

Sensitized principle of virtual work and the single-element strain energy test

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Summary. Determination of sensitizing parameter values in connection with the Galerkin least-squares finite element method in elastic problems is considered. A procedure based on the use of reference solutions and on setting the strain energy expressions in an element evaluated in two different ways equal is used. The procedure is called the single-element strain energy test. The test is applied for a two-node Timoshenko beam element with good results. The sensitized principle of virtual work is shortly described and used to obtain rather directly the Galerkin least-squares formulas needed.

Key words: elastic structural mechanics, single-element strain energy test, sensitized principle of virtual work

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Introduction

A basic theme in the application of the Galerkin least-squares finite element methods is the selection of suitable values for the so-called weighting or sensitizing parameters in the least-squares terms. Here elastic structural problems are considered and a logic based on the consideration of the strain energy for a typical element for the parameter determination is suggested. The procedure will be called as *the single-element strain energy test*. The use of this test is illustrated in connection with Timoshenko beam bending. To achieve generality of formulation, the starting point is taken to be the sensitized principle of virtual work. The purpose of the additional least-squares terms is to hopefully increase the accuracy of the corresponding finite element solution with respect to the pure Galerkin formulation.

Sensitized principle of virtual work

The conventional principle of virtual work can be appended by least-squares type terms. We call the resulting formulation shortly as *sensitized principle of virtual work*. This terminology — and in fact the formulation — is not in common use. We have advocated

this terminology already in [1] and have based the use of the adjective "sensitized" on the important reference by Courant [2], where least-squares terms were appended to functionals and the term "*sensitized functional*" was coined. However, the starting point here is not a functional but a weak form but we still consider the analogous terminology as appropriate. When finite element discretization is applied in connection with the sensitized principle of virtual work, the Galerkin least-squares method emerges directly. We now shortly explain the formulation (for simplicity of presentation in the two-dimensional continuum case). A more detailed presentation is given in [3].

We define matrices (mostly following reference [4])

$$\{\mathbf{u}\} = \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \{\mathbf{b}\} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}, \quad \{\mathbf{t}\} = \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}, \quad \{\boldsymbol{\sigma}\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}, \quad \{\boldsymbol{\varepsilon}\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (1)$$

and further

$$\{\boldsymbol{\varepsilon}\} = [\mathbf{S}]\{\mathbf{u}\}, \quad (2)$$

where

$$[\mathbf{S}] = \begin{bmatrix} \partial / \partial x & 0 \\ 0 & \partial / \partial y \\ \partial / \partial y & \partial / \partial x \end{bmatrix}. \quad (3)$$

Here, $\{\mathbf{u}\}$, $\{\mathbf{b}\}$, $\{\mathbf{t}\}$, $\{\boldsymbol{\sigma}\}$ and $\{\boldsymbol{\varepsilon}\}$ are the displacement, body force, traction, stress and strain in matrix forms, respectively. Further, $[\mathbf{S}]$ is the displacement-strain operator matrix. The standard virtual work equation can be presented now as (with a change of sign for convenience of later development)

$$\int_A \delta\{\boldsymbol{\varepsilon}\}^T \{\boldsymbol{\sigma}\} dA - \int_A \delta\{\mathbf{u}\}^T \{\mathbf{b}\} dA - \int_s \delta\{\mathbf{u}\}^T \{\mathbf{t}\} ds = 0. \quad (4)$$

The integrals are over the plane domain A of the structure and over its boundary line s . To be more precise, the last integral on the left-hand side is to be taken over that boundary part where the traction $\{\mathbf{t}\}$ is given. Additionally, we obtain from (2) by variation the result

$$\delta\{\boldsymbol{\varepsilon}\} = [\mathbf{S}]\delta\{\mathbf{u}\} \quad (5)$$

and (4) transforms to

$$\int_A ([\mathbf{S}]\delta\{\mathbf{u}\})^T \{\boldsymbol{\sigma}\} dA - \int_A \delta\{\mathbf{u}\}^T \{\mathbf{b}\} dA - \int_s \delta\{\mathbf{u}\}^T \{\mathbf{t}\} ds = 0. \quad (6)$$

The local equilibrium equations can be written as

$$\{\mathbf{R}\} \equiv [\mathbf{E}]\{\boldsymbol{\sigma}\} + \{\mathbf{b}\} = \{\mathbf{0}\} \quad (7)$$

with

$$[\mathbf{E}] = [\mathbf{S}]^T = \begin{bmatrix} \partial / \partial x & 0 & \partial / \partial y \\ 0 & \partial / \partial y & \partial / \partial x \end{bmatrix}. \quad (8)$$

Here the left-hand notation $\{\mathbf{R}\}$ in (7) refers to equation *residuals*. A least-squares expression

$$II(\{\boldsymbol{\sigma}\}) = \frac{1}{2} \int_A \{\mathbf{R}\}^T [\boldsymbol{\tau}]\{\mathbf{R}\} dA \quad (9)$$

is formed. Matrix

$$[\boldsymbol{\tau}] = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} \quad (10)$$

can be called here as the *sensitizing parameter matrix* and be taken symmetric ($\tau_{21} = \tau_{12}$) without loss of generality. In the pure least-squares method matrix (10) should be positive definite. Here, in connection with the Galerkin method this is not required and this restriction could be harmful. For instance, in the Timoshenko beam example case presented later in the article the least-squares terms are not "stabilizing terms", they in fact "de-stabilizing terms".

Demanding functional (9) to have a stationary value gives an equation

$$\delta II \equiv \int_A \delta \{\mathbf{R}\}^T [\boldsymbol{\tau}]\{\mathbf{R}\} dA \equiv \int_A ([\mathbf{E}]\delta\{\boldsymbol{\sigma}\})^T [\boldsymbol{\tau}]([\mathbf{E}]\{\boldsymbol{\sigma}\} + \{\mathbf{b}\}) dA = 0. \quad (11)$$

The sensitized virtual work equation is obtained as a linear combination of (6) and (11):

$$\begin{aligned} & \int_A ([\mathbf{S}]\delta\{\mathbf{u}\})^T \{\boldsymbol{\sigma}\} dA - \int_A \delta\{\mathbf{u}\}^T \{\mathbf{b}\} dA - \int_s \delta\{\mathbf{u}\}^T \{\mathbf{t}\} ds + \\ & + \int_A ([\mathbf{E}]\delta\{\boldsymbol{\sigma}\})^T [\boldsymbol{\tau}]([\mathbf{E}]\{\boldsymbol{\sigma}\} + \{\mathbf{b}\}) dA = 0. \end{aligned} \quad (12)$$

Finally, $\delta\{\boldsymbol{\sigma}\}$ must be interpreted as the infinitesimal change due to the virtual displacement $\delta\{\mathbf{u}\}$ (or more properly due to $\delta\{\boldsymbol{\varepsilon}\}$). The constitutive properties of the material under study is needed here.

We consider linear elasticity and thus the stresses are connected to the strains by

$$\{\boldsymbol{\sigma}\} = [\mathbf{D}]\{\boldsymbol{\varepsilon}\} = [\mathbf{D}][\mathbf{S}]\{\mathbf{u}\}, \quad (13)$$

where matrix $[\mathbf{D}]$ contains given elastic constants. Also,

$$\delta\{\boldsymbol{\sigma}\} = [\mathbf{D}]\delta\{\boldsymbol{\varepsilon}\} = [\mathbf{D}][\mathbf{S}]\delta\{\mathbf{u}\}. \quad (14)$$

The sensitized virtual work equation (12) transforms now to

$$\begin{aligned} & \int_A ([\mathbf{S}]\delta\{\mathbf{u}\})^T [\mathbf{D}][\mathbf{S}]\{\mathbf{u}\}dA - \int_A \delta\{\mathbf{u}\}^T \{\mathbf{b}\}dA - \int_s \delta\{\mathbf{u}\}^T \{\mathbf{t}\}ds + \\ & + \int_A ([\mathbf{E}][\mathbf{D}][\mathbf{S}]\delta\{\mathbf{u}\})^T [\boldsymbol{\tau}]([\mathbf{E}][\mathbf{D}][\mathbf{S}]\{\mathbf{u}\} + \{\mathbf{b}\})dA = 0. \end{aligned} \quad (15)$$

Sensitized finite element method

Let us write the finite element displacement approximation in the standard form

$$\{\mathbf{u}\} \approx \{\tilde{\mathbf{u}}\} = [\mathbf{N}]\{\mathbf{a}\}, \quad (16)$$

where the column matrix $\{\mathbf{a}\}$ consists of the nodal displacements of the structure and $[\mathbf{N}]$ is the global shape function matrix. Thus, also,

$$\delta\{\mathbf{u}\} = [\mathbf{N}]\delta\{\mathbf{a}\}. \quad (17)$$

Application of (16) and (17) in (15) and by taking into account the arbitrariness of $\delta\{\mathbf{a}\}$ gives finally the linear system equation set

$$[\mathbf{K}]\{\mathbf{a}\} = \{\mathbf{f}\}, \quad (18)$$

where the stiffness matrix

$$[\mathbf{K}] \equiv [\mathbf{K}_O] + [\mathbf{K}_S] \equiv \int_A [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}]dA + \int_A [\mathbf{C}]^T [\boldsymbol{\tau}][\mathbf{C}]dA \quad (19)$$

and the right-hand side

$$\{\mathbf{f}\} \equiv \{\mathbf{f}_O\} + \{\mathbf{f}_S\} \equiv \int_A [\mathbf{N}]^T \{\mathbf{b}\}dA + \int_s [\mathbf{N}]^T \{\mathbf{t}\}ds - \int_A [\mathbf{C}]^T [\boldsymbol{\tau}]\{\mathbf{b}\}dA. \quad (20)$$

To shorten the above expressions, notations

$$[\mathbf{B}] \equiv [\mathbf{S}][\mathbf{N}] \quad (21)$$

and

$$[\mathbf{C}] \equiv [\mathbf{E}][\mathbf{D}][\mathbf{B}] \quad (22)$$

have been introduced. The subscripts O and S above refer in an obvious way to the terms emerging from the standard and sensitizing parts of the formulas, respectively.

The result (18) can be considered now as an application of the Galerkin least-squares method arrived at here, however, rather directly through the use of the sensitized principle of virtual work.

Single-element strain energy test

The main problem is of course to find a logic for the selection of appropriate values for the sensitizing parameters. For instance, in [1] a kind of patch test approach with so-called reference solutions was used for this purpose. Reference solutions mean some local analytical expressions containing hopefully relevant information about the actual solution behavior around a point of interest. This is explained in more detail in the section "Timoshenko beam element". Here we consider tentatively an alternative possibility based still on the use of reference solutions but applied just on a single element and not on a patch of elements.

Let us consider the finite element expressions on a typical element level. Formula (16) can still be applied (we do not need necessarily to change the notations here) when $\{\mathbf{a}\}$ is considered now to be the element nodal displacement column matrix and $[\mathbf{N}]$ the element shape function matrix. However, we cannot speak any more about element equations but formula (18) must be interpreted as giving the element *contributions*

$$\{\mathbf{F}\} \equiv [\mathbf{K}]\{\mathbf{a}\} - \{\mathbf{f}\} \quad (23)$$

to the global system equations. Of course, the integrations in (19) and (20) must then be performed over the element domain and boundary.

Let us assume that we have available some reference solution $u = u(x, y)$, $v = v(x, y)$ in the element. We can then calculate analytically the corresponding strains $\{\boldsymbol{\varepsilon}\}$ and the corresponding strain energy of the element:

$$U_A = \frac{1}{2} \int_A \{\boldsymbol{\varepsilon}\}^T [\mathbf{D}]\{\boldsymbol{\varepsilon}\} dA. \quad (24)$$

Alternatively, we can evaluate the element nodal column matrix $\{\mathbf{a}\}$ from the reference expressions. The corresponding strain energy using these nodal displacement values is

$$U_N = \frac{1}{2} \{\mathbf{a}\}^T [\mathbf{K}]\{\mathbf{a}\} = \frac{1}{2} \{\mathbf{a}\}^T [\mathbf{K}_O]\{\mathbf{a}\} + \frac{1}{2} \{\mathbf{a}\}^T [\mathbf{K}_S]\{\mathbf{a}\}. \quad (25)$$

It seems rather natural to demand that the two strain energy expressions should have the same value:

$$U_N = U_A. \quad (26)$$

Matrix $[\mathbf{K}_S]$ contains in general the sensitizing parameters and equation (26) gives thus one equation for the determination of the parameters. If the number of suitable reference solutions is large enough, one can hope to obtain in this way values for the parameters. This approach will be called here as the *single-element strain energy test*. In practice constant local material properties are assumed in an element as otherwise the evaluation of the $[\mathbf{C}]$ -matrix (formula (22)) becomes too complicated. Also, the matrix manipulations by hand become tedious and use of programs, say such as Mathematica [5], are of great help.

C^0 -continuous approximations are employed in the Timoshenko beam example case. This means that standard continuity requirements are violated in the evaluation of the $[\mathbf{K}_S]$ -matrix. However, the sensitized stiffness matrix $[\mathbf{K}_S]$ is of such a form that it vanishes compared to the standard stiffness matrix $[\mathbf{K}_O]$ with vanishing mesh size. This fact justifies the apparent mathematical crimes performed.

Timoshenko beam element

Some notations

The Timoshenko beam problem has been considered in some detail in reference [1]. There the patch test with two elements was applied for the sensitizing parameter determination. As the problem is one-dimensional, reference solutions are rather easy to find and we are in position to compare the performance of the suggested single-element strain energy test in a reasonably simple situation.

The beam problem needs some alterations for the contents of the matrices defined above in the two-dimensional continuum case. First, certain notations are given in Figure 1. For the bending moment M the traditional inconsistent sign rule has been used.

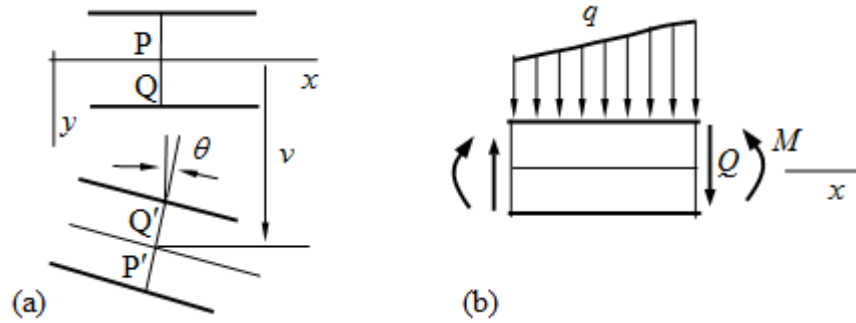


Figure 1. (a) Deflection and cross-sectional rotation, (b) Bending moment, shearing force and transverse loading.

Further, we define following (1) now the interpretations

$$\begin{aligned} \{\mathbf{u}\} &= \begin{Bmatrix} v \\ \theta \end{Bmatrix}, \quad \{\mathbf{b}\} = \begin{Bmatrix} q \\ 0 \end{Bmatrix}, \quad \{\mathbf{t}\} = \begin{Bmatrix} V \\ B \end{Bmatrix}, \quad \{\boldsymbol{\sigma}\} = \begin{Bmatrix} Q \\ M \end{Bmatrix}, \\ \{\boldsymbol{\varepsilon}\} &= \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} = \begin{Bmatrix} dv/dx - \theta \\ -d\theta/dx \end{Bmatrix} = [\mathbf{S}]\{\mathbf{u}\} \end{aligned} \quad (27)$$

with

$$[\mathbf{S}] = \begin{bmatrix} d/dx & -1 \\ 0 & -d/dx \end{bmatrix}. \quad (28)$$

V and B refer to the shearing force and bending moment at the beam ends with such sign changes that V is positive in the positive y -axis direction and B positive in the clockwise direction. For simplicity, we have not included distributed torque loading in the column matrix $\{\mathbf{b}\}$. For an elastic beam

$$[\mathbf{D}] = \begin{bmatrix} kGA & 0 \\ 0 & EI \end{bmatrix}, \quad (29)$$

where GA is the shearing stiffness, k the shear correction factor, and EI the bending stiffness. The equilibrium equations are in matrix notation

$$\begin{bmatrix} d/dx & 0 \\ 1 & -d/dx \end{bmatrix} \begin{Bmatrix} Q \\ M \end{Bmatrix} + \begin{Bmatrix} q \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (30)$$

so the equilibrium operator matrix

$$[\mathbf{E}] = \begin{bmatrix} d/dx & 0 \\ 1 & -d/dx \end{bmatrix} \quad (31)$$

is not any more the transpose of (28) here. The sensitizing parameter matrix can still be written as in (10).

Beam element

The element under study (Figure 2) has two nodes and the simple linear approximation for both the unknowns:

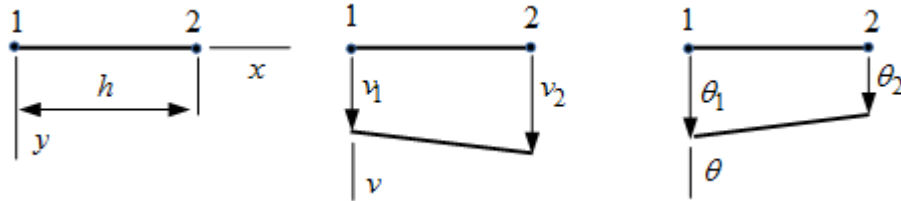


Figure 2. Timoshenko beam element.

$$\{\mathbf{u}\} \approx \{\tilde{\mathbf{u}}\} \equiv \begin{Bmatrix} \tilde{v} \\ \tilde{\theta} \end{Bmatrix} = [\mathbf{N}]\{\mathbf{a}\} \quad (32)$$

with

$$[\mathbf{N}] = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \quad (33)$$

and

$$\{\mathbf{a}\} = [v_1 \quad \theta_1 \quad v_2 \quad \theta_2]^T. \quad (34)$$

The shape functions are

$$N_1 = 1 - x/h, \quad N_2 = x/h, \quad (35)$$

where x is a local elementwise coordinate measured from the left-hand end of the element. Without sensitizing this element demands extremely dense meshes for thin beams to give reasonable results.

We can now apply formula (19) but of course, the integrations must be performed with respect to x . Thus, for the element,

$$[\mathbf{K}_O] = \int_0^h [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}] dx \quad (36)$$

and

$$[\mathbf{K}_S] = \int_0^h [\mathbf{C}]^T [\boldsymbol{\tau}][\mathbf{C}] dx. \quad (37)$$

We record some of the relevant matrices:

$$[\mathbf{B}] = \begin{bmatrix} -1/h & -(1-x/h) & 1/h & -x/h \\ 0 & 1/h & 0 & -1/h \end{bmatrix} \quad (38)$$

and

$$[\mathbf{C}] = \begin{bmatrix} 0 & kGA/h & 0 & -kGA/h \\ -kGA/h & -kGA(1-x/h) & kGA/h & -kGAx/h \end{bmatrix}. \quad (39)$$

It is of some interest to notice that the bending stiffness does not appear at all in the latter matrix and thus from (37) it is seen that the bending stiffness can neither be present in the $[\mathbf{K}_S]$ -matrix. Performing the manipulations, there are finally obtained the element stiffness matrices

$$[\mathbf{K}_O] = \frac{EI}{h} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + kGAh \begin{bmatrix} 1/h^2 & 1/2h & -1/h^2 & 1/2h \\ 1/2h & 1/3 & -1/2h & 1/6 \\ -1/h^2 & -1/2h & 1/h^2 & -1/2h \\ 1/2h & 1/6 & -1/2h & 1/3 \end{bmatrix} \quad (40)$$

and

$D \neq 0$, only $q_0 \neq 0$, ... These give the consecutive specific reference solutions. It is found that the values A etc. cancel in the patch test and we can simply put $A = 1$ etc.

Two-element patch test

We describe for comparison purposes here shortly the two-element patch test. Consider Figure 3. A generic element is considered as "cloned" and a patch of these cloned elements is formed. Here the patch consist just of two elements.

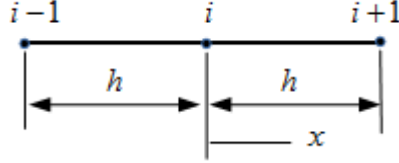


Figure 3. Notations in the two-element patch test.

Using the sensitized formulation, two finite element system equations corresponding to node i are formed. Again, constant local material data is assumed. The nodal values are taken as given from the specific reference solutions. There are finally obtained the following five sets of two-equation systems:

$$0 = 0, \quad (46)$$

$$\tau_{12} - \tau_{21} = 0,$$

$$-\tau_{12} + \tau_{21} = 0, \quad (47)$$

$$0 = 0,$$

$$0 = 0, \quad (48)$$

$$\tau_{21} = 0,$$

$$-EI / (kGA) \cdot \tau_{12} + [h^2 - EI / (kGA)] \tau_{21} = 0, \quad (49)$$

$$-\tau_{11} + EI / (kGA) \cdot \tau_{22} = 0,$$

$$(kGA + 12EI / h^2) \tau_{22} + 1 = 0, \quad (50)$$

$$-\tau_{12} + 2\tau_{21} - 12EI / (kGAh^2) \cdot (\tau_{12} + \tau_{21}) = 0.$$

The equations have been developed without assuming symmetry ($\tau_{12} = \tau_{21}$), however, this follows both from (46) and (47). Further, as full solution there is obtained

$$\tau_{12} = \tau_{21} = 0,$$

$$\tau_{11} = -\frac{EI / (kGA)}{kGA + 12EI / h^2}, \quad (51)$$

$$\tau_{22} = -\frac{1}{kGA + 12EI / h^2}.$$

Thus the sensitizing parameter matrix is found to be diagonal with negative elements and this means physically that sensitizing makes the solution more soft counteracting the so-called "locking" associated with the standard approach. As shown in [1], very accurate results are obtained by the sensitized formulation also for thin beams.

Single-element strain energy test

Figure 4 represents a generic element in the single-element strain energy test. The local origin is now taken at the element midpoint.

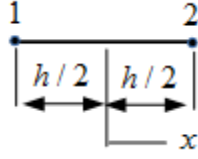


Figure 4. Notations in the single-element strain energy test.

It is found that the four first specific reference solutions according to (45) are enough for obtaining a solution. Corresponding strains and nodal displacements are evaluated and (finally) five equations of type (26) are arrived at. A basic difference is that we have now for each reference solution only one equation contrary to the two-element patch test case which gives two equations. To facilitate the situation, we apply the symmetry condition at the outset by defining $\tau_{21} \equiv \tau_{12}$. The equations corresponding to the four first reference solutions are:

$$0 = 0, \quad (52)$$

$$0 = 0, \quad (53)$$

$$2EIh + \frac{1}{6}kGAh^3 + \frac{1}{6}(kGA)^2 h(12\tau_{11} + h^2\tau_{22}) = 2EIh, \quad (54)$$

$$\frac{h(12EI + kGAh^2)^2}{8kGA} + \frac{1}{8}h(12EI + kGAh^2)^2 \tau_{22} = \frac{3}{2}EIh \left(h^2 + \frac{12EI}{kGA} \right). \quad (55)$$

The equations are recorded as those given by [5] without possible simplifications by hand calculations. As the element interpolants of the first two reference solutions coincide with the reference solutions, identities are obtained. The solutions for τ_{11} and τ_{22} from (54) and (55) are found to *coincide exactly with* (51) obtained by the two-element patch test. However, equations (54) and (55) do not contain the parameter τ_{12} and its value seems thus first to remain undetermined. Some experimentation with the fourth reference solution $v = x^3$, $\theta = 6EI / kGA + 3x^2$ produced rather unexpectedly additional information when the local origin for the reference solution was taken to

differ from the element midpoint. For instance, when the local origin is taken at the left-hand end of the element, the resulting strain energy test equation becomes

$$\begin{aligned} & \frac{1}{2}h \left(15EI h^2 + \frac{36(EI)^2}{kGA} + kGA h^4 \right) + \\ & + \frac{1}{2}h [36(EI)^2 \tau_{22} + 6EI kGA h(6\tau_{12} + h\tau_{22}) + \\ & + (kGA)^2 h^2 (9\tau_{11} + h(3\tau_{12} + h\tau_{22}))] = 6EI h \left(h^2 + \frac{3EI}{kGA} \right). \end{aligned} \quad (56)$$

The solution of (54), (55) and (56) is now completely according to (51). It is found that the same solution is obtained irrespective of the position of the local origin for the additional fourth reference solution if the origin just differs from the midpoint of the element. The above behavior seems to mean that in the single-element strain energy test the possibility to use reference solutions associated to arbitrary coordinate systems should be explored. In all, the results obtained are rather encouraging indicating that the logic behind the single-element strain energy test appears sound.

It should be mentioned that the strain energy test approach is not the only possibility to find appropriate modified beam properties for the two-node element. For instance, in [6] a way to determine modified bending and shear parameters from the anti-symmetric and symmetric bending modes of an element is described. Further, a well-known computationally efficient alternative approach to avoid shear locking is to apply "reduced integration". This is also described in [6].

Numerical example

Numerical examples for a simply supported beam with constant distributed load and point load at the centerpoint are available in [1]. Here, we consider as a sample a cantilever of length L with a constant distributed load q . The exact solution is

$$v / \left(\frac{L^4 q (1 + 4\varepsilon)}{8EI} \right) = \frac{x(-4Lx^2 + x^3 + 6L^2 x(1 - 2\varepsilon) + 24L^3 \varepsilon)}{3L^4(1 + 4\varepsilon)}, \quad (57)$$

$$\theta / \left(\frac{L^3 q}{6EI} \right) = \frac{x(3L^2 - 3Lx + x^2)}{L^3}, \quad (58)$$

in which the scaling is chosen to get the value one for the deflection and rotation at the free end. The dimensionless parameter $\varepsilon = EI / (kGAL^2)$ values 1 and 1/10000 correspond roughly to the very thick and the very thin cases or cases in which deflection due to shear is significant and negligible, respectively. In Figure 5 are shown some results obtained for a 10-element irregular mesh. For the thick beam the accuracy is rather good also without sensitizing. For the thin beam the sensitizing terms change the solution behavior dramatically. Very accurate values are obtained with sensitizing although the mesh is irregular.

It may be mentioned that by applying the reduced integration approach (reference [6]) practically identical results to the ones presented are obtained and in fact with a smaller computational effort. However, our main point here has been just to show in a simple setting the working of the sensitized principle and the strain energy test.

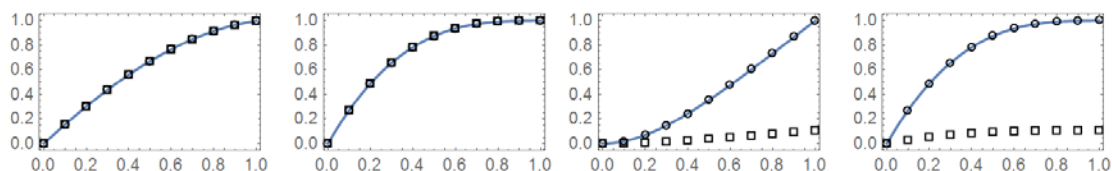


Figure 5. Cantilever and uniform load. The exact solution (solid), standard finite element solution (squares) and the sensitized finite element solution (circles). On the left $\varepsilon = 1$. Deflection and rotation. On the right $\varepsilon = 1/10000$. Deflection and rotation.

Conclusions

The patch test approach for sensitizing parameter determination becomes very complicated in two- or three-dimensional cases with elements having general shapes. The single-element strain energy test may offer in these situations obvious advantages. Although this test worked well for the Timoshenko beam, additional studies are naturally needed to get a more general picture of the applicability of the test. Finally, the sensitized principle of virtual work as such is in theory quite general. However, to find its practical value needs also further studies.

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