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Variational approach for analysis of harmonic vibration and stability of moving panels

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Summary. In this paper, the stability of a simply supported axially moving elastic panel (plate undergoing cylindrical deformation) is considered. A complex variable technique and bifurcation theory are applied. As a result, variational equations and a variational principle are derived. Analysis of the variational principle allows the study of qualitative properties of the bifurcation points. Asymptotic behaviour in a small neighbourhood around an arbitrary bifurcation point is analyzed and presented.

It is shown analytically that the eigenvalue curves in the (ω, V_0) plane cross both the ω and V_0 axes perpendicularly. It is also shown that near each bifurcation point, the dependence $\omega(V_0)$ for each mode approximately follows the shape of a square root near the origin.

The obtained results complement existing numerical studies on the stability of axially moving materials, especially those with finite bending rigidity. From a rigorous mathematical viewpoint, the presence of bending rigidity is essential, because the presence of the fourth-order term in the model changes the qualitative behaviour of the bifurcation points. The results are applicable to both axially moving panels and axially moving beams.

Key words: axially moving panel, axially moving beam, bifurcation theory, complex variable techniques, variational principle

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Introduction

The aim of our studies has been to develop mathematical models representing the behaviour of the paper making process. Previously (see e.g., Banichuk et al., 2013b,a, 2011a,b), we have considered many approaches for modelling moving materials and their stability. Conclusions that have been drawn can be applied for example, the processing of paper or steel, fabric, rubber or some other continuous material, and looping systems such as band saws and timing belts.

Typically systems of axially moving web have been modelled as travelling flexible strings, membranes, beams, and plates. Classical articles in this field are, for example, Mote (1972), Archibald and Emslie (1958), Simpson (1973), Wang et al. (2005), Parker (1998) Kong and Parker (2004), Miranker (1960) Chonan (1986), Wickert and Mote (1990), Bhat et al. (1982), Perkins (1990), Wickert (1994) and Parker (1999).

In the case of beams interacting with external media, one can read e.g. the article by Chang and Moretti (1991), and the articles by Banichuk and Neittaanmäki (2008a,b,c). The study has been extended in Banichuk et al. (2010) for a two-dimensional model of the web, considered as a moving plate under homogeneous tension but without external

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media. The most straightforward and efficient way to study stability is to use linear stability analysis. In a recent article by Hatami et al. (2009), the free vibration of a moving orthotropic rectangular plate was studied at sub- and supercritical speeds, and its flutter and divergence instabilities at supercritical speeds. The study is limited to simply supported boundary conditions at all edges. For the solution of equations of orthotropic moving material, many necessary fundamentals can be found in the book by Marynowski (2008b). An extensive literature review about dynamics of axially moving continua can be found in Marynowski and Kapitaniak (2014) or in the book by Banichuk et al. (2014). However, in this article the effect of surrounding media have been excluded.

The dynamical properties of moving plates have been studied by Shen et al. (1995) and by Shin et al. (2005), and the properties of a moving paper web have been studied in the two-part article by Kulachenko et al. (2007a,b). Critical regimes and other problems of stability analysis have been studied e.g. by Wang (2003) and Sygulski (2007). Moreover, in the articles Marynowski (2002, 2004, 2008a) the dynamical aspects of the axially moving web are discussed extensively. In Yang and Chen (2005), the authors considered transverse vibrations of the axially accelerating viscoelastic beam, and in Pellicano and Vestroni (2000), dynamic behavior of a simply supported beam subjected to an axial transport of mass was studied. An extensive literature review related to areas presented in this paper, can be found for example in Ghayesh et al. (2013). Note also some approaches to bifurcation problems and estimation of critical parameters presented by Nečas et al. (1987) and Neittaanmäki and Ruotsalainen (1985).

The focus of this article is the stability of a simply supported axially moving elastic panel. We have used a complex variable technique and bifurcation theory. Our main task has been the derivation of variational equations and a variational principle. Moreover, we have performed an analysis of the variational principle, which allows the study of qualitative properties of the bifurcation points. Furthermore, asymptotic behaviour around an arbitrary bifurcation point is analyzed and presented. As a result, we show analytically that the eigenvalue curves in the (ω, V_0) plane cross both the ω and V_0 axes perpendicularly. It is also shown that near each bifurcation point, the dependence $\omega(V_0)$ for each mode approximately follows the shape of a square root near the origin. Gained results complement existing numerical studies on the stability of axially moving materials, and especially materials with finite bending rigidity. From a rigorous mathematical viewpoint, the presence of bending rigidity is essential, because the presence of the fourth-order term in the model changes the qualitative behaviour of the bifurcation points.

Basic relations and complex functions

Consider the problem of free harmonic vibrations of an elastic panel, moving axially at a constant velocity V_0. In a stationary orthogonal coordinate system, the transverse vibrations are characterized by the function $w = w(x,t)$, which is determined by the following partial differential equation:

$$w_{,tt} + 2V_0 w_{,xt} + (V_0^2 - C^2) w_{,xx} + \frac{D}{\rho S} w_{,xxxx} = 0 \,, \quad 0 < x < \ell \,, \quad C = \sqrt{\frac{T_0}{\rho S}} \,. \tag{1}$$

At the ends of the considered interval $x \in [0, \ell]$, we take the simply supported boundary conditions

$$w(0,t) = w(\ell,t) = D w_{,xx}(0,t) = D w_{,xx}(\ell,t) = 0 \,. \tag{2}$$

Note that in a realistic paper making scenario, it is expected that supporting rollers will lead to behaviour that is somewhere between the classical simply supported and clamped extremes. The simply supported boundary conditions are here used for the sake of simplicity, and because they are often chosen in fundamental studies on moving materials.

Here x is the axial space coordinate, t time, T_0 tension, ρ material density, S the area of the panel cross section, ℓ the length of the free span, and D the bending rigidity of the panel.

For harmonic vibrations at frequency ω, the transverse displacement can be represented in the form

$$w(x,t) = e^{i\omega t} u(x) \ , \quad i = \sqrt{-1} \ , \tag{3}$$

where $u(x)$ is an amplitude function that satisfies the following boundary value problem:

$$u_{,xxxx} + (V_0^2 - C^2) u_{,xx} + 2i\omega V_0 u_{,x} - \omega^2 u = 0 \ , \tag{4}$$

$$u(0) = u(1) = u_{,xx}(0) = u_{,xx}(1) = 0 \ , \tag{5}$$

which is written in dimensionless variables

$$x = \ell \tilde{x} \ , \quad \frac{\rho S \omega^2 \ell^4}{D} = \tilde{\omega}^2 \ , \quad \frac{\rho S \ell^2}{D} V_0^2 = \tilde{V}_0^2 \ , \quad \frac{\rho S \ell^2}{D} C^2 = \tilde{C}^2 \ . \tag{6}$$

In what follows, the tilde will be omitted.

The amplitude function $u(x)$ determined from the boundary value problem (4)–(5) is a complex-valued function, i.e.

$$u(x) = u^1(x) + i u^2(x) \ , \quad \hat{u}(x) = u^1(x) - i u^2(x) \ , \tag{7}$$

where $u^1(x)$ and $u^2(x)$ are real-valued functions and $\hat{u}(x)$ is the complex conjugate of $u(x)$.

In the following we present a variational formulation of the spectral problem (4)–(5). This formulation allows us to make important conclusions about the frequencies of free vibrations of moving elastic systems without knowing the rigorous solution of the spectral boundary value problem. To derive the variational formulation of (4)–(5), we multiply the differential equation by the complex conjugate (adjoint) amplitude function $\hat{u}(x)$ and integrate the result on the interval $(0, 1)$. We will also take into account the boundary conditions

$$\begin{gathered} u^1(0) = u^2(0) = u^1(1) = u^2(1) = 0 \ , \\ u^1_{,xx}(0) = u^2_{,xx}(0) = u^1_{,xx}(1) = u^2_{,xx}(1) = 0 \ , \end{gathered} \tag{8}$$

which follow from the boundary conditions (5). We obtain the functional equation

$$a\omega^2 + 2bV_0\omega + (V_0^2 - C^2)c - d = 0 \ , \tag{9}$$

where a, b, c and d are integral functional depending on the problem (4)–(5). The functional a is given by

$$a = \int_0^1 u\hat{u} \, \mathrm{d}x = \int_0^1 \left((u^1)^2 + (u^2)^2 \right) \, \mathrm{d}x > 0 \ . \tag{10}$$

Using the boundary conditions (8), we can write the functional b as

$$\int_0^1 u_{,x} \hat{u} \, \mathrm{d}x = i \int_0^1 \left((u^2)_{,x} u^1 - (u^1)_{,x} u^2 \right) \mathrm{d}x = ib \ , \tag{11}$$

where b is real-valued. The functionals c and d are obtained by integrating by parts (once in the case of c and twice for d), and taking into account the corresponding boundary conditions in (8). We have

$$c = -\int_0^1 u_{,xx}\hat{u}\,\mathrm{d}x = \int_0^1 \left([(u^1)_{,x}]^2 + [(u^2)_{,x}]^2\right)\mathrm{d}x > 0\ , \tag{12}$$

$$d = \int_0^1 u_{,xxxx}\hat{u}\,\mathrm{d}x = \int_0^1 \left([(u^1)_{,xx}]^2 + [(u^2)_{,xx}]^2\right)\mathrm{d}x > 0\ . \tag{13}$$

Variational analysis and variational principle in complex variables

Let us write the variation of the functional equation (9). To do this, we take into account the variations of the considered functionals,

$$\begin{aligned}
\delta a &= \int_0^1 (\hat{u}\delta u + u\delta\hat{u})\,\mathrm{d}x\ ,\\
i\delta b &= \int_0^1 (\hat{u}\delta u_{,x} + u_{,x}\delta\hat{u})\,\mathrm{d}x\ ,\\
\delta c &= \int_0^1 (u_{,x}\delta\hat{u}_{,x} + \hat{u}_{,x}\delta u_{,x})\,\mathrm{d}x\ ,\\
\delta d &= \int_0^1 (u_{,xx}\delta\hat{u}_{,xx} + \hat{u}\delta u_{,xx})\,\mathrm{d}x\ ,
\end{aligned} \tag{14}$$

and perform standard transformations in (9), replacing u, $\hat{u}$ and ω with $u + \delta u$, $\hat{u} + \delta\hat{u}$ and $\omega + \delta\omega$, respectively. We will have the variation

$$\begin{aligned}
2(a\omega + bV_0)\delta\omega &+ \int_0^1 \left[-\omega^2 u + 2i\omega V_0 u_{,x} + (V_0^2 - C^2)u_{,xx} + u_{,xxxx}\right]\delta\hat{u}\,\mathrm{d}x\\
&+ \int_0^1 \left[-\omega^2 \hat{u} - 2i\omega V_0 \hat{u}_{,x} + (V_0^2 - C^2)\hat{u}_{,xx} + \hat{u}_{,xxxx}\right]\delta u\,\mathrm{d}x = 0\ .
\end{aligned} \tag{15}$$

For $u(x)$ and $\hat{u}(x)$, which are solutions of the spectral boundary value problem (4)–(5) and its complex conjugate, the integral expressions in (15) are identically zero. Taking this into account, we are left with

$$2(a\omega + bV_0)\,\delta\omega = 0\ . \tag{16}$$

Thus, if $a\omega + bV_0 \neq 0$ for the spectral problem (4)–(5), then the frequency variation for free vibrations $\delta\omega$ is zero. That is,

$$a\omega + bV_0 \neq 0\ , \quad \delta\omega = 0\ . \tag{17}$$

Solving (9) for ω, we arrive at the variational representation for harmonic vibrations, corresponding to each of the two solution branches of equation (9):

$$\omega_\pm(V_0) = \frac{1}{a}\left(-bV_0 \pm \sqrt{(b^2 - ac)V_0^2 + acC^2 + ad}\right) \to \underset{u(x),\hat{u}(x)}{\mathrm{extr}}\ . \tag{18}$$

Analysis of extremum conditions and bifurcation analysis

From (16), the other possibility is

$$a\omega + bV_0 = 0 \tag{19}$$

and $\delta\omega$ free. To perform analysis for this case, we consider equation (9) as an implicit function $F(\omega, V_0)$:

$$F(\omega, V_0) = 0\ , \quad F(\omega, V_0) = a\omega^2 + 2bV_0\omega + (V_0^2 - C^2)c - d\ . \tag{20}$$

Again, we can solve (9) for ω, obtaining the following two solution branches:

$$\omega_\pm(V_0) = \frac{1}{a}\left(-bV_0 \pm \sqrt{(b^2 - ac)V_0^2 + acC^2 + ad}\right) \tag{21}$$

Let (ω^*, V_0^*) denote the bifurcation point, i.e. the values of ω and V_0 at which the solution of (20) branches. At the bifurcation point, the conditions of the implicit function theorem must be violated, i.e. we will have

$$F(\omega, V_0) = 0\ , \quad \frac{\partial F(\omega, V_0)}{\partial \omega} = 0\ . \tag{22}$$

Using (20), these conditions become

$$a\omega^2 + 2bV_0\omega + (V_0^2 - C^2)c - d = 0\ , \quad a\omega + bV_0 = 0\ . \tag{23}$$

As a result, we find the following representation for bifurcation values of the frequency and panel velocity:

$$\omega^* = -\frac{b}{a}V_0^*\ , \quad (ac - b^2)(V_0^*)^2 = acC^2 + ad\ . \tag{24}$$

Alternatively, these values can be obtained from the condition $\omega_+(V_0) = \omega_-(V_0)$ and the representation (21) for $\omega_\pm(V_0)$. Note also that if some solutions have $b = 0$, the corresponding bifurcation points are distributed along the V_0 axis in the (V_0, ω) plane, i.e.

$$\omega^* = 0\ , \quad (V_0^*)^2 = C^2 + \frac{d}{c} \quad \text{(for solutions with } b = 0)\ . \tag{25}$$

Let us differentiate $\omega(V_0)$ with respect to the parameter V_0. To do this, in (20) we replace V_0, u and ω with $V_0 + \delta V_0$, $u + \delta u$ and $\omega + \delta\omega$, respectively. Using the standard transformations (as was done in (16)) we obtain

$$2(a\omega + bV_0)\,\delta\omega + 2(b\omega + cV_0)\,\delta V_0 = 0\ . \tag{26}$$

Consequently,

$$\frac{\mathrm{d}\omega}{\mathrm{d}V_0} = -\frac{b\omega + cV_0}{a\omega + bV_0}\ . \tag{27}$$

In particular, it follows from (27) that for all bifurcation points (ω^*, V_0^*) we have the limit

$$\lim_{V_0 \to V_0^*} \frac{\mathrm{d}\omega_\pm(V_0)}{\mathrm{d}V_0} = \pm\infty\ . \tag{28}$$

In the case $V_0 = 0$, we have $b = 0$, and find that

$$\frac{\mathrm{d}\omega_\pm(V_0 = 0)}{\mathrm{d}V_0} = 0\ . \tag{29}$$

It follows from (28)–(29) that the curves $\omega_\pm(V_0)$ cross the ω and V_0 axes at right angles; see Figure 1.

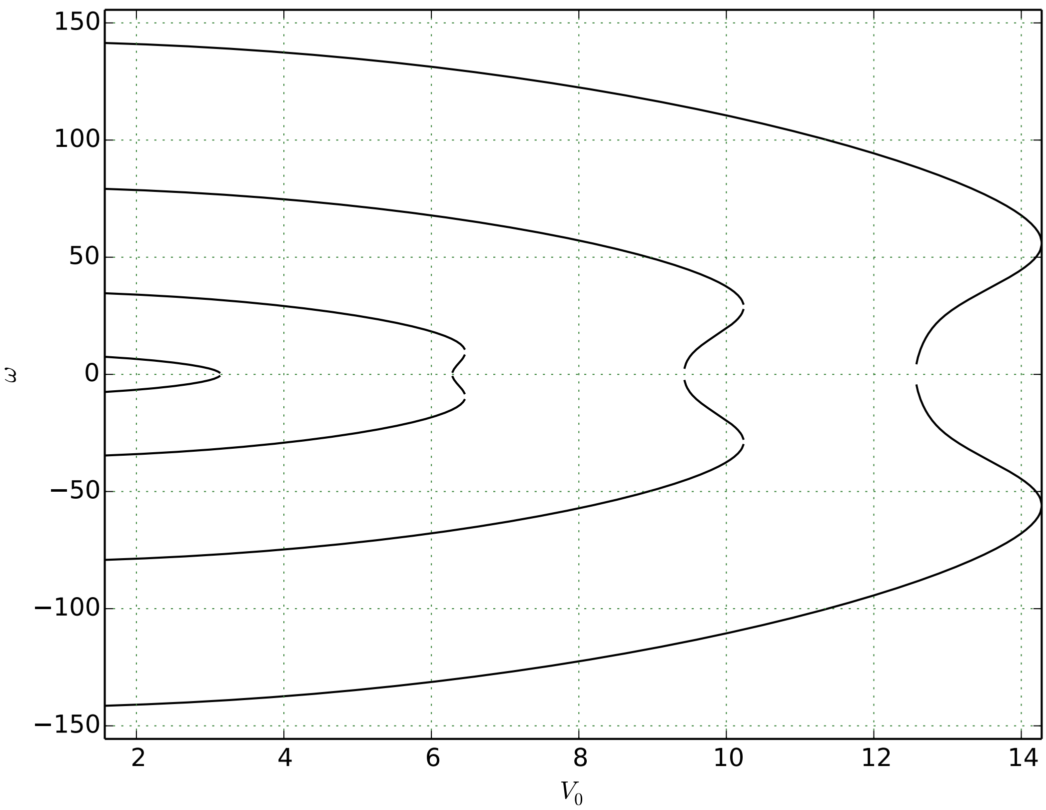

Figure 1. Behaviour of the natural frequencies ω as a function of the panel axial velocity V_0. Numerical solution using finite elements.

Nonlinear analysis of asymptotic behaviour of the frequencies in the vicinity of bifurcation points

Let $(\omega_1^*, V_{01}^*)\,, (\omega_2^*, V_{02}^*)\,, \dots$ be solutions of the system of nonlinear equations (20). Consider the behaviour of the functions $\omega_i(V_0)$ $(i = 1, 2, \dots)$, determined in a small neighbourhood of the bifurcation point (ω_k^*, V_{0k}^*), in implicit form, by the equation $F(\omega, V_0) = 0$. For brevity, we will omit the indices of the functions $\omega_i(V_0)$ and the bifurcation points (ω_k^*, V_{0k}^*).

To study the behaviour of the function $F(\omega, V_0)$, we expand it in series around the bifurcation point (ω^*, V_0^*). We have

$$F(\omega, V_0) = F(\omega^*, V_0^*) + \frac{\partial F(\omega^*, V_0^*)}{\partial \omega}(\omega - \omega^*) + \frac{\partial F(\omega^*, V_0^*)}{\partial V_0}(V_0 - V_0^*) + \frac{1}{2}\frac{\partial^2 F(\omega^*, V_0^*)}{\partial \omega^2}(\omega - \omega^*)^2 + \dots \quad (30)$$

Taking into account that at each bifurcation point (ω^*, V_0^*), relation (22) holds, we have that the first two terms in (30) vanish, obtaining

$$F(\omega, V_0) = \frac{\partial F(\omega^*, V_0^*)}{\partial V_0}(V_0 - V_0^*) + \frac{1}{2}\frac{\partial^2 F(\omega^*, V_0^*)}{\partial \omega^2}(\omega - \omega^*)^2 + \dots \quad (31)$$

Observe that all terms that have been omitted in (30) have a higher order of smallness. The expression (31) thus contains all leading-order terms, and describes completely general behaviour of $F(\omega, V_0)$ in a small neighbourhood of a given bifurcation point (ω^*, V_0^*). This is the general case; the special cases where one or both of $\partial F(\omega^*, V_0^*)/\partial V_0$ and $\partial^2 F(\omega^*, V_0^*)/\partial \omega^2$ are zero must be studied separately.

Without loss of generality, we may represent the function $\omega = \omega(V_0)$ in the small neighbourhood of the bifurcation point (ω^*, V_0^*) as a power series:

$$\omega(V_0) = \omega^* + \alpha_1(V_0 - V_0^*)^{\varepsilon_1} + \alpha_2(V_0 - V_0^*)^{\varepsilon_2} + \dots\,, \quad \text{where } 0 < \varepsilon_1 < \varepsilon_2 < \dots \quad (32)$$

The values of the constants $\alpha_1, \alpha_2, \dots$ and $\varepsilon_1, \varepsilon_2, \dots$ are determined with the help of the condition $F(\omega, V_0) = 0$. After substitution of (32) into (31), the equation $F(\omega, V_0) = 0$ reduces to the corresponding equation

$$\Psi(V - V_0^*) = 0\,, \quad (33)$$

where Ψ is a function of one variable.

In order for (33) to hold, the coefficient of each power of $(V - V_0^*)$ in the expression of Ψ must be equal to zero. This requirement allows us to determine the values of $\alpha_1, \alpha_2, \dots$ and $\varepsilon_1, \varepsilon_2, \dots$ in the power series (32). In the following, for simplicity we consider only the determination of α_1 and ε_1, i.e. we approximate $\omega(V_0)$ as

$$\omega(V_0) \approx \omega^* + \alpha_1(V_0 - V_0^*)^{\varepsilon_1}\,. \quad (34)$$

After substitution of (34) into (31), we obtain

$$\Psi(V_0 - V_0^*) = \frac{\partial F(\omega^*, V_0^*)}{\partial V_0}(V_0 - V_0^*) + \frac{\alpha_1^2}{2}\frac{\partial F(\omega^*, V_0^*)}{\partial \omega^2}(V_0 - V_0^*)^{2\varepsilon_1} + \dots \equiv 0\,. \quad (35)$$

The expression (35) contains the leading-order terms; all omitted terms are of a higher order of smallness.

We will analyze the case where

$$\frac{\partial F(\omega^*, V_0^*)}{\partial V_0} \neq 0 \, , \quad \frac{\partial^2 F(\omega^*, V_0^*)}{\partial \omega^2} \neq 0 \, . \tag{36}$$

A separate analysis is needed if one or both values in (36) are zero.

Consider now the cases $2\varepsilon_1 < 1$, $2\varepsilon_1 = 1$ and $2\varepsilon_1 > 1$, which together cover all possibilities for ε_1. If $2\varepsilon_1 < 1$, then the first term in (35) is of a higher order of smallness with respect to the second term, and consequently in order for (35) to hold, $\partial^2 F(\omega^*, V_0^*)/\partial\omega^2$ must be zero, which contradicts the second condition in (36). Similarly, if $2\varepsilon_1 > 1$, then in order for (35) to hold, $\partial F(\omega^*, V_0^*)/\partial V_0^2$ must be zero, which contradicts the first condition in (36). As a result, the only possible value is $2\varepsilon_1 = 1$, which transforms (35) into

$$\Psi(V_0 - V_0^*) = \left[\frac{\partial F(\omega^*, V_0^*)}{\partial V_0} + \frac{\alpha_1^2}{2} \frac{\partial F(\omega^*, V_0^*)}{\partial \omega^2} \right] (V_0 - V_0^*) + \cdots \equiv 0 \, . \tag{37}$$

The value of α_1 is found from the condition that the coefficient of $(V - V_0^*)$ is zero. We have

$$\alpha_1^2 = -2 \frac{\partial F(\omega^*, V_0^*)/\partial V_0}{\partial^2 F(\omega^*, V_0^*)/\partial \omega^2} \, . \tag{38}$$

Because the functionals (10)–(13) are all real-valued, and thus $F(\omega, V_0)$ is real-valued, it follows from (38) that α_1^2 is real-valued, and thus α_1 is either purely real or purely imaginary.

Thus we find the asymptotic dependence $\omega(V_0)$ in the small neighbourhood of the bifurcation point as

$$\omega(V_0) \approx \omega^* \pm \alpha_1 \sqrt{V_0 - V_0^*} \, , \quad |V_0 - V_0^*| \ll 1 \, . \tag{39}$$

From (39) it follows that in the small neighbourhood around each bifurcation point (ω^*, V_0^*), the frequency of harmonic vibrations ω obtains complex values. If the coefficient α_1 is real, then the frequency becomes complex for $V_0 < V_0^*$; otherwise (α_1 imaginary) the frequency becomes complex for $V_0 > V_0^*$.

The appearance of complex frequencies and their complex conjugates means that according to the model considered, the displacement will grow exponentially, which corresponds to instability in the Lyapunov sense. Thus, the considered elastic system exhibits elastic instability at the bifurcation points, and from a mathematical point of view, the bifurcation points correspond to static (divergence, buckling, $\omega^* = 0$) and dynamic (flutter, $\omega^* \neq 0$) kinds of instability in the Bolotin classification. Both kinds of instabilities are caught by the present analysis, because in both cases ($\omega^* = 0$ and $\omega^* \neq 0$) we have instability in the Lyapunov sense.

Example

As an example, consider the harmonic vibrations of an elastic panel (plate undergoing cylindrical deformation) moving in the axial direction at a constant velocity V_0, and with zero axial tension ($C = 0$). In this case, some of the bifurcation points lie on the V_0

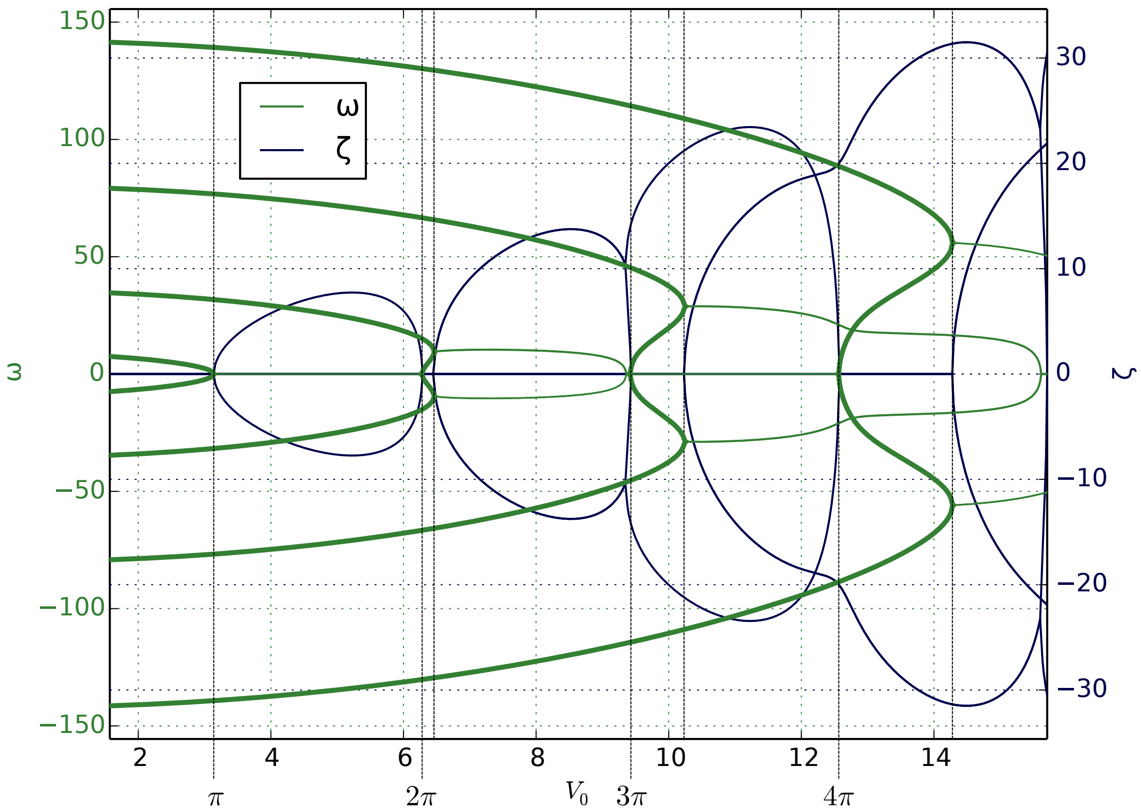

Figure 2. Behaviour of the complex natural frequencies $s = \zeta + i\omega$ as a function of the panel axial velocity V_0, drawn as overlaid projections onto the (V_0, ζ) and (V_0, ω) planes. Numerical solution using finite elements. Parts of the solution corresponding to Figure 1 have been **bolded**.

axis ($\omega = 0$), corresponding to static instabilities (divergence). For this set of points, the bifurcation values of the velocities are

$$V_{0k}^* = k\pi \ , \quad k = 1, 2, 3, \dots \quad \text{(static instabilities)}$$

and the dependences $\omega_k(V_0)$ in the small neighbourhood of the points $(0, V_{0k}^*)$ are given by

$$\omega_k(V_0) \approx \pm\alpha_{1k}\sqrt{V_0 - k\pi} + \dots \ , \quad |V_0 - k\pi| \ll 1 \ , \quad k = 1, 2, \dots \tag{40}$$

where we have used $\varepsilon_1 = 1/2$. Taking into account that $\alpha_{11}^2 < 0$, the first branch $\omega_1(V_0)$ is complex for $V_0 > \pi$, and consequently we will have instability for $V_0 = V_{01}^* = \pi$. It can be shown that the values α_{1k}^2 are positive for all $k \geq 2$, and consequently each branch $\omega_k(V_0)$ takes complex values at $V_0 < k\pi$.

The results of asymptotic analysis of $\omega_k(V_0)$ agree with the numerical solution presented in Figure 1, which was obtained by solving the spectral boundary value problem (4)–(5) as an eigenvalue problem for $(\omega, u(x))$ using finite elements of the Hermite type. Note that at least C^1 continuity is required at the element boundaries due to the term with the fourth derivative in the strong form.

For this picture, we present the bifurcation values for critical points outside the axis

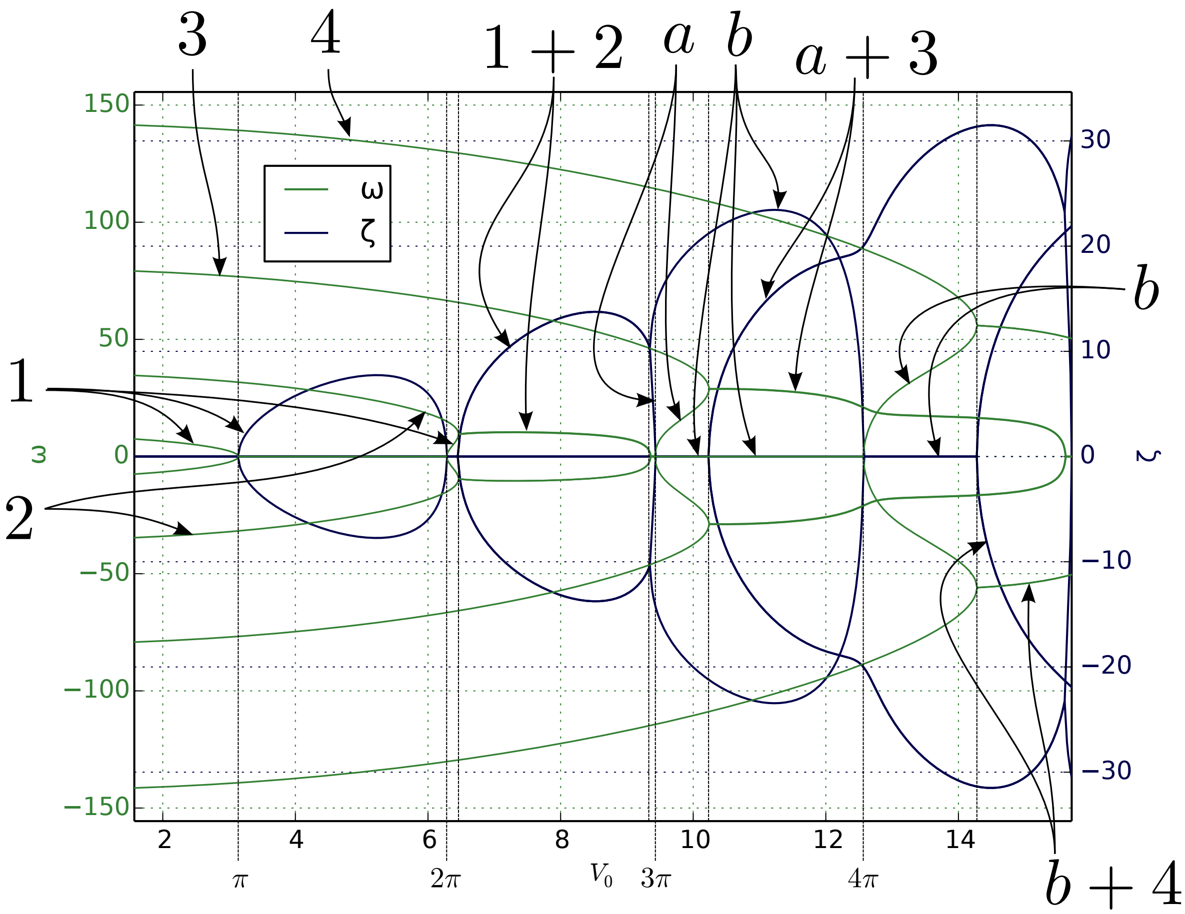

Figure 3. Behaviour of the complex natural frequencies $s = \zeta + i\omega$ as a function of the panel axial velocity V_0, drawn as overlaid projections onto the (V_0, ζ) and (V_0, ω) planes. Numerical solution using finite elements. Same data as in Figure 2, distinct modes labelled. If no real part ζ is labelled for a given mode, its real part is zero. For the mode "b", zero imaginary and real parts have been indicated where appropriate, to emphasize that at first it is a divergence mode, later becoming a stable vibration mode.

$V_0(\omega = 0)$, denoted by two lower indices:

$$(V^*_{021} = 6.45\,,\ \omega^*_{21} = \pm 10.58)$$
$$(V^*_{031} = 10.23\,,\ \omega^*_{31} = \pm 32.01)$$

In the numerical results shown here, 40 elements were used, with C^2 continuity across element boundaries. Each element had six local degrees of freedom, leading to three global degrees of freedom (u, u', u'') per node. With N_e elements, the total number of global nodes is $N_e + 1$. Boundary conditions eliminate four degrees of freedom. The total number of degrees of freedom in the discretization was thus $(N_e + 1) * 3 - 4 = 119$.

Finally, Figures 2 and 3 show the full complex-valued numerical solution, of which Figure 1 shows only those solutions for which the real part is zero. We replace (3) with

$$w(x, t) = e^{s\,t}\, u(x)\ , \quad s = \zeta + i\omega\ , \tag{41}$$

i.e. we now allow complex-valued frequencies to appear. Here the real part ζ represents the stability in the Bolotin sense. If, at any fixed value of V_0, one or more of the natural frequencies has $\zeta > 0$, the system is unstable for that value of V_0. The imaginary part ω is the same as above.

From Figure 2, we see that the model predicts a small supercritical stable range of V_0 (marked by the vertical lines near $V_0 = 2\pi$), beginning after the divergence gap spanning $(\pi, 2\pi)$. In this numerical example, no further stable ranges are seen; at any value of V_0 after this second stable range, there is always at least one solution with a positive real part.

It is an inherent usability drawback in the overlaid projection plot that it may become difficult to identify which real and imaginary parts belong to the same solution. Figure 3 shows the same data as Figure 2, but with distinct modes given labels in order to aid identification.

We see that after the short supercritical stable range above the divergence gap, the system again loses stability, this time by coupled-mode flutter. This is marked by the vertical line just after $V_0 = 2\pi$.

Shortly before $V_0 = 3\pi$, the imaginary part of this coupled mode (labelled as "$1 + 2$" in the Figure) vanishes, and the mode splits into two new modes (labelled "a" and "b"). This bifurcation point is marked by the vertical line just before $V_0 = 3\pi$.

Both "a" and "b" are initially divergence modes (nonzero real part, zero imaginary part). The real part of mode "a" reaches zero at $V_0 = 3\pi$, and at this critical point, mode "a" stabilizes. However, mode "b" remains a divergence mode until $V_0 = 4\pi$, where it stabilizes.

Slightly above $V_0 = 10$, mode "a" combines with mode "3", producing a new flutter mode of the coupled-mode type (labelled "$a+3$"). Later, slightly above $V_0 = 14$, the mode "b" combines with mode "4", producing another coupled-mode flutter mode (labelled "$b + 4$"). Near the right edge of the Figure, the imaginary part of mode "$a + 3$" becomes zero, and it splits into two new divergence modes (not labelled), similar to the earlier split of the coupled mode "$1 + 2$".

In conclusion, while useful information about this system can be derived even ignoring the real parts of the complex-valued natural frequencies, when making stability conclusions, one must look at general complex-valued solutions.

Conclusion

In this paper, the stability of an axially moving elastic panel was considered. The panel was travelling at constant velocity between a system of rollers. Small transverse elastic displacements of the panel were described by a fourth-order differential equation that included the centrifugal and Coriolis effects (induced by the axial motion), axial tension, and bending resistance. The same formulation directly applies also to the small out-of-plane elastic displacements of an axially travelling beam.

To study the stability of the system, a complex variable technique and bifurcation theory were applied. As a result, variational equations and a variational principle were derived. Analysis of the variational principle allowed the study of qualitative properties of the bifurcation points. Asymptotic behaviour in a small neighbourhood around an arbitrary bifurcation point was analyzed and presented. The bifurcation points were found by determining conditions where the conditions of the implicit function theorem (which concerns the uniqueness of a local explicit representation of an implicit function) are violated.

It was shown analytically that the eigenvalue curves in the (ω, V_0) plane cross both the ω and V_0 axes perpendicularly. It was also shown that near each bifurcation point, the dependence $\omega_k(V_0)$, for each mode k, approximately follows the shape of a square root function (considered near the origin). From this analysis it was also seen that, as expected for this class of systems, the eigenvalues appear in conjugate pairs.

The obtained results complement existing numerical studies on the stability of axially moving materials, especially those with finite bending rigidity. From a rigorous mathematical viewpoint, the presence of bending rigidity is essential, because the presence of the fourth-order term in the model changes the qualitative behaviour of the bifurcation points.

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