Waves propagation in an arbitrary direction in heat conducting orthotropic elastic composites

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Summary. Dispersion of thermoelastic harmonic waves propagating in an arbitrary direction in a layered heat conducting orthotropic elastic composite in the context of generalized thermoelasticity is studied. Considering three dimensional field equations of thermoelasticity and to obtain characteristic equation, continuity of displacements, temperature, stresses and thermal gradient at the layers’ interfaces is employed, and the corresponding sixteenth order characteristic determinant is examined. Particular results for the coupled and uncoupled thermoelasticity are obtained as special cases of the obtained results by taking thermal relaxation time and coupling constant equal to zero. Results of previous investigations are derived as particular cases. Finally numerical results are also obtained and represented graphically.

Key words: composites, orthotropic, generalized, thermoelasticity, coupled, thermal relaxation time

Introduction

Advanced high strength, high modulus composite materials must undergo careful inspection to sort out manufacturing errors, in-service degradation, and defect formation due to the influence of elevated temperatures, moisture, cosmic radiation, etc. The growing practical importance of such materials, especially in thermal environment has stimulated many analytical studies. The exact dispersion relations for composite structures can be found in many good books [1] and [2], with many unidirectionally reinforced layers are at least of orthotropic characters and sometimes transversely isotropic.

Wave types occurring in bounded layered anisotropic media are very complicated, and in thermoelasticity, the problem becomes even more complicated, because in thermoelasticity solutions to both the heat conduction and thermoelasticity problems for all the layers are required. These solutions are also to satisfy the thermal and mechanical boundary and interface conditions. As a result, conventional procedure for thermoelastic analysis of a multilayered medium results in having to solve system of two simultaneous equations for a large number of unknown constants as in Refs. [3-7].

By introducing thermal relaxation time constants into the heat conduction equation, new generalized theories of thermoelasticity have been developed in an attempt to eliminate this paradox of infinite velocity of thermal propagation. Of all the non-classical theories, at present, Lord and Shulman [8], Green and Lindsay [9] theories of
the generalized thermoelasticity are mainly considered for engineering applications. Various methods to study the isothermal elasticity problems in heat conducting medium are studied in Ref.[10]. Propagation of plane harmonic thermoelastic waves in an infinitely extended anisotropic solid is considered and investigated in [11]. In Ref. [12] the governing equations of generalized thermoelasticity for anisotropic media were derived.

This paper attempts to study the dispersion of harmonic waves propagating in an arbitrary direction in layered orthotropic elastic composite in the context of generalized thermoelasticity in Ref. [13]. Three dimensional field equations of thermoelasticity are considered, and the corresponding sixteenth order characteristic determinant is examined. The purpose of this paper is to examine the dispersive effects in layered thermoelastic composites, where the direction of the corresponding harmonic waves makes an arbitrary angle with respect to the layers. The results for the coupled and uncoupled thermoelasticity are obtained as particular cases of the obtained results by setting thermal relaxation time and the coupling constant equal to zero. Relevant results of previous investigations are deduced as special cases. A similar type of the approach has also been used in Ref. [14] for the corresponding elastic material.

**Problem Formulation**

Consider a set of Cartesian coordinate system $x = (x_1, x_2, x_3)$ in such a manner that $x_3$-axis is normal to the layering. The basic field equations of generalized thermoelasticity for an infinite generally anisotropic thermoelastic medium with one thermal relaxation time $\tau_0$ at uniform temperature $T_0$ in the absence of body forces and heat sources are

$$
\sigma_{ij,j} = \rho \ddot{u}_i, \quad i, j = 1, 2, 3, \quad (1)
$$

$$
K_0 T_{ij} - \rho C_e (\dot{T} + \tau_0 \ddot{T}) = T_0 \beta_{ij} [\dddot{u}_i + \tau_0 \dddot{u}_{i,j}], \quad (2)
$$

where

$$
\sigma_{ij} = C_{ijkl} e_{kl} - \beta_{ij} T, \quad (3)
$$

$$
\beta_{ij} = C_{ijkl} \theta_{kl}, \quad (4a)
$$

and $\rho$ is the density, $t$ is the time, $u_i$ is the displacement in the $x_i$ direction, $K_0$ are the thermal conductivities, $C_e$ and $\tau_0$ are respectively the specific heat at constant strain, and thermal relaxation time, $\sigma_{ij}$ and $e_{ij}$ are the stress and strain tensor respectively; $\beta_{ij}$ are thermal moduli; $\theta_{ij}$ is the thermal expansion tensor; $T$ is temperature; and the fourth order tensor of the elasticity $C_{ijkl}$ satisfies the (Green) symmetry conditions:

$$
c_{ijkl} = c_{klji} = c_{jkil}, \beta_{ij} = \beta_{ji}, \theta_{ij} = \theta_{ji}. \quad (4b)
$$

Here comma notations are used for spatial derivatives and superposed dot represents differentiation with respect to time. Strain-displacement relation is

$$
e_{ij} = \frac{u_{i,j} + u_{j,i}}{2}. \quad (5)$$
Specializing the equations (1-5) for orthotropic media, the governing equations (1) and (2) via (3) to (5) are

\[
c_1 u_{1,11} + c_{66} u_{2,22} + c_{22} u_{1,33} + (c_{12} + c_{66}) u_{2,12} + (c_{13} + c_{55}) u_{3,33} - \beta_1 T_1 = \rho \ddot{u}_1
\]  

(6a)

\[
(c_{12} + c_{66}) u_{1,12} + c_{66} u_{2,11} + c_{22} u_{2,22} + c_{44} u_{2,33} + (c_{23} + c_{44}) u_{3,23} - \beta_2 T_2 = \rho \ddot{u}_2
\]  

(6b)

\[
(c_{13} + c_{55}) u_{1,13} + (c_{23} + c_{44}) u_{2,23} + c_{55} u_{3,31} + c_{33} u_{3,32} + c_{33} u_{3,33} - \beta_3 T_3 = \rho \ddot{u}_3
\]  

(6c)

\[
K_{11} T_{1,11} + K_{22} T_{2,22} + K_{33} T_{3,33} = \rho C_v (T + \dot{T})
\]  

(6d)

where

\[
\beta_1 = c_1 \alpha_1 + c_{12} \alpha_2 + c_{13} \alpha_3,
\]

\[
\beta_2 = c_{12} \alpha_1 + c_{22} \alpha_2 + c_{23} \alpha_3,
\]

\[
\beta_3 = c_{13} \alpha_1 + c_{32} \alpha_2 + c_{33} \alpha_3.
\]

(7)

On considering equations (6) and (7) for each layer, and at the interface between two layers, the displacements, temperature, thermal stresses and temperature gradient must be continuous.

**Analysis**

For harmonic waves propagating in an arbitrary direction, the displacement components and temperature \( u_1, u_2, u_3 \) and \( T \) are written as

\[
(u_1, u_2, u_3, T) = (U_1(x_1), U_2(x_2), U_3(x_3), U_4(x_4)) e^{i(l_1 x_1 + l_2 x_2 + l_3 x_3 - \omega t)}
\]  

(8)

where \( \xi \) is the wave number, \( c \) is the phase velocity (\( \omega / \xi \)), \( \omega \) is the circular frequency, \( l_1, l_2 \) and \( l_3 \) are the direction cosine defining the propagation direction. Floquet’s theory requires functions \( U_j \) (\( j = 1, 2, 3, 4 \)) to have the same periodicity as that of the layering. Hence the problem is reduced to that of one pair of layers, where

\[
(U_j)(x_3) = \overline{U} e^{-i(l_j + \alpha) x_3}, \quad j = 1, 2, 3, 4,
\]  

(9)

\( \alpha \) is an unknown ratio of the wave number components and \( \overline{U} \) are constants. Upon substitution from (9) into (8) and (6), we have

\[
M_{mn}(\alpha) \overline{U}_n = 0, \quad m, n = 1, 2, 3, 4.
\]  

(10a)

Here

\[
M_{11} = (l_1^2 + l_2^2 \overline{c_{66}} + \alpha^2 \overline{c_{55}} - \xi^2)
\]

\[
M_{12} = (\overline{c_{12}} + \overline{c_{66}}) l_2\alpha
\]

\[
M_{13} = - (\overline{c_{13}} + \overline{c_{23}}) l_3\alpha
\]

\[
M_{14} = l_1\alpha
\]

\[
M_{22} = (l_1^2 \overline{c_{66}} + l_2^2 \overline{c_{22}} + \alpha^2 \overline{c_{44}} - \xi^2)
\]

\[
M_{23} = - (\overline{c_{23}} + \overline{c_{44}}) l_2\alpha
\]

\[
M_{24} = l_1\alpha
\]
\[ M_{24} = \bar{\beta} l_2 \]
\[ M_{33} = (l_i^2 \bar{c}_{35} + l_i^2 \bar{c}_{44} + \alpha^2 \bar{c}_{33} - \xi^2), \]
\[ M_{34} = -\bar{\beta} \alpha \]
\[ M_{44} = \varepsilon \omega^* \xi^2 l_i i/\omega, \]
\[ M_{43} = \varepsilon \omega^* \xi^2 l_i \bar{\beta} i/\omega \]
\[ M_{43} = -\varepsilon \omega^* \xi^2 \alpha \bar{\beta} i/\omega, \]
\[ M_{44} = (l_i^2 + \bar{K}_i l_j^2 + \bar{K}_i \alpha^2 - \omega^* \xi^2 \tau), \quad (10b) \]

where
\[ \xi^2 = \frac{\rho c^2}{c_1}, \quad \omega^* = \frac{c_1 C_e}{K_1}, \quad \varepsilon = \frac{\beta^2 T_0}{\rho C_e c_{11}}, \quad \text{and} \quad \tau = \tau_0 + i/\omega. \]

The existence of nontrivial solutions for \( \bar{U}_j \) \( (j = 1, 2, 3, 4) \) demands the vanishing of the determinant in equations (10a), and yields the eighth degree polynomial equation
\[ \alpha^8 + A_1 \alpha^6 + A_2 \alpha^4 + A_3 \alpha^2 + A_4 = 0, \quad (11) \]
where the coefficients \( A_1, A_2, A_3 \) and \( A_4 \) are given in Appendix-I.

Equations (9) is rewritten as
\[ (U_1, U_2, U_3, T) = \sum_{q=1}^{8} (\bar{U}_{1q}, \bar{U}_{2q}, \bar{U}_{3q}, \bar{U}_{4q}) e^{-i \xi \xi [l_i \alpha_q]} x_3. \quad (12) \]

For each \( \alpha_q, \quad q = 1, 2, \ldots, 8 \), using the relations (10) and expressing the displacements ratios as
\[ \frac{\bar{U}_{2q}}{\bar{U}_{1q}} = \frac{D_1(\alpha_q)}{D(\alpha_q)} = \gamma_q, \quad \frac{\bar{U}_{3q}}{\bar{U}_{1q}} = \frac{D_2(\alpha_q)}{D(\alpha_q)} = \delta_q, \quad \frac{\bar{U}_{4q}}{\bar{U}_{1q}} = \frac{D_3(\alpha_q)}{D(\alpha_q)} = \Theta_q, \quad q = 1, 2, \ldots, 8, \quad (13) \]

\[ D(\alpha_q), \quad D_i(\alpha_q), \quad i = 1, 2, 3, \quad (14) \]

are given in Appendix-II. Therefore, the solution is
\[ (U_1, U_2, U_3, T) = \sum_{q=1}^{8} (1, \gamma_q, \delta_q, \Theta_q) \bar{U}_{1q} e^{-i \xi \xi [l_i \alpha_q]} x_3 \quad (15) \]

In view of the continuity of the displacement components, temperature, tractions and temperature gradient across the interface of the two layers, the following conditions must be satisfied:
\[ u_j^{(1)}(x_3 = \bar{0}) = u_j^{(2)}(x_3 = \bar{0}), \]
\[ T^{(1)}(x_3 = \bar{0}) = T^{(2)}(x_3 = \bar{0}), \]
\[ \sigma_{ij}^{(1)}(x_3 = \bar{0}) = \sigma_{ij}^{(2)}(x_3 = \bar{0}), \]
\[ T^{(1)}(x_3 = \bar{0}) = T^{(2)}(x_3 = \bar{0}), \quad (16) \]
where $T' = \frac{\partial T}{\partial x_3}$ subscripts (1) and (2) refer to layers I and II, respectively, $0^+$ and $0^-$ are values of $x_3$ near zero. Because of periodicity of the deformation and thermoelastic stress fields, additional conditions obtained are:

$$u_j^{(1)}(x_3 = h_1^+) = u_j^{(2)}(x_3 = -h_2^+),$$

$$T^{(1)}(x_3 = h_1^+) = T^{(2)}(x_3 = -h_2^+),$$

$$\sigma_{3j}^{(1)}(x_3 = h_1^+) = \sigma_{3j}^{(2)}(x_3 = -h_2^+), \quad j = 1, 2, 3;$$

$$T'^{(1)}(x_3 = h_1^+) = T'^{(2)}(x_3 = -h_2^+).$$

(17)

Upon substitution of the displacement, temperature, stress and temperature gradient components into (16) and (17), sixteen linear homogeneous equations for sixteen constants $U_{11}^{(1)}, U_{12}^{(1)}, ..., U_{17}^{(2)}$ and $U_{18}^{(2)}$ are obtained. For nontrivial solutions, the determinant of the coefficients must vanish. This yields the following characteristic equation:

$$\det \begin{bmatrix} A_{jk} & -\bar{A}_{jk} \\ B_{jk} & -\bar{B}_{jk} \end{bmatrix} = 0, \quad j, k = 1, 2, ..., 8. \quad (18)$$

The $8 \times 8$ matrices $A_{jk}, \bar{A}_{jk}, B_{jk}$ and $\bar{B}_{jk}$ are given in Appendix-III.

On simplifying equation (18), we have

$$\det \begin{bmatrix} A_{jk} \end{bmatrix} \det \left( \begin{bmatrix} -\bar{B}_{jk} \\ -B_{jk} \end{bmatrix} - \begin{bmatrix} A_{jk} \end{bmatrix}^{-1} \begin{bmatrix} -\bar{A}_{jk} \end{bmatrix} \right) = 0 \quad (19a)$$

which implies that either

$$\det \begin{bmatrix} A_{jk} \end{bmatrix} = 0, \quad (19b)$$

or

$$\det \left( \begin{bmatrix} -\bar{B}_{jk} \\ -B_{jk} \end{bmatrix} - \begin{bmatrix} A_{jk} \end{bmatrix}^{-1} \begin{bmatrix} -\bar{A}_{jk} \end{bmatrix} \right) = 0. \quad (19c)$$

If equation (19a) holds true, then the problem reduces to a free wave propagation in a single thermoelastic plate of thickness $h_j$, and in this case $[-\bar{B}_{jk} - [B_{jk}][A_{jk}]^{-1}[-\bar{A}_{jk}]$ will not exist if $A_{jk}$ is singular. On the hand if $A_{jk}$ is nonsingular then $A_{jk}^{-1}$ exists and accordingly (19c) exists.

In order to solve the problem numerically one has to solve (18), and to solve it is sufficient to consider either equation (19c) for heat conducting composite plates and equation (19b) to solve for free thermoelastic plate.
Particular cases

Uncoupled thermoelasticity

If the coupling constant, \( \varepsilon = 0 \), then thermal and elastic fields decoupled from each other and from equation (10b) we have \( M_{41} = M_{42} = M_{43} = 0 \). In this case equation (10a) reduces to

\[
\left( l_1^2 + K_2 l_2^2 + K_3 \alpha^2 - \omega^* \zeta^2 \right)(\Delta \alpha^6 + F_1 \alpha^4 + F_2 \alpha^2 + F_3) = 0.
\]  

(20)

From the above equation (20), considering the factor

\[
\Delta \alpha^6 + F_1 \alpha^4 + F_2 \alpha^2 + F_3 = 0,
\]

(21)
equation (21) corresponds to the characteristic equation in the uncoupled thermoelasticity where

\[
\Delta = c_{33} c_{44} c_{55},
\]

\[
F_1 = [(c_{22} c_{33} - 2c_{23} c_{44} - c_{23}^2) c_{55} + c_{33} c_{44} c_{66}] l_1^2 + [(c_{33} - 2c_{13} c_{55} - c_{13}^2) c_{44} + c_{33} c_{55} c_{66}] l_1^2,
\]

(22)

\[
F_2 = [(c_{23} - 2c_{13} c_{55} - c_{13}^2) c_{66} + c_{44} c_{55}] l_1^2 + [(c_{22} c_{33} - 2c_{23} c_{44} - c_{23}^2) c_{66} + c_{22} c_{55} c_{44}] l_2^2
\]

\[
+ [(c_{12} c_{33} - 2c_{33} c_{44} - c_{66} c_{23} c_{55} - c_{13} c_{44} c_{55} + c_{13} c_{22} c_{55} - 2c_{44} c_{55} c_{66} - c_{13} c_{44} c_{66}
\]

\[
+ c_{12} c_{33} c_{66} - c_{12} c_{33} c_{44} - c_{13} c_{23} c_{66} - c_{13} c_{23} c_{55} - c_{13} c_{13} c_{23}) - c_{13} c_{22} + c_{22} c_{33} - c_{23}^2] l_2^2 l_2^2,
\]

\[
+ (2c_{13} c_{55} - c_{66} c_{33} - c_{35} c_{55} - c_{44} - c_{33} - c_{66} c_{55} + c_{13}^2) l_1^2 \zeta^2
\]

\[
+ (2c_{23} c_{44} + c_{23}^2 - c_{22} c_{33} - c_{22} c_{55} - c_{66} c_{44} - c_{55} c_{44} - c_{33} c_{66} - c_{33} c_{55} c_{66} l_2^2 + (c_{33} + c_{44} + c_{55}) \zeta^6,
\]

\[
F_3 = (c_{55} l_2^2 + c_{44} l_2^2 - \zeta^2) \left\{ (1 + c_{66}) l_1^2 + (c_{22} + c_{66}) l_2^2 \right\} \zeta^2 - \zeta^4
\]

\[
+ [(2c_{22} c_{66} + c_{12}^2 - c_{22} c_{55}) l_1^2 l_2^2 - c_{22} c_{66} l_2^4 - c_{66} l_1^4) \right\}.
\]

Equation (14) reduces to

\[
D_1(\alpha_q) = M_{13}(\alpha_q) M_{23}(\alpha_q) - M_{12}(\alpha_q) M_{33}(\alpha_q),
\]

\[
D_2(\alpha_q) = M_{12}(\alpha_q) M_{23}(\alpha_q) - M_{13}(\alpha_q) M_{22}(\alpha_q),
\]

\[
D_3(\alpha_q) = 0,
\]

\[
D(\alpha_q) = M_{22}(\alpha_q) M_{33}(\alpha_q) - M_{23}^2(\alpha_q).
\]

(23)

Equations (23) correspond to the orthotropic materials purely elastic composites, which are obtained and discussed by Yamada and Nasser [17], and on the hand, second factor of equation (20) is
Equation (24) corresponds to the purely thermal wave, which is clearly influenced by the thermal relaxation time $\tau$.

**Coupled Thermoelasticity**

In this case when there is no thermal relaxation time, i.e. $\tau_0 = 0$ and hence $\tau = i/\omega$. Following above procedure, we arrived at frequency equation in the coupled thermoelasticity.

**Numerical Results and Discussion**

Using characteristic equation (18), numerical results for phase velocity versus wave number are presented for the first few lower modes to indicate the dependence of dispersion upon the angle of propagation and thermal relaxation times. The material chosen for this purpose is Aluminum epoxy composite / Carbon steel as layer I ($h_1=0.7$) and Layer II ($h_2=0.3$) respectively.

Figure 1 depicts the dispersion curves when the direction cosines of propagation are $l_1=0.259$, $l_2=0.542$, $l_3=0.799$ whereas Figure 2 and Figure 3 depict the dispersion curves when the direction cosines $l_1=0.195$, $l_2=0.515$, $l_3=0.834$ and $l_1=0.125$, $l_2=0.707$, $l_3=0.696$, in all these figures thermal relaxation time $\tau_0=2.10^{-7}$.

It is observed that lowers modes are more influenced with changes in direction cosines whereas little variation is noticed in the upper modes. Further, as $l_1$ increases, the phase velocity of lower modes decreases with wave number.

Each of figure display wave speeds (coupled) corresponding to quasi-longitudinal, quasi-transverse and quasi-thermal, at zero wave number limits, lower modes are found to highly influenced and phase velocity of higher modes have higher values and decreases as wave number increases. One of the thermoelastic modes seems to be associated with quick change in the slope of the mode.

In anisotropic plates the distinction between mode types is somewhat artificial, since the equation for thermal and elastic wave modes i.e. quasi-longitudinal and quasi-transverse and shear horizontal modes will be generally be coupled with quasi-thermal wave modes. For wave propagation in the direction of symmetry some wave types revert to pure modes, leading simple characteristic equation of lower order. Consequence of elastic anisotropy in media is the loss of pure wave modes for general propagation direction. At zero wave number limits, each figure displays wave speeds corresponding to one quasi-longitudinal, two quasi-transverse and a quasi-thermal in generalized thermoelasticity and higher mode. It is apparent that the largest value corresponds to the quasi-longitudinal and the additional mode appears is a quasi-thermal mode. At low wave number limits, modes are found to highly influence and it vary with the direction cosine. At relatively low values of the wave number, little change is seen in these values. As wave number increases others high modes appear, one of the modes seems to be associated with quick change in the slope of the mode. It is also observed that with change in direction cosines, lower modes highly influenced whereas small variation is noticed in the high modes. Thus in generalized thermoelasticity, at low
values of the wave number, only the lower modes get affected at low values of wave number and the little change is seen at relatively high values of wave number. The low value region of the wave number is found to be of more physical interest in generalized thermoelasticity. Further as at high wave number limits, small variation is observed, and so the second sound effects are short lived. Quasi-longitudinal, quasi-transverse (two) and quasi-thermal waves are found coupled with each other due to the thermal and anisotropic effects, also wave-like behavior in the additional quasi-thermal modes. For uncoupled and coupled theory of thermoelasticity the results can be obtained from the present analysis by setting coupling terms and thermal relaxation times equal to zero.

![Graph showing Phase velocity versus wave number for the direction cosine](image)

Figure 1. Phase velocity versus wave number for the direction cosine $l_1 = 0.259$, $l_2 = 0.542$ and $l_3 = 0.799$ in generalized thermoelasticity
Figure 2. Phase velocity versus wave number for the direction cosine $l_1 = 0.195$, $l_2 = 0.515$ and $l_3 = 0.834$ in generalized thermoelasticity.

Figure 3. Phase velocity verses wave number for the direction cosine $l_1 = 0.125$, $l_2 = 0.707$ and $l_3 = 0.696$ in generalized thermoelasticity.
Appendix-I.

\[ A_i = B_i \omega \epsilon \xi^2 i / \omega + P_i K_3 + c_{33} c_{44} c_{55} (l_i^2 + l_i^2 - \omega \tau \xi^2) / \Delta \]

\[ A_2 = B_2 \omega \epsilon \xi^2 i / \omega + P_2 K_3 + (l_i^2 - \omega \tau \xi^2) / \Delta \]

\[ A_3 = B_3 \omega \epsilon \xi^2 i / \omega + P_3 (l_i^2 - \omega \tau \xi^2) / \Delta \]

\[ A_4 = B_4 \omega \epsilon \xi^2 i / \omega + P_4 (l_i^2 + l_i^2 - \omega \tau \xi^2) / \Delta \]

\[ \Delta = c_{33} c_{44} c_{55} K_3 \]

\[ P_i = [(c_{33} c_{55} - 2c_{23} c_{44} - c_{23}^2) c_{55} + c_{33} c_{44} c_{66}] \xi^2 + [(c_{33} - 2c_{13} c_{55} - c_{13}^2) c_{55} + c_{33} c_{44} c_{66}] \xi^2 - (c_{33} c_{44} + c_{33} c_{55} + c_{44} c_{55}) \xi^2 \]

\[ P_2 = [(c_{33} - 2c_{13} c_{55} - c_{13}^2) c_{66} + c_{44} c_{55}] \xi^2 + [(c_{23} c_{44} + c_{23} c_{55} + c_{23} c_{66}) c_{66} - c_{33} c_{44} c_{66}] \xi^2 + [(c_{33} - 2c_{13} c_{44} + c_{13} c_{55} + c_{13} c_{55}) c_{66} - c_{33} c_{44} c_{66}] \xi^2 - (c_{33} c_{44} + c_{33} c_{55} + c_{44} c_{55}) \xi^2 \]

\[ P_3 = (c_{33} c_{55} - 2c_{23} c_{44} - c_{23}^2) \xi^2 + [(c_{33} c_{44} c_{66} c_{66} - c_{33} c_{44} c_{66} - c_{33} c_{44} c_{66}) c_{66} - c_{33} c_{44} c_{66}] \xi^2 + [(c_{33} c_{44} c_{66} c_{66} - c_{33} c_{44} c_{66} - c_{33} c_{44} c_{66}) c_{66} - c_{33} c_{44} c_{66}] \xi^2 - (c_{33} c_{44} + c_{33} c_{55} + c_{44} c_{55}) \xi^2 \]

\[ B_i = c_{44} c_{55} \xi^2 \]

\[ B_2 = (c_{33} c_{44} c_{55}) \xi^2 + 2(c_{13} c_{44} c_{55}) \xi^2 + (c_{33} c_{44} c_{55}) \xi^2 + ((2c_{23} c_{44} + c_{23} c_{55} - c_{23} c_{55}) \xi^2 - c_{33} c_{55}) \xi^2 + (2c_{33} c_{44} + c_{33} c_{55} - c_{33} c_{55}) \xi^2 + (c_{33} c_{44} c_{55} - c_{33} c_{55} - c_{33} c_{55}) \xi^2 - (c_{33} c_{55} + c_{33} c_{55} + c_{33} c_{55}) \xi^2 \]

\[ B_3 = (c_{33} c_{44} c_{55}) \xi^2 + 2(c_{13} c_{44} c_{55}) \xi^2 + (c_{33} c_{44} c_{55}) \xi^2 + ((2c_{23} c_{44} + c_{23} c_{55} - c_{23} c_{55}) \xi^2 - c_{33} c_{55}) \xi^2 + (2c_{33} c_{44} + c_{33} c_{55} - c_{33} c_{55} - c_{33} c_{55}) \xi^2 + (c_{33} c_{44} c_{55} - c_{33} c_{55} - c_{33} c_{55}) \xi^2 - (c_{33} c_{55} + c_{33} c_{55} + c_{33} c_{55}) \xi^2 \]

\[ B_4 = (c_{33} c_{44} c_{55}) \xi^2 + 2(c_{13} c_{44} c_{55}) \xi^2 + (c_{33} c_{44} c_{55}) \xi^2 + ((2c_{23} c_{44} + c_{23} c_{55} - c_{23} c_{55}) \xi^2 - c_{33} c_{55} - c_{33} c_{55}) \xi^2 + (2c_{33} c_{44} + c_{33} c_{55} - c_{33} c_{55} - c_{33} c_{55}) \xi^2 + (c_{33} c_{44} c_{55} - c_{33} c_{55} - c_{33} c_{55}) \xi^2 - (c_{33} c_{55} + c_{33} c_{55} + c_{33} c_{55}) \xi^2 \]

\[ B_4 = (c_{33} c_{44} c_{55}) \xi^2 + 2(c_{13} c_{44} c_{55}) \xi^2 + (c_{33} c_{44} c_{55}) \xi^2 + ((2c_{23} c_{44} + c_{23} c_{55} - c_{23} c_{55}) \xi^2 - c_{33} c_{55} - c_{33} c_{55}) \xi^2 + (2c_{33} c_{44} + c_{33} c_{55} - c_{33} c_{55} - c_{33} c_{55}) \xi^2 + (c_{33} c_{44} c_{55} - c_{33} c_{55} - c_{33} c_{55}) \xi^2 - (c_{33} c_{55} + c_{33} c_{55} + c_{33} c_{55}) \xi^2 \]

\[ B_4 = (c_{33} c_{44} c_{55}) \xi^2 + 2(c_{13} c_{44} c_{55}) \xi^2 + (c_{33} c_{44} c_{55}) \xi^2 + ((2c_{23} c_{44} + c_{23} c_{55} - c_{23} c_{55}) \xi^2 - c_{33} c_{55} - c_{33} c_{55}) \xi^2 + (2c_{33} c_{44} + c_{33} c_{55} - c_{33} c_{55} - c_{33} c_{55}) \xi^2 + (c_{33} c_{44} c_{55} - c_{33} c_{55} - c_{33} c_{55}) \xi^2 - (c_{33} c_{55} + c_{33} c_{55} + c_{33} c_{55}) \xi^2 \]
Appendix-II

\[ D_1(\alpha_q) = M_{23}(\alpha_q)M_{34}(\alpha_q)M_{41}(\alpha_q) + M_{23}(\alpha_q)M_{33}(\alpha_q)M_{44}(\alpha_q) - M_{13}(\alpha_q)M_{24}(\alpha_q)M_{43}(\alpha_q) \]
\[ + M_{12}(\alpha_q)M_{34}(\alpha_q)M_{43}(\alpha_q) + M_{13}(\alpha_q)M_{23}(\alpha_q)M_{44}(\alpha_q) - M_{12}(\alpha_q)M_{33}(\alpha_q)M_{44}(\alpha_q) \]
\[ D_2(\alpha_q) = M_{23}(\alpha_q)M_{24}(\alpha_q)M_{41}(\alpha_q) + M_{12}(\alpha_q)M_{23}(\alpha_q)M_{44}(\alpha_q) + M_{13}(\alpha_q)M_{24}(\alpha_q)M_{42}(\alpha_q) \]
\[ + M_{22}(\alpha_q)M_{34}(\alpha_q)M_{41}(\alpha_q) - M_{13}(\alpha_q)M_{22}(\alpha_q)M_{44}(\alpha_q) - M_{12}(\alpha_q)M_{34}(\alpha_q)M_{42}(\alpha_q) \]
\[ D_3(\alpha_q) = M^2_{23}(\alpha_q)M_{41}(\alpha_q) - M^2_{23}(\alpha_q)M_{33}(\alpha_q)M_{44}(\alpha_q) - M_{12}(\alpha_q)M_{23}(\alpha_q)M_{43}(\alpha_q) \]
\[ + M_{13}(\alpha_q)M_{22}(\alpha_q)M_{43}(\alpha_q) + M_{12}(\alpha_q)M_{33}(\alpha_q)M_{42}(\alpha_q) - M_{13}(\alpha_q)M_{23}(\alpha_q)M_{42}(\alpha_q) \]
\[ D(\alpha_q) = M_{23}(q_j)M_{34}(q_j)M_{42}(q_j) - M_{24}(q_j)M_{33}(q_j)M_{42}(q_j) - M_{22}(q_j)M_{34}(q_j)M_{43}(q_j) \]
\[ + M_{22}(q_j)M_{33}(q_j)M_{44}(q_j) - M^2_{23}(q_j)M_{44}(q_j) + M_{23}(q_j)M_{24}(\alpha_q)M_{43}(q_j) \]

Appendix-III

\[ A_{ij} = 1 \]
\[ A_{2j} = r_j^{(1)} \]
\[ A_{3j} = \delta_j^{(1)} \]
\[ A_{4j} = \theta_j^{(1)} \]
\[ A_{5j} = b_j^{(1)}c_{55} \]
\[ A_{6j} = b_j^{(1)}c_{44} \]
\[ A_{7j} = b_j^{(1)} \]
\[ A_{8j} = -l_j + \alpha_j^{(1)}\theta_j^{(1)} \]
\[ B_{jk} = A_{jk}E_k^+ \]
\[ E_j^- = e^{-\xi_j(l_j + \alpha_j^{(2)})}h_j \]
\[ Q = \xi(h_1 + h_2) \]
\[ \eta = c_{j1}^{(2)} / c_{j1}^{(1)} \]
\[ b_{1j}^{(m)} = l_j\delta_j^{(m)} - \alpha_j^{(m)} \]
\[ b_{2j}^{(m)} = l_j\delta_j^{(m)} - \alpha_j^{(m)} \gamma_j^{(m)} \]
\[ \bar{b}_{3j}^{(m)} = \bar{c}_{13}^{(m)}l_j + \bar{c}_{23}^{(m)}l_j\gamma_j^{(m)} - \bar{c}_{33}^{(m)}\alpha_j^{(m)}\delta_j^{(m)} - \beta_j\theta_j^{(m)} \]
\[ \bar{c}_{jk}^{(m)} = c_{jk}^{(m)} / c_{11}^{(m)} \]

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References


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