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Periodic spectral instability analysis of axially moving beam with elastic supports

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Summary. Problems of dynamics and stability of a moving web, modelled as an elastic beam of unlimited length and axially travelling between an infinite system of rollers (elastic supports) at a constant velocity, are studied using analytical approaches. Transverse elastic displacements of the web are described by a fourth order differential equation that includes the centrifugal force, in-plane tension, bending term and elastic support reaction. The stability of the beam is investigated with the help of studies of small periodic transverse displacements. In this connection the multipoint spectral stability problem of the unlimited length beam with elastic supports is formulated for the periodic interval. In the frame of spectral analysis, it is shown that the onset of instability takes place in the form of divergence (buckling). It is shown that the instability behaviour of the moving beam with elastic supports coincides with that of the same beam with absolutely rigid supports, when the stiffness of the supports exceeds a critical value.

Key words: axially moving beam, stability, elastic supports, periodic systems, multipoint spectral stability problem

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Introduction

The fundamental aim of our research has been to create mathematical models related to the paper making process, simplifying the problems of moving materials sufficiently, while still providing an understanding of the phenomena both qualitatively and quantitatively. A key point is that the productivity of the paper mill is strongly dependent on the efficiency and reliability of the running paper web. If it is possible to increase the axial velocity of the web, production increases, and the paper mill will have more paper to sell. In this paper, we will demonstrate the fundamental idea of combining process and product to the machinery and its properties. This approach will provide new insight for controlling and analyzing paper machines and similar systems and their processes using mathematical modelling efficiently. This combination process may reveal unexpected properties which would not be seen by considering just one part of the system. For example, in the present study it will be seen that the stability reaches its maximum after the stiffness of the supports exceeds a critical value. Observations such as this, made possible by the combined approach, open the possibility to improve the quality of manufacturing by using less resources than before.

In our previous studies (see e.g., Banichuk et al. (2013b), Banichuk et al. (2013a), Banichuk et al. (2011), Banichuk et al. (2010b)), we have considered many aspects of

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mathematical modelling of the paper making process. The studies themselves however are general, and the results can be used in many different practical applications. Examples include the processing of paper or steel, fabric, rubber or some other continous material, and looping systems such as band saws and timing belts.

The most often used models for systems of axially travelling material have been travelling flexible strings, membranes, beams, and plates. The dynamic and stability considerations discussed here were first reviewed in the article by Mote (1972). Natural frequencies are commonly analyzed together with the stability. It was realized early on that the vibration problem for an axially moving continuum is not the conventional one. Because of the longitudinal continuity of the material, the equation of motion for transverse vibration will contain additional terms, representing a Coriolis force and a centripetal force acting on the material. As a consequence, the resonant frequencies will be dependent on the longitudinal velocity of the axially moving continuum, as was noted by Archibald and Emslie (1958), as well as Swope and Ames (1963), Simpson (1973), and Mujumdar and Douglas (1976).

The effects of axial motion of the web on its frequency spectrum and eigenfunctions were investigated in the papers by Archibald and Emslie (1958) and by Simpson (1973). It was shown that the natural frequency of each mode decreases when the transport speed increases, and that the travelling string and beam both experience divergence instability at a sufficiently high speed. However, in the case of the string, this result was recently contrasted by Wang et al. (2005), who showed using Hamiltonian mechanics that the ideal string remains stable at any speed. Travelling beams have been further analyzed by Parker (1998) in his study on gyroscopic continua, and by Kong and Parker (2004), where an approximate analytical expression was derived for the eigenfrequencies of a moving beam with small flexural stiffness. Response predictions have been made for particular cases where the excitation assumes special forms, such as harmonic support motion (Miranker (1960)) or a constant transverse point force (Chonan (1986)). Arbitrary excitations and initial conditions were analyzed with the help of modal analysis and a Green's function method in the article by Wickert and Mote (1990). As a result, the critical speeds for travelling strings and beams were explicitly determined. Travelling strings and beams on an elastic foundation have been investigated by, e.g., Bhat et al. (1982), Perkins (1990), Wickert (1994) and Parker (1999).

The loss of stability was studied with an application of dynamic and static approaches in the article by Wickert (1992). It was shown by means of numerical analysis that in all cases instability occurs when the frequency is zero and the critical velocity coincides with the corresponding velocity obtained from static analysis. The dynamical properties of moving plates have been studied by Shen et al. (1995) and by Shin et al. (2005), and the properties of a moving paper web have been studied in the two-part article by Kulachenko et al. (2007a,b). Critical regimes and other problems of stability analysis have been studied e.g. by Wang (2003) and Sygulski (2007).

In Yurddas et al. (2013) the nonlinear vibrations of an axially moving multi-supported string have been investigated. They have studied very similar case as us, with non-ideal supports allowing minimal deflections between ideal supports at both ends of the string, but used Hamiltonian approach and concentrate on nonlinear dynamics. Moreover, in Chen (2005) there are extensive literature review for nondynamical studies related to moving strings.

Results that axially moving beams experience divergence instability at a sufficiently high beam velocity have been obtained also for beams interacting with external media;

see, e.g., Chang and Moretti (1991), Banichuk and Neittaanmäki (2008b), Banichuk and Neittaanmäki (2008c) and Banichuk and Neittaanmäki (2008a). The study has been extended in Banichuk et al. (2010a), for a two-dimensional model of the web, considered as a moving plate under homogeneous tension but without external media. The most straightforward and efficient way to study stability is to use linear stability analysis.

In a recent article by Hatami et al. (2009), the free vibration of a moving orthotropic rectangular plate was studied at sub- and supercritical speeds, and its flutter and divergence instabilities at supercritical speeds. The study is limited to simply supported boundary conditions at all edges. For the solution of equations of orthotropic moving material, many necessary fundamentals can be found in the book by Marynowski (2008).

In the present study, we will limit our focus to moving beams. The loss of stability of an elastic infinite beam travelling between a system of elastic supports at a constant velocity, will be studied. We will describe transverse elastic displacements of the beam by a fourth order differential equation. The centrifugal force, axial tension, bending rigidity term and elastic support reaction will be taken into account. The stability of the beam will be investigated using analysis of small periodic transverse displacements. We will perform the studies mainly using analytical approaches. We will formulate a multipoint spectral stability problem with elastic supports, and use the periodic interval and Floquet's representation of solution. As a result, the basic relations characterizing the behaviour of the web at the onset of instability are found in an analytical form. The critical velocity, which corresponds to the onset of instability in the form of divergence (buckling), will be estimated in the frame of spectral analysis. We also analyse the dependence of the critical velocity on the support rigidity parameter.

Governing equations of elastic instability of axially moving web interacting with elastic supports

The equation of unforced small transverse vibrations of a web travelling with a constant velocity V_0 along the axis x and interacting with elastic supports at $x_n = n\ell$ $(n = 0, \pm 1, \pm 2, ...)$ has the form

$$m\frac{\partial^2 w}{\partial t^2} + 2mV_0\frac{\partial^2 w}{\partial x \partial t} + mV_0^2\frac{\partial^2 w}{\partial x^2} - T_0\frac{\partial^2 w}{\partial x^2} + D\frac{\partial^4 w}{\partial x^4} = 0\,, \tag{1}$$

where m is the mass per unit length of the beam, $D = EI$ is the bending rigidity of the beam (E is Young's modulus, I is moment of inertia), T_0 is tension along the $x-$axis, w is the small displacement in the $z-$direction (see Figure 1). The equation (1) of the travelling web, modelled as a beam of unlimited length with elastic supports (elastic roller-supports), is written with respect to the fixed reference frame xz.

Let us consider static form of instability (divergence or buckling). In this case time derivatives in the equation (1) vanishes. We will have

$$\left(mV_0^2 - T_0\right)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + D\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = 0\,. \tag{2}$$

Introducing the following notations

$$\lambda = \gamma^2 = \frac{1}{D}\left(mV_0^2 - T_0\right), \qquad \varkappa = \frac{k}{EI}\,, \tag{3}$$

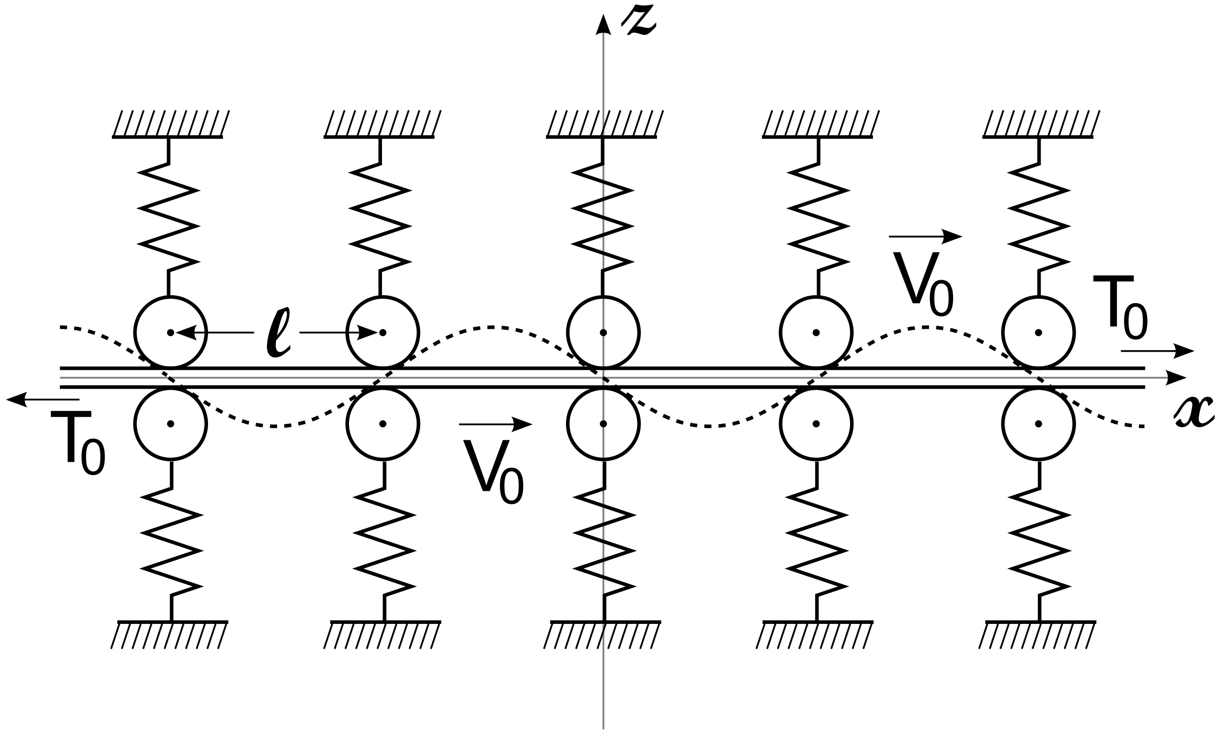

Figure 1. Beam of unlimited length, axially travelling at velocity V_0, supported by an infinite system of elastic supports with spacing ℓ, and subjected to constant axial tension T_0. The straight lines represent the beam in the trivial equilibrium configuration, while the dashed line shows one possible deformed shape.

we formulate the following multipoint spectral problem of elastic instability for the ordinary differential equation

$$\lambda \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = 0\,, \qquad x_{j-1} < x < x_j\,, \tag{4}$$

where $x_j = j\ell,\ j = 0, \pm 1, \pm 2, ...$ with conjuction conditions

$$\begin{gathered} (w)^+_{x_j} = (w)^-_{x_j}\,, \qquad \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^+_{x_j} = \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^-_{x_j}\,, \\ \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^+_{x_j} = \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^-_{x_j}\,, \qquad j = 0, \pm 1, \pm 2... \\ \left(\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\right)^+_{x_j} - \left(\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\right)^-_{x_j} = -\varkappa\,(w)_{x_j}\,. \end{gathered} \tag{5}$$

where the upper symbols + and − denote right and left limiting values, respectively, and k is a constant usually called the modulus of the support (elastic foundation). This constant denotes the reaction per unit length when the deflection w is equal to unity. The parameter λ in the equation (4) plays the role of the eigenvalue of the considered spectral problem. This problem is described by the ordinary differential equation (4) with constant coefficients and periodic boundary conditions (5), and consequently its solution can be represent with the help of Floquet's theorem as Floquet (1883); Jkubovich and Starjinsky (1972)

$$w\,(x, \alpha) = w_0\,(x)\, e^{i\alpha x}\,, \qquad -\infty < x < \infty\,, \qquad \alpha \in [\,0,\ 2\pi/\ell\,]\,. \tag{6}$$

Here $i \equiv \sqrt{-1}$ is the imaginary unit, α is some real parameter and w_0 is a periodic function with period ℓ, i.e.

$$w_0\,(x + s\ell) \equiv w_0\,(x)\,, \qquad s = \pm 1, \pm 2, \ldots \tag{7}$$

Using the representation (6) and (7) it is required to find $w_0\,(x)$ at the interval $[0, \ell]$. Considering $w(x, \alpha)$ at $x = 0$ and $x = \ell$, and using the equality $w\,(0, \alpha) = w_0\,(0)$ we will have

$$w\,(\ell, \alpha) = w_0\,(\ell)\, e^{i\alpha\ell} = w_0\,(0)\, e^{i\alpha\ell} = w\,(0, \alpha)\, e^{i\alpha\ell}\,. \tag{8}$$

In a similar way we obtain

$$\left(\frac{\mathrm{d}^s w}{\mathrm{d}x^s}\,(\ell, \alpha)\right)^{\pm} = \left(\frac{\mathrm{d}^s w}{\mathrm{d}x^s}\,(0, \alpha)\right)^{\pm} e^{i\alpha\ell}\,, \qquad s = 0, 1, 2, \ldots, \qquad \alpha \in [\,0,\, 2\pi/\ell\,]\,. \tag{9}$$

Thus the multipoint periodic spectral problem (4),(5) is reduces to the eigenvalue problem formulated on the interval $[0, \ell]$:

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + \lambda \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0, \qquad 0 < x < \ell\,, \tag{10}$$

$$w\,(\ell, \alpha) = w\,(0, \alpha)\, e^{i\alpha\ell}\,, \qquad \frac{\mathrm{d}w}{\mathrm{d}x}\,(\ell, \alpha) = \frac{\mathrm{d}w}{\mathrm{d}x}\,(0, \alpha)\, e^{i\alpha\ell}\,, \tag{11}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,(\ell, \alpha) = \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,(0, \alpha)\, e^{i\alpha\ell}\,, \qquad 0 \le \alpha\ell \le 2\pi\,, \tag{12}$$

$$\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\,(\ell, \alpha) = e^{i\alpha\ell}\left[\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\,(0, \alpha) - \varkappa w\,(0, \alpha)\right]. \tag{13}$$

In the case of absolutely rigid supports we have $\varkappa = \infty$, and the condition (13) is excluded from the consideration. It is seen from the presented formulation (10) – (13) that the values of λ and w depend continously on α in the interval $\alpha \in [\,0,\, 2\pi/\ell\,]$, i.e.

$$\lambda = \lambda\,(\alpha)\,, \qquad w = w\,(x, \alpha)\,. \tag{14}$$

Note also the following variational formulation corresponding to the spectral problem (10) – (13). It is required to find a minimal value

$$\lambda\,(\alpha) = \min_{w, w^*} \frac{\int_0^\ell \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,(x, \alpha)\, \frac{\mathrm{d}^2 w^*}{\mathrm{d}x^2}\,(x, \alpha)\, \mathrm{d}x + \varkappa w\,(0, \alpha)\, w^*\,(0, \alpha)}{\int_0^\ell \frac{\mathrm{d}w}{\mathrm{d}x}\,(x, \alpha)\, \frac{\mathrm{d}w^*}{\mathrm{d}x}\,(x, \alpha)\, \mathrm{d}x} \tag{15}$$

under the constraints

$$w = w\,(x, \alpha) \in \Lambda\,, \qquad w^* = w^*\,(x, \alpha) \in \Lambda^*\,, \tag{16}$$

where Λ and Λ^* are sets of admissible functions satisfying, respectively boundary conditions (11) – (13) for w and complex conjugate boundary conditions for w^*.

In the limiting case of absolutely rigid supports, when $\varkappa = \infty$, the corresponding boundary conditions take the form

$$w\,(0, \alpha) = w\,(\ell, \alpha) = 0\,, \qquad \frac{\mathrm{d}w}{\mathrm{d}x}\,(\ell, \alpha) = \frac{\mathrm{d}w}{\mathrm{d}x}\,(0, \alpha)\, e^{i\alpha\ell}\,, \tag{17}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(\ell,\alpha) = \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(0,\alpha)\, e^{i\alpha\ell}\,, \qquad 0 \le \alpha\ell \le 2\pi\,.$$

In this case the variational formulation (15), (16) is reduced to the minimization

$$\lambda(\alpha) = \min_{w,w^*} \frac{\int_0^\ell \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(x,\alpha)\, \frac{\mathrm{d}^2 w^*}{\mathrm{d}x^2}(x,\alpha)\,\mathrm{d}x}{\int_0^\ell \frac{\mathrm{d}w}{\mathrm{d}x}(x,\alpha)\, \frac{\mathrm{d}w^*}{\mathrm{d}x}(x,\alpha)\,\mathrm{d}x} \tag{18}$$

under constraints that $w(x,\alpha)$ and $w^*(x,\alpha)$ satisfy, respectively, boundary conditions (17) for w and complex conjugate boundary conditions for w^*.

It follows from the considered formulation (10) – (13) that for each real eigenvalue $\lambda = \lambda(\alpha)$, beside the eigenfunction $w(x,\alpha)$ also the complex conjugate function w^* satisfies the differential equation. The boundary condition for w^* are derived by passing on to complex conjugate values in (11) – (13). Taking into account that

$$e^{-i\alpha\ell} = e^{i2\pi} e^{-i\alpha\ell} = e^{i(2\pi/\ell - \alpha)\ell}\,, \tag{19}$$

we obtain

$$w^*(x,\alpha) = w\left(x, {}^{2\pi}/_{\ell} - \alpha\right)\,. \tag{20}$$

Thus, for $\alpha = \pi/\ell$ we have

$$w^*\left(x, {}^{\pi}/_{\ell}\right) = w\left(x, {}^{\pi}/_{\ell}\right)\,, \tag{21}$$

and in this case $w\left(x, {}^{\pi}/_{\ell}\right)$ is a real solution.

Let us use the property (20) and the variational representation (15) for $\lambda\left({}^{2\pi}/_{\ell} - \alpha\right)$, and pass from the functions $w\left(x, {}^{2\pi}/_{\ell} - \alpha\right)$ to the functions $w(x,\alpha)$ in the integrands. As a result we will have the equality

$$\lambda(\alpha) = \lambda\left({}^{2\pi}/_{\ell} - \alpha\right)\,, \tag{22}$$

and that the following two functions

$$w(x,\alpha)\,,\, w^*(x,\alpha) = w\left(x, {}^{2\pi}/_{\ell} - \alpha\right) \tag{23}$$

correspond to one and the same eigenvalue $\lambda = \lambda(\alpha)$.

Thus, in the case when $\alpha \ne \pi/\ell$, the eigenvalues are double. Taking into account that the differential equation (10) is invariant with respect to the operation $x \to \ell - x$, and introducing the notation

$$\widetilde{w}(x,\alpha) = w(\ell - x,\alpha)\,, \tag{24}$$

we obtain that the functions $\widetilde{w}(x,\alpha)$ and $w(x,\alpha)$ satisfy the differential equation for one and the same eigenvalue $\lambda(\alpha)$. In addition the boundary conditions for the functions $\widetilde{w}(x,\alpha)$ correspond to $\widetilde{\alpha} = {}^{2\pi}/_{\ell} - \alpha$, and according to (22), (23) we will have $\widetilde{w}(x,\alpha) \equiv w^*(x,\alpha)$. As a result we obtain

$$w(\ell - x,\alpha) \equiv w^*(x,\alpha)\,, \qquad 0 \le x \le \ell, \qquad 0 \le \alpha\ell < 2\pi\,. \tag{25}$$

Consequently the complete analysis can be restricted to the interval $0 \le \alpha \le {}^{\pi}/_{\ell}$.

Taking into account the condition (25) and performing double integration we represent the equation (4) in the form

$$\lambda w + \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 2iD_1\left(x - \frac{\ell}{2}\right) + D_2\,, \qquad 0 < x < \ell\,. \tag{26}$$

Here $D_1,\ D_2$ are unknown real constants of integration.

Instability analysis

Let us use the following dimensionless variables

$$\tilde{x} = \frac{x}{\ell}, \qquad \tilde{\lambda} = \ell^2 \lambda, \qquad \tilde{\varkappa} = \varkappa \ell^3 = \frac{k\ell^3}{EI}\,. \tag{27}$$

In these dimensionless variables we will have

$$\lambda\,(2\pi - \alpha) = \lambda\,(\alpha)\,, \qquad w\,(x,\, 2\pi - \alpha) = w^*\,(x, \alpha)\ , \tag{28}$$

and the boundary value problem (10) – (13) is written as

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + \lambda \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0\,, \qquad 0 < x < 1\,, \tag{29}$$

$$(w)_{x=1} = e^{i\alpha}\,(w)_{x=0}\ , \qquad \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)_{x=1} = e^{i\alpha}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)_{x=0}\,,$$

$$\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)_{x=1} = e^{i\alpha}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)_{x=0}\,, \tag{30}$$

$$\left(\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\right)_{x=1} = e^{i\alpha}\left(\left(\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\right)_{x=0} + \varkappa\,(w)_{x=0}\right), \qquad 0 \le \alpha \le \pi\,. \tag{31}$$

Here and in what follows the tilde is omitted. General solutions of (29) have the form

$$w = C_1 \cos\left(\sqrt{\lambda}x\right) + C_2 \sin\left(\sqrt{\lambda}x\right) + C_3 x + C_4\,, \tag{32}$$

where the eigenvalue λ and the arbitrary constants $C_1,\ C_2,\ C_3,\ C_4$ are determined with the help of the boundary conditions (30), (31).

Let us analyse separately the solutions of the considered problem in the cases of elastic supports and absolute rigid supports. In the case of rigid supports, $\varkappa = \infty$, and the boundary conditions are written as

$$(w)_{x=0} = (w)_{x=1} = 0\,, \qquad 0 \le \alpha \le \pi\,, \tag{33}$$

$$\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)_{x=1} = e^{i\alpha}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)_{x=0}\,, \qquad \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)_{x=1} = e^{i\alpha}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)_{x=0}\,.$$

Using the general solution (32) and the boundary conditions (33) we will obtain the following transcendental equation:

$$\cos\lambda = \varphi\,(\lambda)\,, \qquad \varphi\,(\lambda) \equiv \frac{\sqrt{\lambda}\cos\sqrt{\lambda} - \sin\sqrt{\lambda}}{\sqrt{\lambda} - \sin\sqrt{\lambda}}, \qquad 0 \le \alpha \le \pi\,, \tag{34}$$

for finding the eigenvalues in the case of rigid supports. The eigenvalues are determined taking into account the condition $-1 \le \cos\alpha \le 1$, and consequently, the eigenvalues fill the continuous interval with the boundaries corresponding to the conditions $\cos\alpha = \pm 1$.

The function $\varphi\,(\lambda)$ tends to -2 (i.e. $\varphi\,(\lambda) \to -2$) when λ tends to zero ($\lambda \to 0$), and consequently, the lower bound of the spectrum is determined with the help of the equation $\cos\alpha = -1$, which corresponds to $\alpha = \pi$. Then the equation (34) is reduced to the following equations:

$$\cos\left(\frac{\sqrt{\lambda}}{2}\right) = 0, \qquad \tan\left(\frac{\sqrt{\lambda}}{2}\right) = \frac{\sqrt{\lambda}}{2}\,. \tag{35}$$

The solution of the first equation in (35),

$$\lambda_k = \pi^2\left(1+2k\right)^2, \qquad k = 0, 1, 2, ... \tag{36}$$

determines the lower bound of the continuous spectrum, when $\alpha = \pi$. The values $\lambda_k = \pi^2\left(1+2k\right)^2$ are simple eigenvalues, and the corresponding eigenfunctions

$$w_k\left(x, \pi\right) = C\sin\left[\left(1+2k\right)\pi x\right], \qquad 0 \le x \le 1\,, \tag{37}$$

are even with respect to the midpoint of the interval $[0, 1]$. Note here that the presented eigenvalues and eigenfunctions correspond to the solution of the stability problem for the simply supported beam.

Consider the case of elastic supports, in which case $\varkappa \neq \infty$. In this case we will use general solution (32) and boundary conditions (30),(31) to find unknown constants C_1, C_2, C_3, C_4. Substituting (32) into (30), (31) we derive a homogeneous system of equations with respect to the arbitrary constants C_1, C_2, C_3, C_4. To obtain a nontrivial solution of this system, we require that its determinant $\psi\left(\lambda, \kappa, \alpha\right)$ become zero. Thus we arrive at the following equation to determine $\lambda = \lambda\left(\varkappa, \alpha\right)$:

$$\psi\left(\lambda, \varkappa, \alpha\right) = 0\,, \tag{38}$$

where

$$\psi\left(\lambda, \varkappa, \alpha\right) \equiv \tag{39}$$

$$\frac{\varkappa}{\sqrt{\lambda^3}}\left(\cos\sqrt{\lambda} - \cos\alpha\right) + \frac{\left(1-\cos\alpha\right)}{\sqrt{\lambda}}\left[2\left(\cos\alpha - \cos\sqrt{\lambda}\right) - \frac{\varkappa\sin\sqrt{\lambda}}{\sqrt{\lambda^3}}\right].$$

For a fixed value of $\varkappa$ (the stiffness of the supports) the loss of elastic stability of the beam occurs when the eigenvalue $\lambda(\varkappa, \alpha)$ achieves a minimum value, i.e.

$$\lambda_* = \min_{\alpha}\lambda\left(\varkappa, \alpha\right), \qquad 0 \le \alpha \le \pi\,. \tag{40}$$

Thus the critical values $\lambda = \lambda_*$ are found with the help of the equations

$$\psi\left(\lambda, \varkappa, \alpha\right) = 0\,, \qquad \frac{\partial\psi\left(\lambda, \varkappa, \alpha\right)}{\partial\alpha} = 0\,, \tag{41}$$

and the supposition

$$\frac{\partial\psi\left(\lambda, \varkappa, \alpha\right)}{\partial\lambda} \neq 0\,. \tag{42}$$

Using the expression (39) for $\psi\left(\lambda, \varkappa, \alpha\right)$ we write the second equation in (41) as

$$\frac{\partial\psi\left(\lambda, \varkappa, \alpha\right)}{\partial\alpha} = \frac{\sin\alpha}{\sqrt{\lambda}}\left[4\cos\alpha - 2\left(1+\cos\sqrt{\lambda}\right) + \frac{\varkappa}{\lambda}\left(1 - \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\right)\right] = 0\,. \tag{43}$$

This equation satisfied either when $\alpha = 0$ or $\alpha = \pi$ (i.e. $\sin\alpha = 0$), or when the expression in square brackets vanishes, i.e.

$$\cos\alpha = \frac{1}{2}\left(1 + \cos\sqrt{\lambda}\right) - \frac{\varkappa}{4\lambda}\left(1 - \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\right). \tag{44}$$

Let us investigate all three cases. First, if $\alpha = 0$, then the system of equations is reduced to a single equation

$$\psi\left(\lambda, \varkappa, 0\right) = \frac{\varkappa}{\sqrt{\lambda^3}}\left(\cos\sqrt{\lambda} - 1\right) = 0\,. \tag{45}$$

This equation has double roots $\lambda_j = 4j^2\pi^2$, ($j = 1, 2, ...$), and corresponding two-parametric eigenmodes

$$w_j = C_{1j}\cos 2j\pi x + C_{2j}\sin 2j\pi x\,.$$

The minimal eigenvalue $\lambda_1 = 4\pi^2$ determines the upper bound of the first band of the continuous distribution of eigenvalues in the problem of elastic instability of the beam with rigid supports ($\varkappa = \infty$).

Next, consider the case $\alpha = \pi$. The system of equations (41) is reduced to the equation

$$\psi\left(\lambda, \varkappa, \pi\right) = \frac{\varkappa\left(1 + \cos\sqrt{\lambda}\right)}{\sqrt{\lambda^3}} - \frac{2}{\sqrt{\lambda}}\left[2\left(1 + \cos\sqrt{\lambda}\right) + \frac{\varkappa\sin\sqrt{\lambda}}{\sqrt{\lambda^3}}\right] = 0\,, \tag{46}$$

which can be written as

$$\left[\frac{\varkappa}{\lambda\sqrt{\lambda}}\left(\cos\frac{\sqrt{\lambda}}{2} - \frac{2}{\sqrt{\lambda}}\sin\frac{\sqrt{\lambda}}{2}\right) - \frac{4}{\sqrt{\lambda}}\cos\frac{\sqrt{\lambda}}{2}\right]\cos\frac{\sqrt{\lambda}}{2} = 0\,. \tag{47}$$

The first set of solutions of the equation (47) is given by $\cos(\sqrt{\lambda}/2) = 0$, and is written as $\lambda_j = (2j+1)^2\pi^2$, $j = 0, 1, 2, ...$ This set of solutions does not depend on the rigidities of the elastic supports. The second set of solutions of the equation (47) is obtained if we assume that $\cos(\sqrt{\lambda}/2) \neq 0$, and the expression in square brackets in (47) is equal to zero. For these solutions we have

$$\varkappa = \frac{4\lambda}{1 - \dfrac{2}{\sqrt{\lambda}}\tan\left(\dfrac{\sqrt{\lambda}}{2}\right)}\,. \tag{48}$$

Since $\lambda \geq 0$ and $\varkappa \geq 0$, then (as follows from the equation (48)) the admissible values of λ satisfy the inequality $\tan(\sqrt{\lambda}/2) \leq 0$. From here we derive the following estimates for the minimal admissible λ: it must hold that $\pi/2 \leq \sqrt{\lambda}/2 \leq 3\pi/2$. Consequently we will have $\lambda \geq \pi^2$ for the minimal values of λ. Since these values exceed the value which corresponds to the beam of unlimited length with rigid supports, we exclude this case from the consequent analysis.

Finally, consider now the system (41) assuming that $\alpha \neq 0$ and $\alpha \neq \pi$. In this case, as it was noted before, the equation (43) will be fulfilled if (44) is satisfied. Substituting (44) into (38), we derive a nonlinear equation for finding λ as a function of $\varkappa$:

$$p_1^2 - \frac{p_1}{\lambda}\left(1 + p_3\right)\varkappa + \frac{1}{4\lambda^2}\left(1 - p_3\right)\varkappa^2 = 0\,. \tag{49}$$

Here and in what follows we will use, for brevity, the notations

$$p_1 = 1 - \cos\sqrt{\lambda}, \quad p_2 = 1 + \cos\sqrt{\lambda}, \quad p_3 = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}, \quad z = \frac{\varkappa}{2\lambda}\left(1 - p_3\right) \geq 0\,. \tag{50}$$

This equation is solved under the constraint

$$\left|\frac{1}{2}p_2 - \frac{\varkappa}{4\lambda}\left(1 - p_3\right)\right| \leq 1\,, \tag{51}$$

which is a consequence of the inequality $-1 \leq \cos\alpha \leq 1$ and the representation (44) for $\cos\alpha$.

The equation (49) and the constraint (51) can be transformed to the following form:

$$z^2 - \frac{2p_1\left(1 + p_3\right)}{1 - p_3}z + p_1^2 = 0\,, \tag{52}$$

$$0 \leq z \leq \chi\left(\lambda\right) \equiv 2 + p_2\,. \tag{53}$$

Consider the two solutions z_1 and z_2 of the equation (52),

$$z_1 = \frac{p_1\left(1 + \sqrt{p_3}\right)}{1 - \sqrt{p_3}}\,, \qquad z_2 = \frac{p_1\left(1 - \sqrt{p_3}\right)}{1 + \sqrt{p_3}}\,, \tag{54}$$

on the interval $\lambda \in [0, \pi^2]$. Only the solution z_2 satisfies the inequality (53).

For the solution z_1, we observe $z_1\left(\lambda = 0\right) = 12$, $z_1\left(\lambda = \pi^2\right) = 2$, and that $z_1\left(\lambda\right)$ is a monotonically decreasing function in the interval $\lambda \in [0, \pi^2]$. Hence the inequality (53) cannot be satisfied in this interval for the solution $z_1\left(\lambda\right)$. The functions $z_1\left(\lambda\right)$ and $\chi\left(\lambda\right) = 2 + p_2 = 3 + \cos\sqrt{\lambda}$ (where p_2 is defined in (50)) are shown in Figure 2.

Taking into account the definition (50) for z, and the expression (54) for the admissible solution z_2, we obtain

$$\frac{p_1\left(1 - \sqrt{p_3}\right)}{\left(1 + \sqrt{p_3}\right)} = \frac{\varkappa\left(1 - p_3\right)}{2\lambda}\,.$$

Performing simplifications we can rewrite this the following form, solved with respect to $\varkappa$:

$$\varkappa = \frac{2\lambda p_1}{\left(1 + \sqrt{p_3}\right)^2} = 2\lambda\left(1 - \cos\sqrt{\lambda}\right)\left(1 + \sqrt{\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}}\right)^{-2}\,. \tag{55}$$

Because the support stiffness parameter $\varkappa$ is a real number, we see from (55) that p_3 must be positive. The requirement of positiveness of p_3 is reduced to the condition $\sin\sqrt{\lambda} \geq 0$, and consequently we obtain the following intervals of admissible eigenvalues:

$$4j^2\pi^2 \leq \lambda_j \leq \left(2j + 1\right)^2\pi^2\,, \qquad j = 0, 1, 2, ... \tag{56}$$

The loss of stability of moving continuous beam of unlimited length is realised for the minimal values of critical velocity, and thus the corresponding eigenvalues belong to the interval

$$0 < \lambda \leq \pi^2\,. \tag{57}$$

It is important to note that in accordance with the relation (55), we have $\varkappa = \varkappa_* = 4\pi^2$ for $\lambda = \lambda_* = \pi^2$. As it is seen from (57), the value $\lambda_* = \pi^2$ is a maximal possible value.

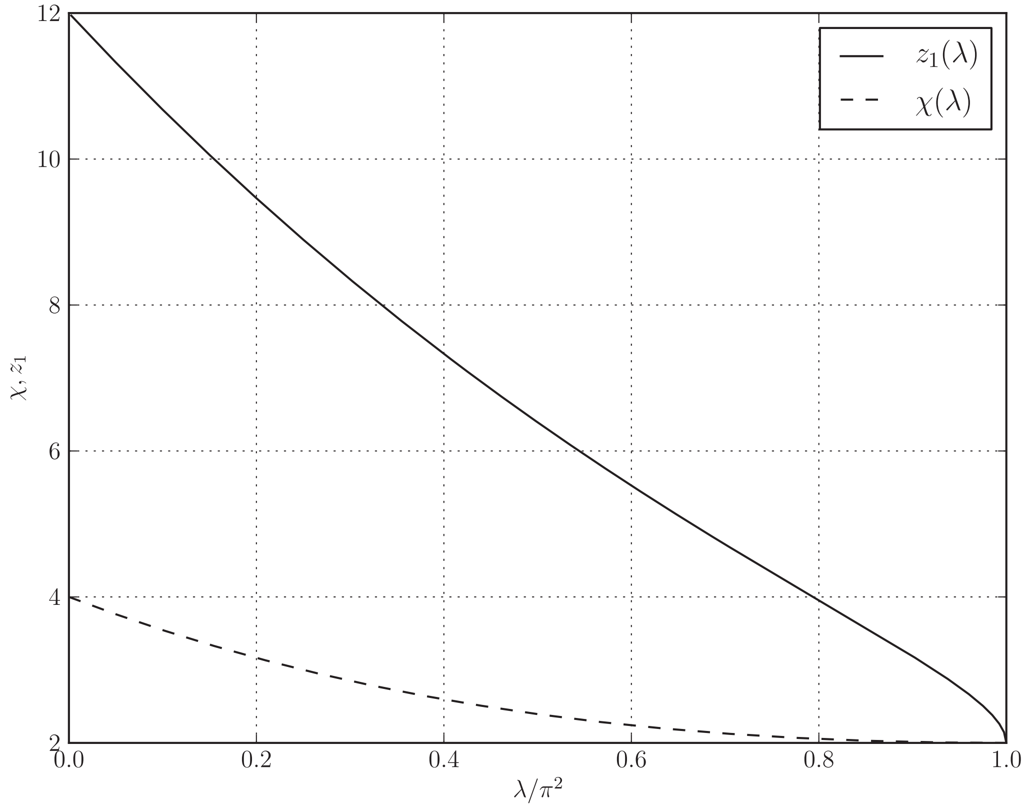

Figure 2. The solution $z_1(\lambda)$, defined in equation (54), and the auxiliary function $\chi(\lambda) = 3 + \cos\sqrt{\lambda}$, from equation (53), are represented.

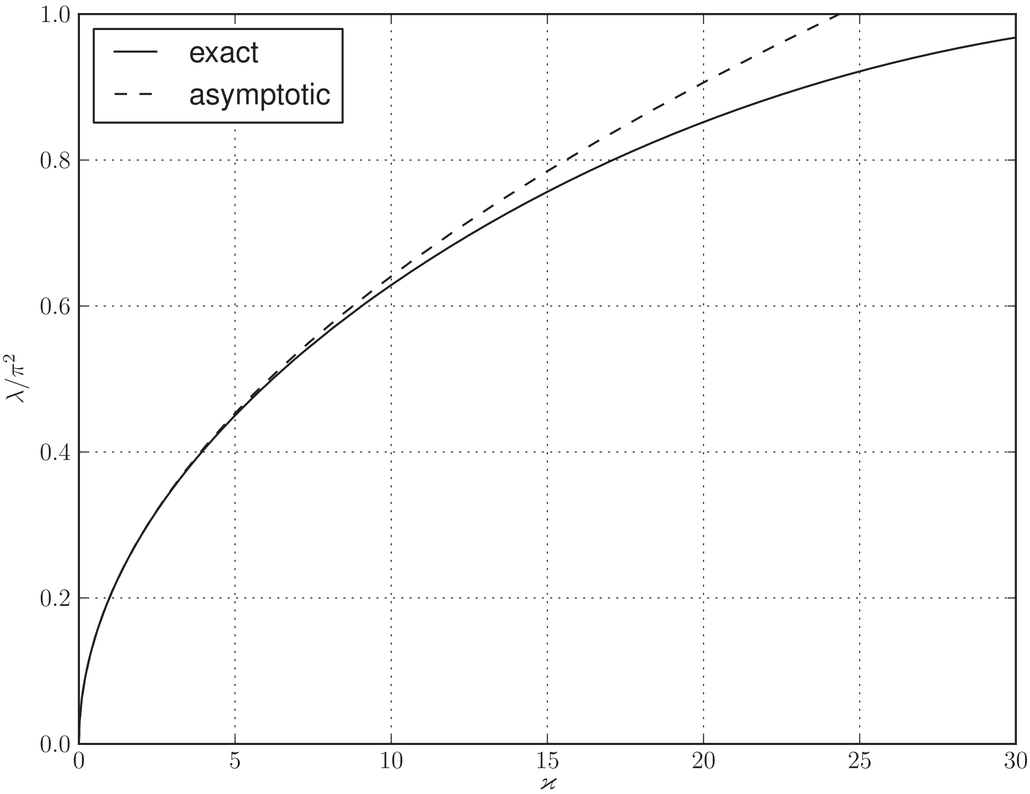

Figure 3. The dependence of the critical eigenvalue λ on the elastic support rigidity parameter $\varkappa$, obtained in accordance with the exact relation (55) and the asymptotic formula (58).

Taking into account the variational formulation of the considered spectral problem, we note that $\lambda(\varkappa)$ is a monotonically increasing function (i.e. $\mathrm{d}\lambda(\varkappa)/\mathrm{d}\varkappa \geq 0$). Consequently, the critical eigenvalue λ must remain the same when $\varkappa \geq \varkappa_*$. On the other hand, for absolutely rigid supports, $\varkappa = \infty$. Thus the infinite moving beam with elastic supports loses stability in the same manner as the moving beam with rigid supports, when it holds that $\varkappa \geq \varkappa_*$. This means, in particular, that the instability of the moving beam with elastic supports is characterised by zero transverse (vertical) displacements at the points of elastic supports when $\varkappa \geq \varkappa_* = 4\pi^2$, as also takes place in the case of absolutely rigid supports with $\varkappa = \infty$.

The basic relation (55) between $\varkappa$ and λ can be simplified and written in an asymptotic manner. For small elastic support rigidities ($\varkappa \ll 1$) we have $\lambda \ll 1$, and the following asymptotic formula holds:

$$\lambda(\varkappa) = \sqrt{4\varkappa}\,, \qquad \varkappa \ll 1\,. \tag{58}$$

The dependence of the critical eigenvalue λ on the rigidity parameter $\varkappa$, obtained in accordance with the exact relation (55) and the asymptotic formula (58), are shown in Figure 3. The figure illustrates good precision of the asymptotic formula (58) in the interval $0 < \varkappa \leq 10$.

The dependence of the minimal eigenvalue λ on the parameter $\alpha \in [0, \pi]$ is shown in Figure 4 for some fixed values of the rigidity parameter $\varkappa$. The critical values of the parameter $\alpha = \alpha_*$ correspond to the minimal values of $\lambda = \lambda_*$. Comparing the locations

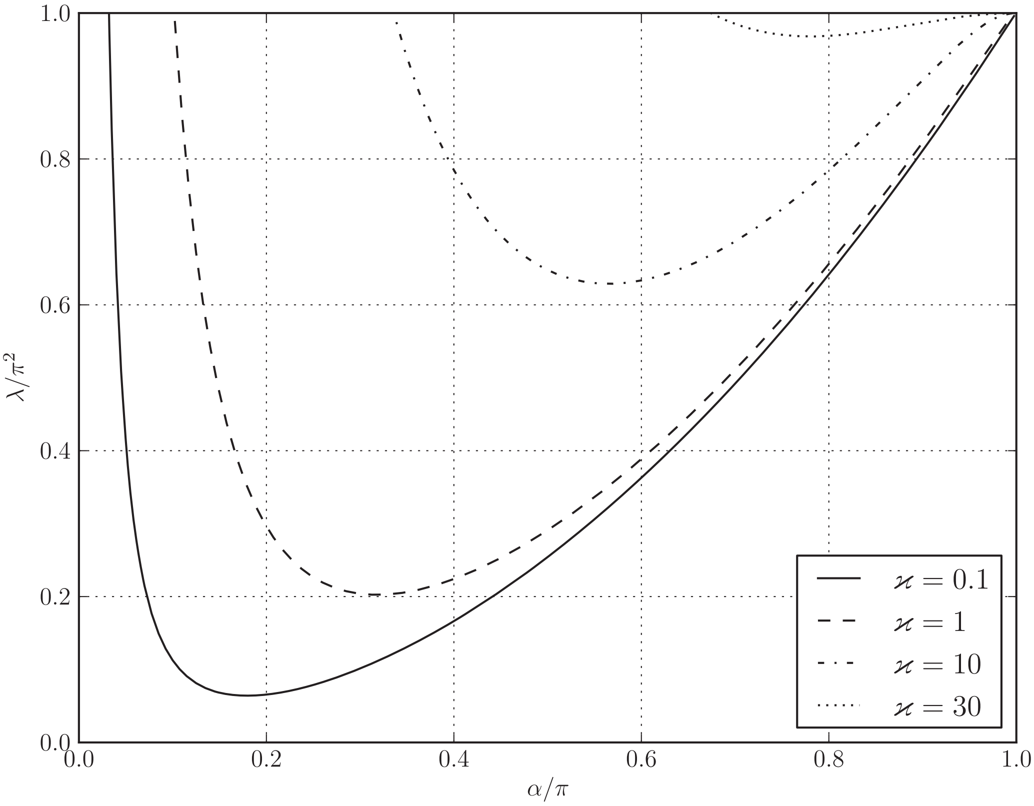

Figure 4. The dependence of the minimal eigenvalue λ on the parameter $\alpha \in [0, \pi]$ for some fixed values of the elastic support rigidity parameter $\varkappa$.

of the minima in Figure 4, it is observed that α_* is a monotonically increasing function of $\varkappa$.

Conclusions

The loss of stability of the moving web, modelled as an elastic beam (panel) of unlimited length, and travelling between an infinite system of rollers (elastic supports) at a constant velocity, was investigated. Transverse elastic displacements of the web were described by a fourth order differential equation that included the centrifugal force, in-plane tension (axial tension), bending term and elastic support reaction.

The stability of the beam was investigated with the help of analysis of small periodic transverse displacements. The studies performed were mainly based on analytical approaches. In this connection the multipoint spectral stability problem for the beam of unlimited length, with an infinite system of elastic supports, was formulated for the periodic interval, and Floquet's representation of solution was used. As a result, the basic relations characterizing the behaviour of the web at the onset of instability were found in an analytical form.

The critical velocity, that corresponds to the onset of instability in the form of divergence (buckling), was estimated in the frame of the performed spectral analysis. The obtained dependence of critical velocity on the support rigidity parameter $\varkappa$ was analysed, and in particular, it was shown that the instability of the moving beam with elastic supports coincides with the instability of the same beam with absolutely rigid supports (having $\varkappa = \infty$), when the rigidity of the elastic supports exceeds a critical value, $\varkappa \geq \varkappa_* = 4\pi^2$.

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