanding the parentheses in the standard manner by the binomial theorem

$$(\alpha+\beta)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k}\beta^k$$

and collecting the result in powers of z, we have

$$z^3 + (-3h + a')z^2 + (3h^2 - 2a'h + b')z + (-h^3 + a'h^2 - b'h + c') = 0\,. \tag{5}$$

From (5) we see that the second-highest degree term is eliminated if we choose the shift constant h as

$$h = \frac{a'}{3} = \frac{a}{3e}\,. \tag{6}$$

We thus obtain the depressed cubic corresponding to equation (1):

$$z^3 + pz + q = 0\,, \tag{7}$$

where

$$p \;=\; b' + \frac{1}{3}a'^2 - \frac{2}{3}a'^2 \;=\; b' - \frac{1}{3}a'^2 \;=\; \frac{b}{e} - \frac{a^2}{3e^2}\,, \tag{8}$$

$$q \;=\; c' - \frac{1}{27}a'^3 + \frac{1}{9}a'^3 - \frac{1}{3}a'b' \;=\; c' + \frac{2}{27}a'^3 - \frac{1}{3}a'b' \;=\; \frac{c}{e} + \frac{2}{27}\frac{a^3}{e^3} - \frac{ab}{3e^2}\,. \tag{9}$$

Now, let us scale the coordinates. Let us define a second auxiliary variable x:

$$z = rx\,, \tag{10}$$

for r a constant to be determined later. Equation (7) becomes

$$r^3x^3 + prx + q = 0\,. \tag{11}$$

Provided that we choose r such that $r \neq 0$, we may divide (11) by r^3, obtaining

$$x^3 + \frac{p}{r^2}x + \frac{q}{r^3} = 0\,. \tag{12}$$

By the choice

$$r = \sqrt[3]{q}\,, \tag{13}$$

which is valid (for use in (12)) whenever $q \neq 0$, the problem is reduced to the *canonical form*

$$x^3 + \gamma x + 1 = 0\,, \qquad \gamma = \frac{p}{\sqrt[3]{q^2}}\,, \tag{14}$$

where p and q are as above. In terms of the original coefficients of (1), we have

$$p = \frac{b}{e} - \frac{a^2}{3e^2}\,, \qquad q = \frac{c}{e} - \frac{ab}{3e^2} + \frac{2}{27}\frac{a^3}{e^3}\,. \tag{15}$$

In the special case $q = 0$, equation (7) becomes

$$z(z^2 + p) = 0\,, \tag{16}$$

where z and p are as defined above; this problem is solved trivially. One solution is $z = 0$, valid for any p. The other solutions are $z = \pm\sqrt{-p}$. These solutions are real when $p \leq 0$; for $p > 0$, they are complex conjugates. The only bifurcation point exists at $p = 0$, $z = 0$. For the rest of this study, we will concentrate on the general case $q \neq 0$.

Taking into account that with the exception of the special case $q = 0$, all cubic equations with real coefficients can be transformed to the canonical form (14), we conclude that all qualitative singularities of the solutions of the cubic equation (in the general case) are determined by the only real parameter γ.

Bifurcation analysis of the canonical cubic equation

In the following, we perform bifurcation analysis using the canonical equation (14). Let us denote the solutions of the cubic equation (14) by $x_1(\gamma)$, $x_2(\gamma)$ and $x_3(\gamma)$. These solutions represent the different branches of the function $x(\gamma)$, defined in implicit form by the equation

$$F\,(x, \gamma) \equiv x^3 + \gamma x + 1 = 0\,.$$

The solutions are visualized in Figure 1.

Following the approach presented in Banichuk et al. [1] (for a summary, see the section titled Bifurcation of continuous systems, below), we note that according to the implicit function theorem, the bifurcation points of the equation $F(x, \gamma) = 0$ are given by the conditions

$$F\,(x, \gamma) = x^3 + \gamma x + 1 = 0\,, \qquad \frac{\partial F(x, \gamma)}{\partial x} = 3x^2 + \gamma = 0\,. \tag{17}$$

In other words, the point being sought is a solution of the problem, i.e. $F(x, \gamma) = 0$, while it also violates the uniqueness criterion of the theorem. This implies that a bifurcation takes place, because at that point, a unique local representation of $x = x(\gamma)$ does not exist.

The only solution of equations (17) is

$$x_* = \frac{1}{\sqrt[3]{2}}\,, \qquad \gamma_* = -\frac{3}{\sqrt[3]{2^2}}\,. \tag{18}$$

Thus we have a unique value $\gamma = \gamma_*$ of the problem parameter, for which bifurcation takes place. Consequently, we have one (or three) real solutions of equation (1) when $\gamma < \gamma_*$, and for $\gamma > \gamma_*$, we respectively have three (or one) real solutions.

Because the cubic equation (14) has a unique real solution $x = -1$ for $\gamma = 0 > \gamma_*$, we conclude that there exists only one real solution for $\gamma > \gamma_*$, denoted $x_1(\gamma)(\gamma > \gamma_*)$. For $\gamma < \gamma_*$, there exist three real solutions denoted $x_1(\gamma)$, $x_2(\gamma)$ and $x_3(\gamma)$.

Note also that $x = 0$ is not a solution of equation $F(x, \gamma) = 0$ in (17) for any value of γ, and consequently the curves $x_i(\gamma)$, $i = 1, 2, 3$, never cross the line $x = 0$. Each of them must thus lie in one of the half-planes $x > 0$ or $x < 0$ of the plane (x, γ). In particular, we have $x_1(\gamma) < 0$ in the interval $-\infty < \gamma < \infty$. At the same time

$$x_2\,(\gamma_*) = x_3\,(\gamma_*) = \frac{1}{\sqrt[3]{2}} \tag{19}$$

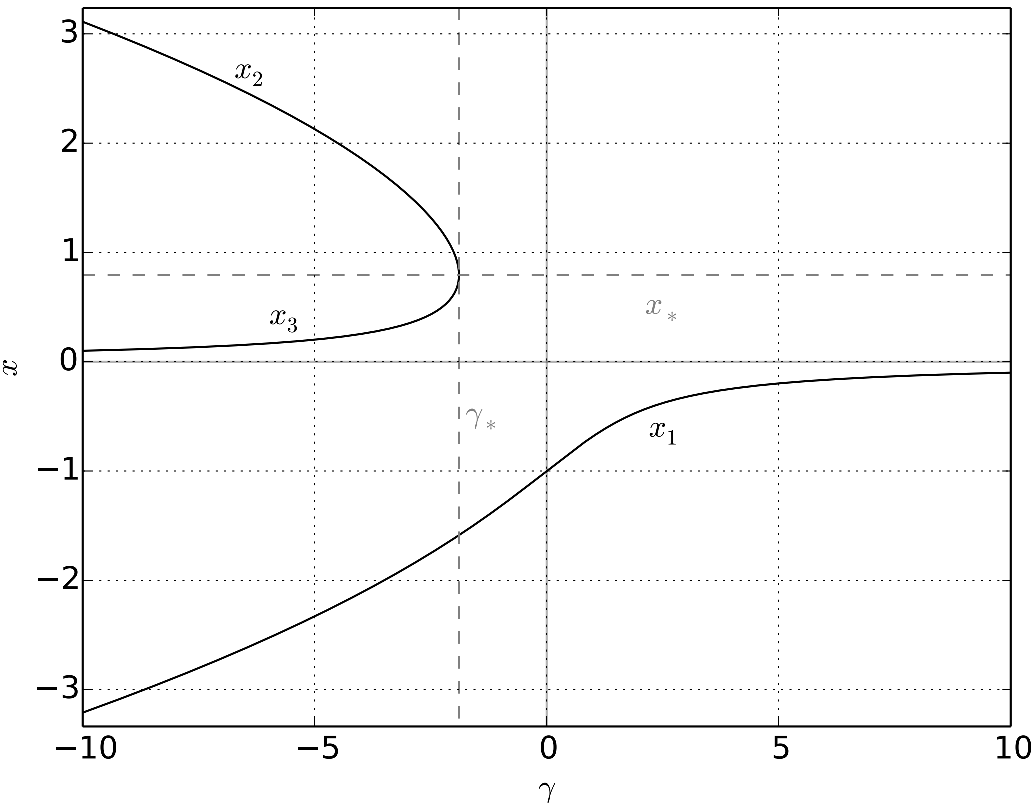

Figure 1. The real-valued solutions of the canonical cubic equation $x^3 + \gamma x + 1 = 0$ as a function of the (real) parameter γ, computed numerically using the trigonometric algorithm given in [9]. The unique bifurcation point (γ_*, x_*) given by equation (18) is indicated. Note that the curves never cross the line $x = 0$. The asymptotic behaviour far away from the origin is given by expressions (25).

at the bifurcation value $\gamma = \gamma_*$, and consequently, the functions $x_2(\gamma)$ and $x_3(\gamma)$ are always positive. As real-valued functions, they exist in the interval $-\infty < \gamma < \gamma_*$, ending at the bifurcation point.

Qualitative analysis and asymptotic expressions

To present qualitative analysis of the solutions $x_i(\gamma)$, we use the derivative of an implicitly defined function, obtaining

$$\frac{\mathrm{d}x_i(\gamma)}{\mathrm{d}\gamma} = -\frac{\dfrac{\partial F}{\partial \gamma}}{\dfrac{\partial F}{\partial x}} = -\frac{x_i}{3x_i^2 + \gamma}\,, \qquad i = 1, 2, 3\,. \tag{20}$$

Note that in accordance with (17) and (18), the interval $\gamma \in (-\infty, \infty)$ is divided by the value $\gamma = \gamma_*$ into two parts $(-\infty, \gamma_*)$ and (γ_*, ∞), in each of which the sign of $\partial F/\partial x$ does not vary (due to continuity, and the only zero of the derivative being located at γ_*). Observe that

$$\frac{\partial F}{\partial x} = 3x^2 > 0\,, \qquad \forall x \in \mathbb{R}$$

for $\gamma = 0 > \gamma_*$. Consequently, $\partial F/\partial x > 0$ when $\gamma \in (\gamma_*, \infty)$. But in the interval $(-\infty, \gamma_*)$, the sign of $\partial F/\partial x$ depends on the considered function $x_i(\gamma)$.

In particular, the value $\partial F/\partial x$ cannot be equal to zero for the function $x_1(\gamma)$ (one of the three solutions does not participate in the bifurcation; we denote that one as $x_1(\gamma)$), and for this function, the inequality $\partial F/\partial x > 0$ holds for any value of γ, because $\partial F/\partial x = 3x^2 > 0$ for $\gamma = 0 > \gamma_*$.

Because $x_1(\gamma) < 0$ and $\partial F/\partial x > 0$ for $x_1(\gamma)$, from equation (20) we arrive at the conclusion that $x_1(\gamma)$ is a monotonically increasing function of γ in the interval $-\infty < \gamma < \infty$. In a similar manner, we see that the function $x_2(\gamma)$ is a monotonically decreasing function, and $x_3(\gamma)$ is a monotonically increasing function of the parameter γ when $-\infty < \gamma < \gamma_*$.

Finally, let us consider the asymptotic behaviour of equation (14). We will find that in the asymptotic range, short, explicit analytical expressions for the solution curves x_1, x_2 and x_3 can be obtained as functions of γ. Let us first look for solutions where $|x| \ll 1$. In this case, we may drop the x^3 term (now being the cube of a small quantity), leading to the asymptotic equation

$$\gamma x + 1 = 0\,, \quad |x| \ll 1\,, \tag{21}$$

whence the asymptotic solution is

$$x \approx -\frac{1}{\gamma}\,, \quad |\gamma| \gg 1\,. \tag{22}$$

In rewriting the condition of validity, we have used the fact that by the form of the solution (22), the original assumption $|x| \ll 1$ is equivalent with $|\gamma| \gg 1$.

Equation (22) is valid for both positive and negative large γ. For $\gamma > 0$, this must be the only asymptotic solution, because by the bifurcation analysis already performed, we know that (14) has only one real solution in this range.

Alternatively, it is possible that $|x| \gg 1$. Let us look for the corresponding asymptotic solutions. In this case, we may drop the constant term in (14), leading to

$$x(x^2 + \gamma) = 0\,, \quad |x| \gg 1\,, \tag{23}$$

which has the solutions $x = 0$ (inconsistent with the condition of validity of (23), hence rejected) and

$$x \approx \pm\sqrt{-\gamma}\,, \quad -\gamma \gg 1\,. \tag{24}$$

We have again rewritten the condition of validity, this time using the fact that by the form of (24), the assumption $|x| \gg 1$ (and real) is equivalent with $\gamma < 0$, $-\gamma \gg 1$. Hence these two additional asymptotic solutions are valid for large negative γ.

Three asymptotic solutions have been found. Because the original equation is a cubic, no further solutions exist.

In conclusion, to summarize the asymptotic results (and identifying the representations against Figure 1), we have

$$\begin{gathered} x_1(\gamma) \approx -\frac{1}{\gamma}\,, \qquad \gamma \gg 1\,, \\ x_1(\gamma) \approx -\sqrt{-\gamma}\,, \qquad x_2(\gamma) \approx \sqrt{-\gamma}\,, \qquad x_3(\gamma) \approx -\frac{1}{\gamma}\,, \qquad -\gamma \gg 1\,. \end{gathered} \tag{25}$$

Parametric representation of the solutions

In this section, we will derive exact, analytical parametric representations $\gamma(\tau)$, $x_1(\tau)$, $x_2(\tau)$ and $x_3(\tau)$ as functions of a real parameter τ. As is well known, the cubic equation (14) can be factored as

$$x^3 + \gamma x + 1 = (x - x_1)(x - x_2)(x - x_3) = 0\,, \tag{26}$$

where x_1, x_2 and x_3 are the roots. Equating equal powers of x, we have the equations

$$\begin{cases} x_1 + x_2 + x_3 = 0\,, \\ x_1(x_2 + x_3) + x_2 x_3 = \gamma\,, \\ x_1 x_2 x_3 = -1\,. \end{cases} \tag{27}$$

As was previously shown, the function $x_1(\gamma)$ is negative and monotonically increasing in the interval $\gamma \in (-\infty, \infty)$.

The functions $x_2(\gamma)$ and $x_3(\gamma)$ are, respectively, positive monotone decreasing and increasing in $\gamma \in (-\infty, \gamma_*)$, and complex conjugate in $\gamma \in (\gamma_*, \infty)$.

The complex conjugate property follows from the last equation in (27), our observations above, and the fundamental theorem of algebra. It is known that a polynomial of the kth degree always admits exactly k complex-valued roots (which in general need not be distinct). Above, we already observed that in $\gamma \in (\gamma_*, \infty)$, there is only one real root, x_1. Hence in this interval, x_2 and x_3 must be complex, with a nonzero imaginary part. Then, because x_1 is always real, the only way the last equation in (27) can hold is to have $x_3 = \mathrm{conj}(x_2)$, so that Im $(x_1 x_2 x_3) = 0$; which is required because the imaginary part of the right-hand side is zero.

Let us introduce the new variables σ and s, defined by

$$s = x_2 + x_3\,, \qquad \sigma = x_2 x_3\,, \qquad \sigma, s \in \mathbb{R}\ \forall\, \gamma \in (-\infty, \infty)\,. \tag{28}$$

Using (28) we can rewrite (27) as

$$\begin{cases} x_1 + s = 0\,, \\ x_1 s + \sigma = \gamma\,, \\ x_1 \sigma = -1\,. \end{cases} \tag{29}$$

As x_1 is always negative, let us introduce a new positive variable $\tau = -x_1$ ($\tau > 0$) and eliminate x_1 from equations (29). From the first equation we immediately have $s = \tau$, and the other two equations become

$$\sigma = \tau^2 + \gamma\,, \qquad \sigma = \frac{1}{\tau}\,, \qquad \tau > 0\,. \tag{30}$$

We thus obtain $\tau^2 + \gamma = 1/\tau$, and find the following parametric representations for σ and γ:

$$\sigma(\tau) = \tau^2 + \gamma\,, \qquad \gamma(\tau) = \frac{1 - \tau^3}{\tau}\,, \qquad \tau > 0\,. \tag{31}$$

Note that $\mathrm{d}\gamma/\mathrm{d}\tau < 0$, and thus $\gamma(\tau)$ is a monotonically decreasing function of the parameter τ. In addition, the values $\tau \ll 1$ correspond to $\gamma \gg 1$, and $\gamma \to -\infty$ when $\tau \gg 1$ (i.e. when $x_1 = -\tau \to -\infty$).

We also obtain a parametric representation for $x_1(\gamma)$ for $\gamma \in (-\infty, \infty)$:

$$x_1(\tau) = -\tau\,, \qquad \gamma(\tau) = \frac{1-\tau^3}{\tau}\,, \qquad \tau > 0\,. \tag{32}$$

It follows from (32) that $\tau = 1$ for $\gamma = 0$, and $x_1(\gamma = 0) = -1$. Thus the curve $x_1(\gamma)$ passes through the point $(0, -1)$ in the plane (γ, x). Taking the derivative of a parametrically defined function (via the chain rule), we have

$$\frac{\mathrm{d}x_1}{\mathrm{d}\gamma} = \frac{\dfrac{\mathrm{d}x_1}{\mathrm{d}\tau}}{\dfrac{\mathrm{d}\gamma}{\mathrm{d}\tau}} = \frac{\tau^2}{1+2\tau^3} > 0\,, \qquad \tau > 0\,, \tag{33}$$

and consequently, $x_1(\gamma)$ is a monotonically increasing function of the parameter γ for $\gamma \in (-\infty, \infty)$.

It is possible to use the parametric representation for an alternative starting point for asymptotic analysis. Using (32), we study the asymptotic behaviour of the solution $x_1(\gamma)$ for asymptotic values of parameter γ, i.e. $|\gamma| \gg 1$. When $\gamma \to +\infty$, the right-hand side of the second equation in (32) must be dominated by the first term (i.e. τ is small), and hence dropping the second term and using the result in the first equation, we immediately have

$$x_1(\gamma) = -\frac{1}{\gamma}\,, \qquad \gamma \gg 1\,. \tag{34}$$

Similarly, when $\gamma \to -\infty$, the right-hand side must be dominated by the second term (i.e. τ is large), and hence $\gamma(\tau) = -\tau^2 < 0$ ($\tau \approx \sqrt{-\gamma}$, $\gamma < 0$) and we find asymptotic behaviour for $x_1(\gamma)$ at $\gamma \to -\infty$ in the form

$$x_1(\gamma) \approx -\sqrt{-\gamma}, \qquad \gamma \to -\infty \tag{35}$$

These asymptotic expressions coincide with the corresponding asymptotic expressions in equation (25). The present analysis has the additional advantage that the solution curves are identified automatically.

Let us now return to the general analysis, and construct parametric representations also for $x_2(\gamma)$ and $x_3(\gamma)$. To this purpose, we rewrite the relations (28) taking into account $s = \tau$, $x_1 = -\tau$ and $\sigma = 1/\tau$, $\tau > 0$, from equation (30). We have the following relations:

$$x_2 + x_3 = \tau\,, \qquad x_2 x_3 = \frac{1}{\tau}\,, \qquad \tau > 0\,, \tag{36}$$

which are considered as a system of equations with respect to the variables x_2 and x_3. The solution of the system can be written in the form

$$x_{2,3}(\tau) = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - \frac{4}{\tau}}\right)\,. \tag{37}$$

Using the parametric representation for γ from equation (31), we find the parametric representations for $x_2(\gamma)$ and $x_3(\gamma)$:

$$x_{2,3}(\tau) = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - \frac{4}{\tau}}\right), \qquad \gamma(\tau) = \frac{1-\tau^3}{\tau}, \qquad \tau > 0. \tag{38}$$

For brevity we write $x_{2,3}$ instead of x_2 and x_3 in equations (37) and (38), where the plus and minus signs in the $\pm$ correspond, respectively, to x_2 and x_3.

Note that $x_2(\tau_*) = x_3(\tau_*)$ at $\tau = \tau_* = \sqrt[3]{2^2}$. At this point, the square root is equal to zero in the equations (37) and (38). As follows from (38), at this point we have

$$\gamma_* = \gamma(\tau_*) = -\frac{3}{\sqrt[3]{2^2}}, \qquad x_2(\tau_*) = x_3(\tau_*) = \frac{1}{\sqrt[3]{2}} \tag{39}$$

If $\tau > \tau_*$, then $\tau^2 - 4/\tau > 0$, and the functions $x_2(\gamma)$ and $x_3(\gamma)$ are real when $\gamma < \gamma_*$. In the case $\tau < \tau_*$, the inequalities $\gamma > \gamma_*$, $\tau^2 - 4/\tau < 0$ are satisfied, and the functions $x_2(\gamma)$ and $x_3(\gamma)$ are complex-conjugate when $\gamma > \gamma_*$.

The value γ_* in the equation (39) coincides with the value γ_* obtained with the help of bifurcation analysis of the cubic equation, given in equation (18).

Finally, using the parametric representations (38), we can evaluate the asymptotic behaviour of the functions $x_2(\gamma)$ and $x_3(\gamma)$ when $\gamma \to -\infty$. As was already pointed out above for equation (32), it follows that $\gamma \to -\infty$ when $\tau \to \infty$.

In the case of the plus sign in (38), the result is obtained trivially. For $\tau \gg 1$, we may drop the second term inside the square root, immediately obtaining

$$x_2(\tau) = \tau\,, \quad \tau \gg 1\,. \tag{40}$$

The case with the minus sign is more complicated, because the obvious leading-order term cancels out. In order to access the next term, we develop the square root into a Taylor series at $\tau \to \infty$. To do this, we change variables into

$$k = \frac{1}{\tau}\,, \tag{41}$$

develop the series at $k = k_0$, take the limit $k_0 \to 0^+$, and then transform back. In terms of k, equation (38) reads

$$x_{2,3}(k) = \frac{1}{2}\left(\frac{1}{k} \pm \sqrt{\frac{1}{k^2} - 4k}\right)\,. \tag{42}$$

After developing the square root into a Taylor series around $k = k_0$, and then letting $k_0 \to 0^+$, we have

$$x_{2,3}(k) = \frac{1}{2}\left(\frac{1}{k} \pm \frac{1}{k} \mp 2k^2 \mp 2k^5 + O(k^6)\right)\,. \tag{43}$$

The upper and lower signs correspond to each other. The representation for x_2 (upper signs) coincides with the results obtained above.

For x_3, we pick the lower signs. Taking just the leading-order term, we have

$$x_3(\tau) = \frac{1}{\tau^2}\,, \quad \tau \gg 1\,. \tag{44}$$

Summarizing, for $\tau \to \infty$, we have obtained the following asymptotic representations:

$$x_2(\tau) \approx \tau\,, \qquad x_3(\tau) \approx \frac{1}{\tau^2}\,, \qquad \tau \gg 1\,. \tag{45}$$

For $\tau \gg 1$, we have $\gamma \approx -\tau^2$ (again, $\tau \approx \sqrt{-\gamma}$, $\gamma < 0$), leading to

$$x_2(\gamma) \approx \sqrt{-\gamma}\,, \qquad x_3(\gamma) \approx -\frac{1}{\gamma}\,, \qquad \gamma \to -\infty\,. \tag{46}$$

Bifurcation of continuous systems

In this section, we will briefly look at the use of the implicit function theorem as a tool for bifurcation analysis for continuous mechanical systems. A more detailed treatment can be found in Banichuk at al. [1].

Many problems in continuous mechanical systems can be represented in the following general form:

$$\mathcal{L}(u(x), \lambda, \gamma) = \sum_{k=0}^{m} \sum_{\ell=0}^{n} \lambda^k \gamma^\ell \mathcal{L}_{k\ell}(u(x)) = 0 \,, \tag{47}$$

where γ is a real-valued loading parameter, λ is a spectral parameter (typically, eigenvalue to be determined), and $\mathcal{L}_{k\ell}$ are given differential operators applied to the behaviour function $u(x)$, defined in the domain Ω ($x \in \Omega$). Boundary conditions are considered as included in the differential operator $\mathcal{L}(u(x))$.

For example, γ may represent applied tension or compression in a vibration problem, or the axial drive speed in problems of axially moving materials. The eigenvalue λ is then the complex-valued eigenfrequency, and a representation of the form (47) is obtained after inserting the standard time-harmonic trial function into the original partial differential equation. The operators $\mathcal{L}_{k\ell}$ include only space differentiation, the time dependency having been factored out in the choice of the trial function.

Let the function $v(x)$ be the eigenfunction (corresponding to an eigenvalue λ) of the spectral problem

$$\mathcal{L}^*(v(x), \lambda, \gamma) = 0 \,, \tag{48}$$

which is adjoint to the problem (47). For simplicity, let us focus on the case where the eigenvalues are distinct.

Multiplying equation (47) by the adjoint eigenfunction $v(x)$ and integrating over Ω, we obtain a functional equation

$$\Phi(\lambda, J_{00}, \ldots, J_{mn}, \gamma) = \sum_{k=0}^{m} \sum_{\ell=0}^{n} \lambda^k \gamma^\ell J_{k\ell} = 0 \,, \tag{49}$$

where the functionals $J_{k\ell}$, $k = 0, 1, 2 \ldots, m$; $\ell = 0, 1, 2, \ldots, n$ are defined as

$$J_{k\ell}(v, u) = \int_\Omega v(x) \mathcal{L}_{k\ell}(u(x))\, \mathrm{d}\Omega \,. \tag{50}$$

Note that (49), which states that $\Phi(\lambda, \gamma) = 0$, can be interpreted as an implicit relation for $\lambda = \lambda(\gamma)$, determining a set of functions $\lambda_1(\gamma), \ldots, \lambda_m(\gamma)$ corresponding to the eigenvalue curves of the problem under study. (There are m (possibly complex-valued) eigenvalues, because (49), understood as a polynomial in the variable λ, is of degree m.)

It turns out that for the purposes of the following analysis, the fact that the values of the $J_{k\ell}$ depend on γ and λ (because they depend on the eigenfunctions u and v), does not matter (this is shown in Banichuk et al. [1]). Hence we may write $\Phi = \Phi(\lambda, \gamma)$.

A bifurcation of $\lambda = \lambda(\gamma)$ can occur for some values

$$\lambda = \lambda^* \,, \quad \gamma = \gamma^* \,, \tag{51}$$

at which the condition for the uniqueness of the local representation, in the theorem on implicit functions, is violated. The theorem on implicit functions (see e.g. Rektorys [10]) states that a unique solution of

$$\Phi(\lambda, \gamma) = 0 \tag{52}$$

exists in a small neighbourhood of the point $(\lambda, \gamma) = (\widetilde{\lambda}, \widetilde{\gamma})$, if $\partial\Phi/\partial\lambda \neq 0$ at the point $(\widetilde{\lambda}, \widetilde{\gamma})$.

Therefore, in other words, the bifurcation values λ^* and γ^* can be found with the help of the equations

$$\Phi(\lambda^*, \gamma^*) = 0\,, \qquad \frac{\partial\Phi\,(\lambda^*, \gamma^*)}{\partial\lambda} = 0\,. \tag{53}$$

The first equation in (53) requires that the point being sought is a solution of the problem. The second equation requires that the uniqueness of the local representation of $\lambda = \lambda(\gamma)$ is violated: in other words, the solution experiences a bifurcation.

Let us denote by

$$(\lambda_1^*, \gamma_1^*)\,,\ (\lambda_2^*, \gamma_2^*)\,, \ldots \tag{54}$$

the solutions of the system of equations (53), representing points on the (λ, γ) plane, and investigate the behaviour of the functions $\lambda_i = \lambda_i(\gamma)$ (where $i = 1, 2, \ldots, m$) in a small vicinity of the bifurcation points $(\lambda_k^*, \gamma_k^*)$. For brevity, the subscript indices of the considered functions and points will be omitted.

Let us represent the function $\Phi(\lambda, \gamma)$ as a series expansion developed at an arbitrary bifurcation point $(\lambda, \gamma) = (\lambda^*, \gamma^*)$:

$$\begin{gathered}\Phi(\lambda, \gamma) = \Phi\,(\lambda^*, \gamma^*) + \frac{\partial\Phi\,(\lambda^*, \gamma^*)}{\partial\lambda}\,[\lambda - \lambda^*] \\ + \frac{\partial\Phi\,(\lambda^*, \gamma^*)}{\partial\gamma}\,[\gamma - \gamma^*] + \frac{1}{2}\frac{\partial^2\Phi\,(\lambda^*, \gamma^*)}{\partial\lambda^2}\,[\lambda - \lambda^*]^2 \\ + \frac{\partial^2\Phi\,(\lambda^*, \gamma^*)}{\partial\lambda\partial\gamma}\,[\lambda - \lambda^*]\,[\gamma - \gamma^*] + \frac{1}{2}\frac{\partial^2\Phi\,(\lambda^*, \gamma^*)}{\partial\gamma^2}\,[\gamma - \gamma^*]^2 + \ldots\end{gathered} \tag{55}$$

A bifurcation point satisfies the relations (53), i.e. $\Phi\,(\lambda^*, \gamma^*) = 0$, and $\partial\Phi/\partial\lambda = 0$. This eliminates the first two terms on the right-hand side. Retaining only the lowest order non-zero terms, we are left with

$$\begin{gathered}\Phi\,(\lambda, \gamma) = \frac{\partial\Phi\,(\lambda^*, \gamma^*)}{\partial\gamma}\,[\gamma - \gamma^*] + \frac{\partial^2\Phi\,(\lambda^*, \gamma^*)}{\partial\lambda\partial\gamma}\,[\lambda - \lambda^*]\,[\gamma - \gamma^*] + \\ + \frac{1}{2}\frac{\partial^2\Phi\,(\lambda^*, \gamma^*)}{\partial\lambda^2}\,[\lambda - \lambda^*]^2 + \ldots\end{gathered} \tag{56}$$

Let us now represent the behaviour of the function $\lambda = \lambda(\gamma)$ in the vicinity of the bifurcation point (λ^*, γ^*) in the form

$$\lambda\,(\gamma) = \lambda^* + \alpha\,[\gamma - \gamma^*]^\varepsilon + \ldots \tag{57}$$

where α and ε are unknown constants to be determined with the help of the condition $\Phi\,(\lambda, \gamma) = 0$. By substituting (57) into (56), equation (56) is transformed into

$$\widetilde{\Phi} = \widetilde{\Phi}\,(\gamma - \gamma^*) \equiv 0\,, \tag{58}$$

which must be satisfied identically. Here $\widetilde{\Phi}(\gamma - \gamma^*)$ denotes the series expansion (56), expressed in terms of the only remaining perturbation variable $\gamma - \gamma^*$, after $\lambda - \lambda^*$ has been eliminated using the representation (57). Explicitly, we have

$$\widetilde{\Phi}\,(\gamma - \gamma^*) = \frac{\partial\Phi\,(\lambda^*, \gamma^*)}{\partial\gamma}\,[\gamma - \gamma^*] + \alpha\frac{\partial^2\Phi\,(\lambda^*, \gamma^*)}{\partial\lambda\partial\gamma}\,[\gamma - \gamma^*]^{1+\varepsilon} +$$

$$\frac{\alpha^2}{2}\frac{\partial^2\Phi\left(\lambda^*,\gamma^*\right)}{\partial\lambda^2}\left[\gamma-\gamma^*\right]^{2\varepsilon}+\dots\equiv 0\,. \tag{59}$$

To find an approximation in the lowest nonzero order, we try picking different values for ε to match the orders of different terms in (59). Once a value is chosen, we omit any remaining higher-order terms and analyze the result.

There are three possibilities. First, ε can be chosen to match the order of the first two terms by taking $\varepsilon = 0$. This however eliminates them in favour of the third term, which becomes a constant. If this constant is nonzero, this is not a solution of (59). The second possibility is to match the orders of the last two terms with $\varepsilon = 1$, eliminating them and leaving only the first term. If the coefficient $\partial\Phi/\partial\gamma \neq 0$, this is not a solution of (59).

The final possibility is to match the orders of the first and third terms with $2\varepsilon = 1$, eliminating the second term. This is the typical general case. It is valid when

$$\frac{\partial\Phi\left(\lambda^*,\gamma^*\right)}{\partial\gamma}\neq 0\,,\qquad \frac{\partial^2\Phi\left(\lambda^*,\gamma^*\right)}{\partial\lambda^2}\neq 0\,. \tag{60}$$

If either or both of these terms vanish, the analysis must be repeated retaining the lowest-order terms for that particular case. Inserting $\varepsilon = 1/2$ into (59) and dropping the higher-order terms obtains

$$\Phi\left(\lambda,\gamma\right)=\frac{\partial\Phi\left(\lambda^*,\gamma^*\right)}{\partial\gamma}\left[\gamma-\gamma^*\right]+\frac{\alpha^2}{2}\frac{\partial^2\Phi\left(\lambda^*,\gamma^*\right)}{\partial\lambda^2}\left[\gamma-\gamma^*\right]+\dots=0\,, \tag{61}$$

which is satisfied identically in the lowest nonzero order by

$$\alpha^2=-2\left(\frac{\partial\Phi\left(\lambda^*,\gamma^*\right)}{\partial\gamma}\right)\left(\frac{\partial^2\Phi\left(\lambda^*,\gamma^*\right)}{\partial\lambda^2}\right)^{-1}\,. \tag{62}$$

With $\varepsilon = 1/2$ equation (57) gives the asymptotic representation

$$\lambda(\gamma)=\lambda^*+\alpha\sqrt{\gamma-\gamma^*}\,,\qquad |\gamma-\gamma^*|\ll 1\,, \tag{63}$$

provided that the inequalities (60) are satisfied. As is pointed out in Banichuk et al. [1], this square root shape of $\lambda = \lambda(\gamma)$, valid in a small neighbourhood of any bifurcation point, holds for all systems in the considered class.

Transverse free vibrations of an axially compressed beam

Let us finish with a concrete example from mechanics. We consider the problem of small transverse free vibrations of an axially compressed Euler–Bernoulli beam of length ℓ, with its ends resting on simple supports. Refer to Figure 2.

This is the classical prototype of a family of problems whose gyroscopic versions appear in the stability analysis of process industry applications, such as paper making. See e.g. the book by Banichuk et al. [2].

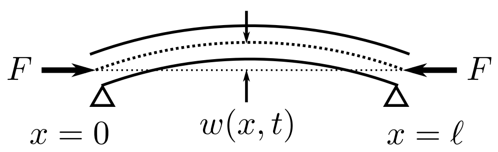

Figure 2. Axially compressed Euler–Bernoulli beam of length ℓ, with ends resting on simple supports.

For simplicity, we restrict our consideration to the classical case where the beam is made of homogeneous material having Young's modulus E, and has a cross-section of constant shape and area. The partial differential equation describing the dynamics of small transverse displacement $w = w(x, t)$ of the beam mid-surface in this case reads

$$\rho \frac{\partial^2 w}{\partial t^2} + F \frac{\partial^2 w}{\partial x^2} + EI \frac{\partial^4 w}{\partial x^4} = 0 \qquad \text{in } 0 < x < \ell \,, \tag{64}$$

where ρ is the linear density of the beam ($[\rho] = \mathrm{kg/m}$), I is the moment of inertia (second moment of area) of its cross-section, and $F > 0$ is the compressive axial force.

From the viewpoint of the transverse displacement, the compressive axial force appears as a transverse projection term that tends to amplify any existing local curvature of the beam, leading to elastic instability at a critical value of the compressive loading.

Let us transform the problem into dimensionless coordinates. Let $x' \equiv x/\ell$ and $t' \equiv t/\tau$, where τ is a characteristic time (a dimensional arbitrary constant whose value can be chosen later, $[\tau] = \mathrm{s}$).

Let us represent also the transverse displacement in a dimensionless form, $w' = w/h$, where h is an arbitrary constant. Since each term in the equation is linear in w, we may immediately cancel h from the equation. In other words, for free vibrations (no transverse loading), transverse scaling (and hence the maximum amplitude of the small vibrations) does not affect the dynamics.

In terms of the dimensionless variables, equation (64) transforms into

$$\frac{\rho}{\tau^2} \frac{\partial^2 w'}{\partial t'^{\,2}} + \frac{F}{\ell^2} \frac{\partial^2 w'}{\partial x'^{\,2}} + \frac{EI}{\ell^4} \frac{\partial^4 w'}{\partial x'^{\,4}} = 0 \qquad \text{in } 0 < x' < 1 \,. \tag{65}$$

Multiplying the equation by ℓ^4/EI, we have

$$\frac{\rho\, \ell^4}{EI\, \tau^2} \frac{\partial^2 w'}{\partial t'^{\,2}} + \frac{F \ell^2}{EI} \frac{\partial^2 w'}{\partial x'^{\,2}} + \frac{\partial^4 w'}{\partial x'^{\,4}} = 0 \qquad \text{in } 0 < x' < 1 \,. \tag{66}$$

Inserting the standard time-harmonic trial function

$$w'(x', t') = \exp(i \omega_0 t')\, u(x') \,, \tag{67}$$

where $i = \sqrt{-1}$, ω_0 is a dimensionless angular frequency, and $u(x')$ is dimensionless, we obtain

$$\exp(i\omega_0 t') \left[\frac{\mathrm{d}^4 u}{\mathrm{d}x'^{\,4}} + \frac{F\ell^2}{EI} \frac{\mathrm{d}^2 u}{\mathrm{d}x'^{\,2}} - \frac{\rho\, \ell^4}{EI\, \tau^2} \omega_0^2 u \right] = 0 \qquad \text{in } 0 < x' < 1 \,. \tag{68}$$

This equation holds for all t' if and only if the expression in the brackets vanishes:

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x'^{\,4}} + \gamma \frac{\mathrm{d}^2 u}{\mathrm{d}x'^{\,2}} - \omega^2 u = 0 \qquad \text{in } 0 < x' < 1 \,, \tag{69}$$

where we have defined the dimensionless compressive loading γ, and the scaled dimensionless angular frequency ω:

$$\gamma \equiv \frac{F\ell^2}{EI} \,, \qquad \omega \equiv \frac{\rho\, \ell^4}{EI\, \tau^2} \omega_0^2 \,. \tag{70}$$

The simply supported (pinned, hinged) boundary conditions are expressed as

$$\left(\frac{\mathrm{d}^2 u}{\mathrm{d}x'^{\,2}} \right)_{x'=0} = \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x'^{\,2}} \right)_{x'=1} = 0 \,, \qquad u(0) = u(1) = 0 \,. \tag{71}$$

Let us now investigate the asymptotic behaviour of ω as a function of the loading parameter γ, i.e. $\omega = \omega(\gamma)$, using the perturbation method explained in the previous section. In this example, ω plays the role of the spectral parameter λ.

Let us develop the problem into a variational form. To do this, we take the complex-valued L^2 inner product of equation (69) and u. Explicitly, we multiply equation (69) by $u^*(x')$, which is the complex conjugate of $u(x')$, and integrate over the domain $\Omega = \{x' : 0 < x' < 1\}$. We have

$$\int_0^1 u^* \frac{\mathrm{d}^4 u}{\mathrm{d}x'^4}\,\mathrm{d}x' + \int_0^1 u^* \frac{\mathrm{d}^2 u}{\mathrm{d}x'^2}\,\mathrm{d}x' - \omega^2 \int_0^1 u u^*\,\mathrm{d}x' = 0\,. \tag{72}$$

In the considered case, $u^* = u$ (i.e. u is real-valued), because the problem (69), (71) is self-adjoint with respect to the complex-valued L^2 inner product on Ω. This is easily shown by applying integration by parts to (72) until all differentiations have been moved to operate on u^*, and using the boundary conditions (71) to note that the boundary terms vanish.

We note in passing that this is the main difference between this prototype problem and those encountered in process industry applications, since problems involving gyroscopic terms are not self-adjoint; there u will typically be complex-valued. In this example, for simplicity we have chosen to examine the classical self-adjoint case.

We rewrite the left-hand side of (72) as an implicit function Ψ:

$$\Psi\,(\omega, a, c, d, \gamma) = -a\omega^2 - \gamma c + d = 0\,, \tag{73}$$

where the functionals a, c and d are given by

$$a = \int_0^1 u u^*\,\mathrm{d}x' > 0\,,$$

$$c = -\int_0^1 u^* \frac{\mathrm{d}^2 u}{\mathrm{d}x'^2}\,\mathrm{d}x' = \int_0^1 \frac{\mathrm{d}u}{\mathrm{d}x'}\frac{\mathrm{d}u^*}{\mathrm{d}x'}\,\mathrm{d}x' > 0\,,$$
$$d = \int_0^1 u^* \frac{\mathrm{d}^4 u}{\mathrm{d}x'^4}\,\mathrm{d}x' = \int_0^1 \frac{\mathrm{d}^2 u}{\mathrm{d}x'^2}\frac{\mathrm{d}^2 u^*}{\mathrm{d}x'^2}\,\mathrm{d}x' > 0\,.$$

The last forms of c and d follow by integration by parts and the boundary conditions (71). Each integrand is the squared complex norm of a quantity, $\|z\|^2 = zz^*$; this gives the result that a, c and d are positive for any $u \not\equiv 0$.

It turns out that in general, when applying the method explained in the previous section, for bifurcation analysis purposes we may treat Ψ as a function of only two variables, $\Psi = \Psi(\omega, \gamma)$; details can be found in Banichuk et al. [1].

The functionals a, c and d can be expressed with the help of eigenmodes of free vibrations of the beam,

$$u_k(x) = B_k \sin(k\pi x)\,, \qquad k = 1, 2, \dots\,, \tag{74}$$

where B_k is an arbitrary constant. To obtain (74), we have used the boundary conditions (71). For the kth mode, we have

$$a_k = \int_0^1 (u_k(x))^2\,\mathrm{d}x = \frac{1}{2}B_k^2\,,$$

$$c_k = \int_0^1 \left(\frac{\mathrm{d}u_k(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x = \frac{1}{2}k^2\pi^2 B_k^2\,, \tag{75}$$
$$d_k = \int_0^1 \left(\frac{\mathrm{d}^2u_k(x)}{\mathrm{d}x^2}\right)^2 \mathrm{d}x = \frac{1}{2}k^4\pi^4 B_k^2\,.$$

Inserting the expressions (75) into (73) and multiplying the equation by $2/B_k^2$, we have

$$\frac{2\Psi}{B_k^2} = -\omega^2 - \gamma k^2\pi^2 + k^4\pi^4 = 0\,. \tag{76}$$

From the general explanation in the previous section, the bifurcation points $(\omega, \gamma) = (\omega_k^*, \gamma_k^*)$ must satisfy $\Psi = 0$ and $\partial\Psi/\partial\omega = 0$. From (76), the condition on the derivative gives $-2\omega = 0$ and thus $\omega_k^* = 0$. Inserting this into (76) and solving for $\Psi = 0$ gives $\gamma_k^* = k^2\pi^2$. Summarizing, the bifurcation points are located at

$$\omega_k^* = 0\,, \qquad \gamma_k^* = k^2\pi^2\,. \tag{77}$$

Using the general formulas (62) and (63), the asymptotic behaviour of ω in the vicinity of the bifurcation points is then described by the expressions

$$\omega_k = \omega_k(\gamma) = \pm\alpha\sqrt{\gamma - k^2\pi^2}\,, \qquad \left|\gamma - k^2\pi^2\right| \ll 1\,, \tag{78}$$

where the value of the coefficient α is given by the relation

$$\alpha^2 = -2\left(\frac{\partial\Psi\left(\omega_k^*, \gamma_k^*\right)}{\partial\gamma}\right)\left(\frac{\partial^2\Psi\left(\omega_k^*, \gamma_k^*\right)}{\partial\omega^2}\right)^{-1} = -k^2\pi^2\,, \tag{79}$$

and thus

$$\alpha = ik\pi\,. \tag{80}$$

Inserting (80) into (78), we obtain the final result

$$\omega_k = \pm k\pi\sqrt{k^2\pi^2 - \gamma}\,. \tag{81}$$

Equation (81) tells us that in terms of the loading parameter γ, when $\gamma < k^2\pi^2$, the scaled dimensionless angular frequencies ω for the eigenmode k, i.e. ω_k, are purely real. This corresponds to stable vibrations of the system (for the mode k). When $\gamma > k^2\pi^2$, the frequencies become purely imaginary. One solution in the pair then correponds to an exponentially damped mode, while the other corresponds to an exponentially growing mode; in other words, elastic stability is lost.

Finally, note that this example is one of those cases where the angular frequencies ω are initially purely real. As Bolotin [3], pp. 99–100, cautions, in such cases one needs to be careful when applying a free-vibration analysis to make conclusions about stability.

In reality, most if not all mechanical systems exhibit some finite amount of dissipation. If the system is acted upon by conservative forces only (such as in this example), the Kelvin–Tait–Chataev theorem (see [5], p. 163) guarantees that the addition of small but finite dissipation to the idealized model (which does not account for dissipation) does not change the stability conclusions that are obtained by a free vibration analysis.

However, if the system is non-conservative, no analogous theorem can exist; perhaps the most famous counterexample demonstrating this is the double pendulum loaded by a follower force, investigated by Ziegler [17]. For non-conservative systems, the addition of small but finite dissipation may significantly decrease the critical value of the loading.

Conclusion

In this paper we analyzed some bifurcation problems. In problems of mechanics, especially in the context of stability analysis, polynomial equations and implicit functions are encountered; these were the focus of this study.

Cubic polynomial equations were analyzed, presenting new analytical insights concerning parametric analytical representation of the solutions, and asymptotic behaviour of the solutions for large values of the parameter in the canonical form.

The implicit function theorem was considered for bifurcation analysis for continuous mechanical systems. An asymptotic property for the behaviour of the natural frequency curves in the small vicinity of each bifurcation point was shown, covering all systems in the considered class.

The free vibration problem of a stationary compressed beam was presented as an example. This is the classical prototype of a family of problems whose gyroscopic versions appear in the stability analysis of process industry applications, such as paper making.

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References

[1] Banichuk, N., Barsuk, A., Jeronen, J., Neittaanmäki, P., & Tuovinen, T. (2016). On bifurcation analysis of implicitly given functionals in the theory of elastic stability. In P. Neittaanmäki, S. Repin, & T. Tuovinen (Eds.), *Mathematical Modeling and Optimization of Complex Structures, dedicated to Prof. Nikolay Banichuk for his 70th anniversary* (pp. 175–188).: Springer. ISBN 978-3-319-23563-9.

[2] Banichuk, N., Jeronen, J., Neittaanmäki, P., Saksa, T., & Tuovinen, T. (2014). *Mechanics of moving materials*, volume 207 of *Solid mechanics and its applications*. Springer. ISBN: 978-3-319-01744-0 (print), 978-3-319-01745-7 (electronic).

[3] Bolotin, V. V. (1963). *Nonconservative Problems of the Theory of Elastic Stability*. New York: Pergamon Press.

[4] Bolotin, V. V. (1964). *The Dynamic Stability of Elastic Systems*. Holden–Day, Inc. Translated from the Russian (1956) and German (1961) editions.

[5] Kirillov, O. N. (2013). *Nonconservative Stability Problems of Modern Physics*. de Gruyter. ISBN 978-3-11-027043-3.

[6] Lacarbonara, W. (2013). *Nonlinear structural mechanics: theory, dynamical phenomena and modeling*. Springer Science & Business Media. ISBN 978-1-4419-1275-6.

[7] Neittaanmäki, P. & Ruotsalainen, K. (1985). On the numerical solution of the bifurcation problem for the sine-Gordon equation. *Arab Journal of Mathematics*, 6(1 and 2).

[8] Nečas, J., Lehtonen, A., & Neittaanmäki, P. (1987). On the construction of Lusternik–Schnirelmann critical values with application to bifurcation problems. *Applicable Analysis*, 25(4), 253–268.

[9] Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. (2007). *Numerical Recipes: The Art of Scientific Computing*. New York: Cambridge University Press, 3rd edition. ISBN 978-0-521-88068-8.

[10] Rektorys, K. (1969). *Survey of Applicable Mathematics*. Iliffe Books London Ltd.

[11] Seydel, R. (2009). *Practical bifurcation and stability analysis*, volume 5 of *Interdisciplinary Applied Mathematics*. Springer Science & Business Media. ISBN 978-1-4419-1739-3.

[12] Thompson, J. M. T. (1979). Stability predictions through a succession of folds. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 292(1386), 1–23.

[13] Thompson, J. M. T. (1982). *Instabilities and catastrophes in science and engineering*. Wiley.

[14] Thomsen, J. J. (2013). *Vibrations and stability: advanced theory, analysis, and tools*. Springer Science & Business Media. ISBN 978-3-540-40140-7.

[15] Troger, H. & Steindl, A. (2012). *Nonlinear stability and bifurcation theory: an introduction for engineers and applied scientists*. Springer Science & Business Media. ISBN 978-3-211-82292-0.

[16] Viète, F. (1615). De emendatione.

[17] Ziegler, H. (1952). Die stabilitätskriterien der elastomechanik. *Ingenieur Archiv*, 20, 49–56.

Nikolay Banichuk, Juha Jeronen, Pekka Neittaanmäki and Tero Tuovinen
University of Jyväskylä, Department of Mathematical Information Technology
Mattilanniemi 2, 40014 University of Jyväskylä, Finland
juha.jeronen@jyu.fi, pekka.neittaanmaki@jyu.fi, tero.tuovinen@jyu.fi

Nikolay Banichuk
Institute for Problems in Mechanics RAS
Prospect Vernadskogo 101, Bld. 1, 119526 Moscow, Russian Federation
banichuk@ipmnet.ru

Alexander Barsuk
Moldova State University
Kishinev, Moldova
a.a.barsuk@mail.ru