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Waves propagation in an arbitrary direction in heat conducting orthotropic elastic composites

K. L. Verma

Summary. Dispersion of thermoelastic harmonic waves propagating in an arbitrary direction in a layered heat conducting orthotropic elastic composite in the context of generalized thermoelasticity is studied. Considering three dimensional field equations of thermoelasticity and to obtain characteristic equation, continuity of displacements, temperature, stresses and thermal gradient at the layers' interfaces is employed, and the corresponding sixteenth order characteristic determinant is examined. Particular results for the coupled and uncoupled thermoelasticity are obtained as special cases of the obtained results by taking thermal relaxation time and coupling constant equal to zero. Results of previous investigations are derived as particular cases. Finally numerical results are also obtained and represented graphically.

Key words: composites, orthotropic, generalized, thermoelasticity, coupled, thermal relaxation time

Introduction

Advanced high strength, high modulus composite materials must undergo careful inspection to sort out manufacturing errors, in-service degradation, and defect formation due to the influence of elevated temperatures, moisture, cosmic radiation, etc. The growing practical importance of such materials, especially in thermal environment has stimulated many analytical studies. The exact dispersion relations for composite structures can be found in many good books [1] and [2], with many unidirectionally reinforced layers are at least of orthotropic characters and sometimes transversely isotropic.

Wave types occurring in bounded layered anisotropic media are very complicated, and in thermoelasticity, the problem becomes even more complicated, because in thermoelasticity solutions to both the heat conduction and thermoelasticity problems for all the layers are required. These solutions are also to satisfy the thermal and mechanical boundary and interface conditions. As a result, conventional procedure for thermoelastic analysis of a multilayered medium results in having to solve system of two simultaneous equations for a large number of unknown constants as in Refs. [3-7].

By introducing thermal relaxation time constants into the heat conduction equation, new generalized theories of thermoelasticity have been developed in an attempt to eliminate this paradox of infinite velocity of thermal propagation. Of all the non-classical theories, at present, Lord and Shulman [8], Green and Lindsay [9] theories of the generalized thermoelasticity are mainly considered for engineering applications. Various methods to study the isothermal elasticity problems in heat conducting medium are studied in Ref.[10]. Propagation of plane harmonic thermoelastic waves in an infinitely extended anisotropic solid is considered and investigated in [11]. In Ref. [12] the governing equations of generalized thermoelasticity for anisotropic media were derived.

This paper attempts to study the dispersion of harmonic waves propagating in an arbitrary direction in layered orthotropic elastic composite in the context of generalized thermoelasticity in Ref. [13]. Three dimensional field equations of thermoelasticity are considered, and the corresponding sixteenth order characteristic determinant is examined. The purpose of this paper is to examine the dispersive effects in layered thermoelastic composites, where the direction of the corresponding harmonic waves makes an arbitrary angle with respect to the layers. The results for the coupled and uncoupled thermoelasticity are obtained as particular cases of the obtained results by setting thermal relaxation time and the coupling constant equal to zero. Relevant results of previous investigations are deduced as special cases. A similar type of the approach has also been used in Ref. [14] for the corresponding elastic material.

Problem Formulation

Consider a set of Cartesian coordinate system $x_i = (x_1, x_2, x_3)$ in such a manner that x_3-axis is normal to the layering. The basic field equations of generalized thermo- elasticity for an infinite generally anisotropic thermoelastic medium with one thermal relaxation time τ_0 at uniform temperature T_0 in the absence of body forces and heat sources are

$$\sigma_{ij,j} = \rho \ddot{u}_i, \quad i, j = 1, 2, 3, \tag{1}$$

$$K_{ij} T_{,ij} - \rho C_e (\dot{T} + \tau_0 \ddot{T}) = T_0 \beta_{ij} [\dot{u}_{i,j} + \tau_0 \ddot{u}_{i,j}], \tag{2}$$

where

$$\sigma_{ij} = C_{ijkl} e_{kl} - \beta_{ij} T \quad , \tag{3}$$

$$\beta_{ij} = C_{ijkl} \theta_{kl}, \tag{4a}$$

and ρ is the density, t is the time, u_i is the displacement in the x_i direction, K_{ij} are the thermal conductivities, C_e and τ_0 are respectively the specific heat at constant strain, and thermal relaxation time, σ_{ij} and e_{ij} are the stress and strain tensor respectively; β_{ij} are thermal moduli; θ_{ij} is the thermal expansion tensor; T is temperature; and the fourth order tensor of the elasticity C_{ijkl} satisfies the (Green) symmetry conditions:

$$c_{ijkl} = c_{klij} = c_{ijlk} = c_{jikl}, \beta_{ij} = \beta_{ji}, \ \theta_{ij} = \theta_{ji}. \tag{4b}$$

Here comma notations are used for spatial derivatives and superposed dot represents differentiation with respect to time. Strain-displacement relation is

$$e_{ij} = \frac{u_{i,j} + u_{j,i}}{2}. \tag{5}$$

Specializing the equations (1-5) for orthotropic media, the governing equations (1) and (2) via (3) to (5) are

$$c_{11}u_{1,11}+c_{66}u_{1,22}+c_{55}u_{1,33}+(c_{12}+c_{66})u_{2,12}+(c_{13}+c_{55})u_{3,13}-\beta_1 T_{,1}=\rho\ddot{u}_1 \tag{6a}$$

$$(c_{12}+c_{66})u_{1,12}+c_{66}u_{2,11}+c_{22}u_{2,22}+c_{44}u_{2,33}+(c_{23}+c_{44})u_{3,23}-\beta_2 T_{,2}=\rho\ddot{u}_2 \tag{6b}$$

$$(c_{13}+c_{55})u_{1,13}+(c_{23}+c_{44})u_{2,23}+c_{55}u_{3,11}+c_{44}u_{3,22}+c_{33}u_{3,33}-\beta_{,3}T_{,3}=\rho\ddot{u}_3 \tag{6c}$$

$$\begin{aligned}&K_{11}T_{,11}+K_{22}T_{,22}+K_{33}T_{,33}-\rho C_e(\dot{T}+\tau_0\ddot{T})\\&=T_0[\beta_1(\dot{u}_{1,1}+\tau_0\ddot{u}_{1,1})\beta_2(\dot{u}_{2,2}+\tau_0\ddot{u}_{2,2})+\beta_3(\dot{u}_{3,3}+\tau_0\ddot{u}_{3,3})]\end{aligned} \tag{6d}$$

where

$$\begin{aligned}\beta_1&=c_{11}\alpha_1+c_{12}\alpha_2+c_{13}\alpha_3,\\ \beta_2&=c_{12}\alpha_1+c_{22}\alpha_2+c_{23}\alpha_3,\\ \beta_3&=c_{13}\alpha_1+c_{32}\alpha_2+c_{33}\alpha_3.\end{aligned} \tag{7}$$

On considering equations (6) and (7) for each layer, and at the interface between two layers, the displacements, temperature, thermal stresses and temperature gradient must be continuous.

Analysis

For harmonic waves propagating in an arbitrary direction, the displacement components and temperature u_1, u_2, u_3 and T are written as

$$(u_1,u_2,u_3,T)=(U_1(x_3),U_2(x_3),U_3(x_3),U_4(x_3))e^{i\xi(l_1x_1+l_2x_2+l_3x_3-ct)} \tag{8}$$

where ξ is the wave number, c is the phase velocity (= ω/ξ), ω is the circular frequency, l_1, l_2 and l_3 are the direction cosine defining the propagation direction. Floquet's theory requires functions U_j ($j=1,2,3,4.$) to have the same periodicity as that of the layering. Hence the problem is reduced to that of one pair of layers, where

$$(U_j,T)(x_3)=\bar{U}_j e^{-i\xi(l_3+\alpha)x_3}\ ,\ j=1,2,3,4, \tag{9}$$

α is an unknown ratio of the wave number components and $\bar{U}_j$ are constants.Upon substitution from (9) into (8) and (6), we have

$$M_{mn}(\alpha)\bar{U}_n=0,\ m,\ n=1,2,3,4. \tag{10a}$$

Here

$$\begin{aligned}M_{11}&=(l_1^2+l_2^2\bar{c}_{66}+\alpha^2\bar{c}_{55}-\zeta^2)\\ M_{12}&=(\bar{c}_{12}+\bar{c}_{66})l_1l_2,\\ M_{13}&=-(\bar{c}_{13}+\bar{c}_{55})l_1\alpha\\ M_{14}&=l_1,\\ M_{22}&=(l_1^2\bar{c}_{66}+l_2^2\bar{c}_{22}+\alpha^2\bar{c}_{44}-\zeta^2)\\ M_{23}&=-(\bar{c}_{23}+\bar{c}_{44})l_2\alpha,\end{aligned}$$

$$
\begin{aligned}
&M_{24}=\bar{\beta}_2 l_2\\
&M_{33}=(l_1^2\bar{c}_{55}+l_2^2\bar{c}_{44}+\alpha^2\bar{c}_{33}-\zeta^2)\,,\\
&M_{34}=-\bar{\beta}_3\alpha\\
&M_{41}=\varepsilon\omega^*\zeta^2 l_1\, i/\omega\,,\\
&M_{42}=\varepsilon\omega^*\zeta^2 l_2\bar{\beta}_2\, i/\omega\\
&M_{43}=-\varepsilon\omega^*\zeta^2\alpha\bar{\beta}_3\, i/\omega\,,\\
&M_{44}=(l_1^2+\bar{K}_2 l_2^2+\bar{K}_3\alpha^2-\omega^*\zeta^2\tau\,),
\end{aligned}
\tag{10b}
$$

where

$$
\zeta^2=\frac{\rho c^2}{c_{11}},\omega^*=\frac{c_{11}C_e}{K_1},\varepsilon=\frac{\beta_1^2 T_0}{\rho C_e c_{11}},\quad \text{and}\quad \tau=\tau_0+i/\omega\,.
$$

The existence of nontrivial solutions for $\bar{U}_j$ (j = 1, 2, 3, 4) demands the vanishing of the determinant in equations (10a), and yields the eighth degree polynomial equation

$$
\alpha^8+A_1\alpha^6+A_2\alpha^4+A_3\alpha^2+A_4=0\,, \tag{11}
$$

where the coefficients A_1, A_2, A_3 and A_4 are given in Appendix-I.

Equations (9) is rewritten as

$$
(U_1,U_2,U_3,T)=\sum_{q=1}^{8}(\bar{U}_{1q},\bar{U}_{2q},\bar{U}_{3q},\bar{U}_{4q})e^{-i\xi\left(l_3+\alpha_q\right)x_3}\,. \tag{12}
$$

For each α_q, q = 1, 2,...8, using the relations (10) and expressing the displacements ratios as

$$
\frac{\bar{U}_{2q}}{\bar{U}_{1q}}=\frac{D_1(\alpha_q)}{D(\alpha_q)}=\gamma_q\,,\frac{\bar{U}_{3q}}{\bar{U}_{1q}}=\frac{D_2(\alpha_q)}{D(\alpha_q)}=\delta_q\,,\frac{\bar{U}_{4q}}{\bar{U}_{1q}}=\frac{D_3(\alpha_q)}{D(\alpha_q)}=\Theta_q\,,\quad q=1,2,\ldots,8, \tag{13}
$$

$$
D(\alpha_q),\ \ D_i(\alpha_q),\ \ i=\ 1,\ 2,\ 3\,, \tag{14}
$$

are given in Appendix-II. Therefore, the solution is

$$
(U_1,U_2,U_3,T)=\sum_{q=1}^{8}(1,\gamma_q,\delta_q,\theta_q)\bar{U}_{1q}e^{-i\xi\left(l_3+\alpha_q\right)x_3} \tag{15}
$$

In view of the continuity of the displacement components, temperature, tractions and temperature gradient across the interface of the two layers, the following conditions must be satisfied:

$$
\begin{aligned}
&u_j^{(1)}\left(x_3=\bar{0}\right)=u_j^{(2)}\left(x_3=\overset{+}{0}\right),\\
&T^{(1)}\left(x_3=\bar{0}\right)=T^{(2)}\left(x_3=\overset{+}{0}\right),\\
&\sigma_{3j}^{(1)}\left(x_3=\overset{-}{0}\right)=\sigma_{3j}^{(2)}\left(x_3=\overset{+}{0}\right),\\
&T'^{(1)}\left(x_3=\overset{-}{0}\right)=T'^{(2)}\left(x_3=\overset{+}{0}\right),
\end{aligned}
\tag{16}
$$

where $T' = \partial T/\partial x_3$ subscripts (1) and (2) refer to layers I and II, respectively, $\overset{+}{0}$ and $\bar{0}$ are values of x_3 near zero. Because of periodicity of the deformation and thermoelastic stress fields, additional conditions obtained are:

$$u_j^{(1)}(x_3 = h_1^-) = u_j^{(2)}(x_3 = -h_2^+),$$

$$T^{(1)}(x_3 = h_1^-) = T^{(2)}(x_3 = -h_2^+),$$

$$\sigma_{3j}^{(1)}(x_3 = h_1^-) = \sigma_{3j}^{(2)}(x_3 = -h_2^+), \quad j = 1, 2, 3;$$

$$T'^{(1)}(x_3 = h_1^-) = T'^{(2)}(x_3 = -h_2^+)\,. \tag{17}$$

Upon substitution of the displacement, temperature, stress and temperature gradient components into (16) and (17), sixteen linear homogeneous equations for sixteen constants $U_{11}^{(1)}, U_{12}^{(1)}, \ldots, U_{17}^{(2)}$ and $U_{18}^{(2)}$ are obtained. For nontrivial solutions, the determinant of the coefficients must vanish. This yields the following characteristic equation:

$$\det\begin{pmatrix} \left[A_{jk}\right] & \left[-\bar{A}_{jk}\right] \\ \left[B_{jk}\right] & \left[-\bar{B}_{jk}\right] \end{pmatrix} = 0, \quad j,k = 1,2,\ldots,8\,. \tag{18}$$

The 8×8 matrices $A_{jk}, \bar{A}_{jk}, B_{jk}$ and $\bar{B}_{jk}$ are given in Appendix-III.

On simplifying equation (18), we have

$$\det\left[A_{jk}\right]\det\left(\left[-\bar{B}_{jk}\right] - \left[B_{jk}\right]\left[A_{jk}\right]^{-1}\left[-\bar{A}_{jk}\right]\right) = 0 \tag{19a}$$

which implies that either

$$\det\left[A_{jk}\right] = 0, \tag{19b}$$

or

$$\det\left(\left[-\bar{B}_{jk}\right] - \left[B_{jk}\right]\left[A_{jk}\right]^{-1}\left[-\bar{A}_{jk}\right]\right) = 0\,. \tag{19c}$$

If equation (19a) holds true, then the problem reduces to a free wave propagation in a single thermoelastic plate of thickness h_1, and in this case $[-\bar{B}_{jk}] - [B_{jk}][A_{jk}]^{-1}[-\bar{A}_{jk}]$ will not exist if A_{jk} is singular. On the hand if A_{jk} is nonsingular then A_{jk}^{-1} exists and accordingly (19c) exists.

In order to solve the problem numerically one has to solve (18), and to solve it is sufficient to consider either equation (19c) for heat conducting composite plates and equation (19b) to solve for free thermoelastic plate.

Particular cases

Uncoupled thermoelasticity

If the coupling constant, $\varepsilon = 0$, then thermal and elastic fields decoupled from each other and from equation (10b) we have $M_{41} = M_{42} = M_{43} = 0$. In this case equation (10a) reduces to

$$\left(l_1^2 + \bar{K}_2 l_2^2 + \bar{K}_3 \alpha^2 - \omega^* \zeta^2 \tau\right)\left(\Delta\alpha^6 + F_1\alpha^4 + F_2\alpha^2 + F_3\right) = 0 \ . \tag{20}$$

From the above equation (20), considering the factor

$$\Delta\alpha^6 + F_1\alpha^4 + F_2\alpha^2 + F_3 = 0, \tag{21}$$

equation (21) corresponds to the characteristic equation in the uncoupled thermoelasticity where

$$\begin{aligned}
\Delta &= c_{33}c_{44}c_{55}\,,\\
F_1 &= [(c_{22}c_{33} - 2c_{23}c_{44} - c_{23}^2)c_{55} + c_{33}c_{44}c_{66}]l_2^2 + [(c_{33} - 2c_{13}c_{55} - c_{13}^2)c_{44} + c_{33}c_{55}c_{66}]l_1^2\\
&\quad -(c_{33}c_{44} + c_{33}c_{55} + c_{44}c_{55})\zeta^2,\\
F_2 &= [(c_{33} - 2c_{13}c_{55} - c_{13}^2)c_{66} + c_{44}c_{55}]l_1^4 + [(c_{22}c_{33} - 2c_{23}c_{44} - c_{23}^2)c_{66} + c_{22}c_{55}c_{44}]l_2^4\\
&\quad + [(-c_{12}^2 c_{33} - 2(c_{33}c_{44} - c_{66}c_{23}c_{55} - c_{12}c_{44}c_{55} + c_{13}c_{22}c_{55} - 2c_{44}c_{55}c_{66} - c_{13}c_{44}c_{66}\\
&\quad + c_{12}c_{33}c_{66} - c_{12}c_{13}c_{44} - c_{13}c_{23}c_{66} - c_{12}c_{23}c_{55} - c_{12}c_{13}c_{23}) - c_{13}^2 c_{22} + c_{22}c_{33} - c_{23}^2]l_1^2 l_2^2\\
&\quad + (2\mathrm{c}_{13}\mathrm{c}_{55} - c_{66}\mathrm{c}_{33} - \mathrm{c}_{55}\mathrm{c}_{44} - \mathrm{c}_{44} - \mathrm{c}_{33} - c_{66}c_{55} + c_{13}^2)l_1^2\zeta^2\\
&\quad + (2c_{23}c_{44} + c_{23}^2 - c_{22}c_{33} - c_{22}c_{55} - c_{66}c_{44} - \mathrm{c}_{55}\mathrm{c}_{44} - c_{33}c_{66})l_2^2 + (c_{33} + c_{44} + c_{55})\zeta^6,\\
F_3 &= \left(c_{55}l_1^2 + c_{44}l_2^2 - \zeta^2\right)\Big\{\Big((1+c_{66})l_1^2 + (c_{22}+c_{66})l_2^2\Big)\zeta^2 - \zeta^4\\
&\quad + [(2c_{22}c_{66} + c_{12}^2 - c_{22})c_{55}]l_1^2 l_2^2 - c_{22}c_{66}l_2^4 - c_{66}l_1^4\Big\}.
\end{aligned} \tag{22}$$

Equation (14) reduces to

$$\begin{aligned}
D_1(\alpha_q) &= M_{13}(\alpha_q)M_{23}(\alpha_q) - M_{12}(\alpha_q)M_{33}(\alpha_q),\\
D_2(\alpha_q) &= M_{12}(\alpha_q)M_{23}(\alpha_q) - M_{13}(\alpha_q)M_{22}(\alpha_q),\\
D_3(\alpha_q) &= 0,\\
D(\alpha_q) &= M_{22}(\alpha_q)M_{33}(\alpha_q) - M^2{}_{23}(\alpha_q).
\end{aligned} \tag{23}$$

Equations (23) correspond to the orthotropic materials purely elastic composites, which are obtained and discussed by Yamada and Nasser [17], and on the hand, second factor of equation (20) is

$$l_1^2 + \bar{K}_2 l_2^2 + \bar{K}_3 \alpha^2 - \omega^* \zeta^2 \tau = 0. \tag{24}$$

Equation (24) corresponds to the purely thermal wave, which is clearly influenced by the thermal relaxation time τ .

Coupled Thermoelasticity

In this is case when there is no thermal relaxation time, i.e. $\tau_0 = 0$ and hence $\tau=i/\omega$. Following above procedure, we arrived at frequency equation in the coupled thermoelasticity.

Numerical Results and Discussion

Using characteristic equation (18), numerical results for phase velocity versus wave number are presented for the first few lower modes to indicate the dependence of dispersion upon the angle of propagation and thermal relaxation times. The material chosen for this purpose is Aluminum epoxy composite / Carbon steel as layer I (h_1=0.7) and Layer II (h_2= 0.3) respectively.

Figure 1 depicts the dispersion curves when the direction cosines of propagation are l_1=0.259, l_2=0.542, l_3=0.799 whereas Figure 2 and Figure 3 depict the dispersion curves when the direction cosines l=0.195, l_2=0.515, l_3=0.834 and l_1=0.125, l_2=0.707, l_3=0.696, in all these figures thermal relaxation time τ_0=2.10^{-7}.

It is observed that lowers modes are more influenced with changes in direction cosines whereas little variation is noticed in the upper modes. Further, as l_1 increases, the phase velocity of lower modes decreases with wave number.

Each of figure display wave speeds (coupled) corresponding to quasi-longitudinal, quasi-transverse and quasi-thermal, at zero wave number limits, lower modes are found to highly influenced and phase velocity of higher modes have higher values and decreases as wave number increases. One of the thermoelastic modes seems to be associated with quick change in the slope of the mode.

In anisotropic plates the distinction between mode types is somewhat artificial, since the equation for thermal and elastic wave modes i.e. quasi-longitudinal and quasi-transverse and shear horizontal modes will be generally be coupled with quasi-thermal wave modes. For wave propagation in the direction of symmetry some wave types revert to pure modes, leading simple characteristic equation of lower order. Consequence of elastic anisotropy in media is the loss of pure wave modes for general propagation direction. At zero wave number limits, each figure displays wave speeds corresponding to one quasilongitudinal, two quasitransverse and a quasithermal in generalized thermoelasticity and higher mode. It is apparent that the largest value corresponds to the quasi-longitudinal and the additional mode appears is a quasi-thermal mode. At low wave number limits, modes are found to highly influence and it vary with the direction cosine. At relatively low values of the wave number, little change is seen in these values. As wave number increases others high modes appear, one of the modes seems to be associated with quick change in the slope of the mode. It is also observed that with change in direction cosines, lower modes highly influenced whereas small variation is noticed in the high modes. Thus in generalized thermoelasticity, at low values of the wave number, only the lower modes get affected at low values of wave number and the little change is seen at relatively high values of wave number. The low value region of the wave number is found to be of more physical interest in generalized thermoelasticity. Further as at high wave number limits, small variation is observed, and so the second sound effects are short lived. Quasi-longitudinal, quasi-transverse (two) and quasi-thermal waves are found coupled with each other due to the thermal and anisotropic effects, also wave-like behavior in the additional quasi-thermal modes. For uncoupled and coupled theory of thermoelasticity the results can be obtained from the present analysis by setting coupling terms and thermal relaxation times equal to zero.

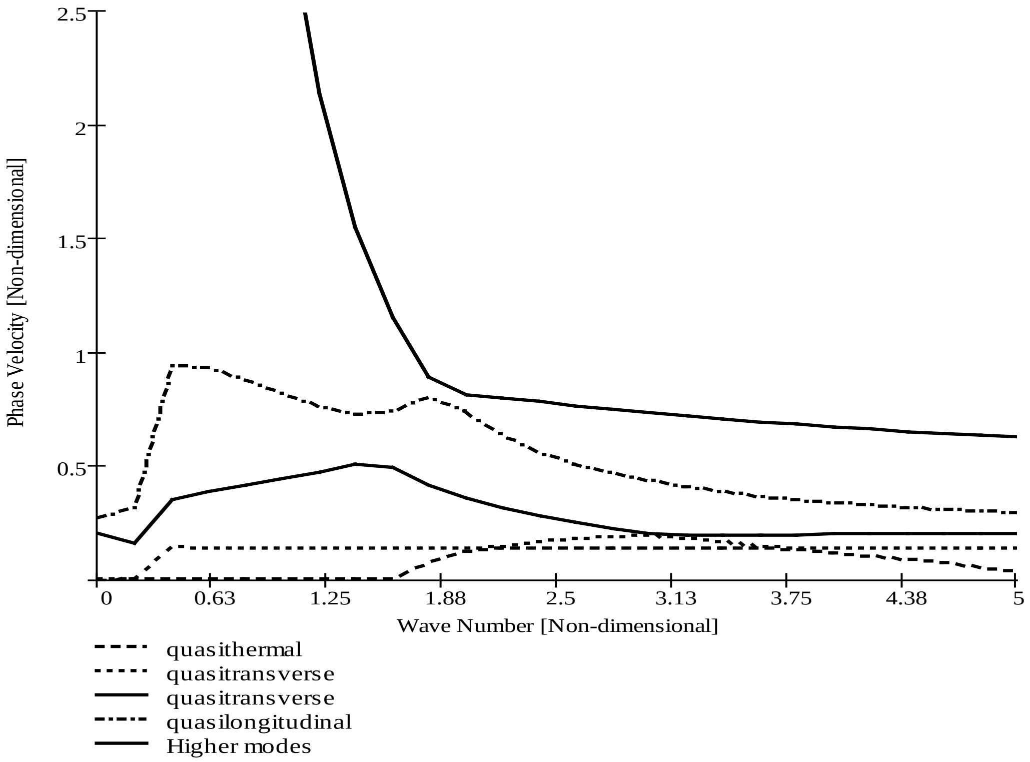

Figure 1. Phase velocity versus wave number for the direction cosine $l_1 = 0.259$, $l_2 = 0.542$ and $l_3 = 0.799$ in generalized thermoelasticity

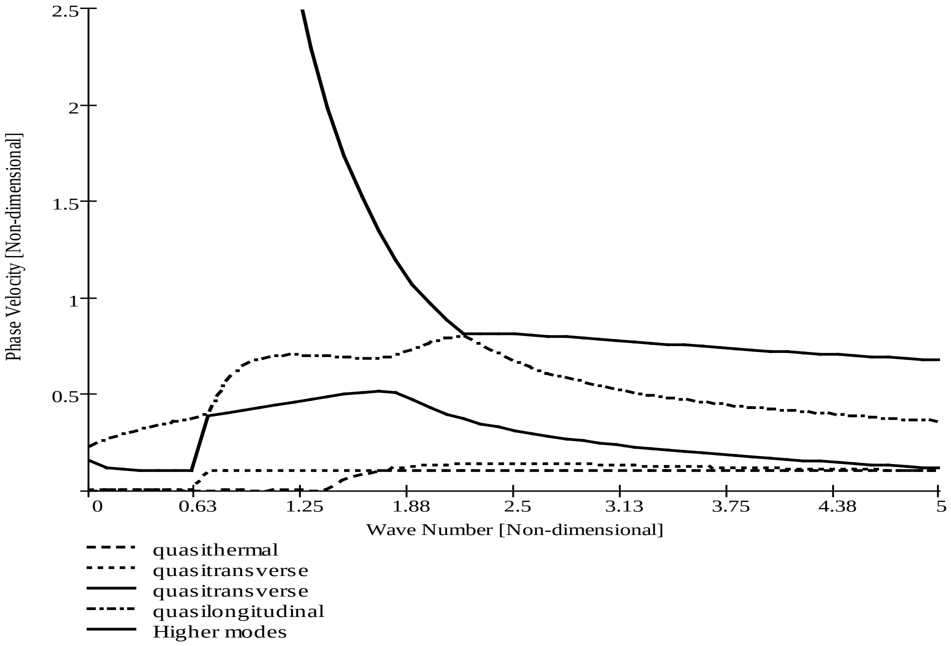

Figure 2. Phase velocity versus wave number for the direction cosine $l_1 = 0.195$, $l_2 = 0.515$ and $l_3 = 0.834$ in generalized thermoelasticity

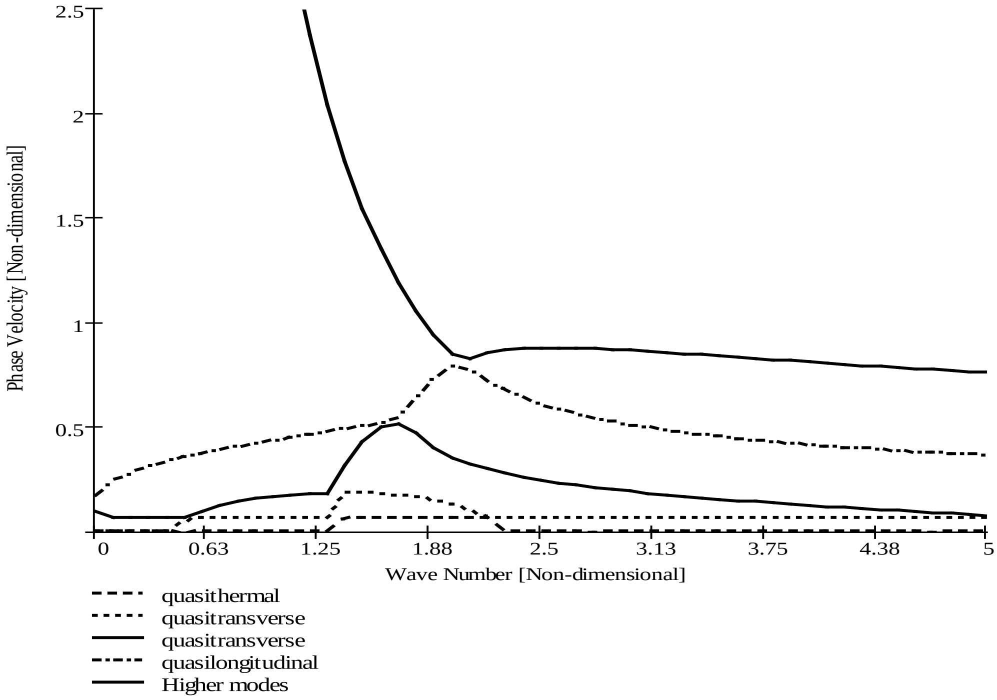

Figure 3. Phase velocity verses wave number for the direction cosine $l_1 = 0.125$, $l_2 = 0.707$ and $l_3 = 0.696$ in generalized thermoelasticity.

Appendix-I.

$$A_1 = B_1\omega^* \varepsilon\zeta^2\, i/\omega + P_1\bar{K}_3 + c_{33}c_{44}c_{55}(l_1^2 + l_2^2\bar{K}_2 - \omega^*\tau\zeta^2)\big/\Delta$$
$$A_2 = B_2\omega^{\bullet}\zeta^2\, i/\omega + P_2\bar{K}_3 + P_1(l_1^2 + \bar{K}_2 l_2^2 - \omega^{\bullet}\tau\zeta^2)\big/\Delta$$
$$A_3 = B_3\omega^* \varepsilon\zeta^2\, i/\omega + P_3\bar{K}_3 + P_2(l_1^2 + l_2^2\bar{K}_2 - \omega^*\tau\zeta^2)\big/\Delta$$
$$A_4 = B_4\omega^* \varepsilon\zeta^2\, i/\omega + P_3(l_1^2 + l_2^2\bar{K}_2 - \omega^*\tau\zeta^2)\big/\Delta$$

$$\Delta = c_{33}\ c_{44}\ c_{55}\ \bar{K}_3$$

$$P_1 = [(c_{22}c_{33} - 2c_{23}c_{44} - c_{23}^2)c_{55} + c_{33}c_{44}c_{66}]l_1^2 + [(c_{33} - 2c_{13}c_{55} - c_{13}^2)c_{55} + c_{33}c_{55}c_{66}]l_2^2 \text{-} (c_{33}c_{44} + c_{33}c_{55} + c_{44}c_{55})\zeta^2$$

$$\begin{aligned} P_2 = {} & [(c_{33} - 2c_{13}c_{55} - c_{13}^2)c_{66} + c_{44}c_{55}]l_1^4 + [(c_{22}c_{33} - 2c_{23}c_{44} - c_{23}^2)c_{66} + c_{22}c_{55}c_{44}]l_2^4 + [(-c_{12}^2c_{33} - 2(c_{33}c_{44} + c_{66}c_{23}c_{55} \\ & + c1_2\, c_{44}c_{55} - c_{13}c_{22}c_{55} + 2c_{44}c_{55}c_{66} + c_{13}c_{44}c_{66} - c_{12}c_{33}c_{66} + c_{12}c_{13}c_{44} + c_{13}c_{23}c_{66} + c_{12}c_{23}c_{55} + c_{12}c_{13}c_{23}) - c_{13}^2c_{22} \\ & + c_{22}c_{33} - c_{23}^2]l_1^2l_2^2 \ + [(2c_{13}c_{55} - c_{66}c_{33} - c_{55}c_{44} - c_{44} - c_{33} - c_{66}c_{55} - c_{13}^2)l_1^2 \ + (2c_{23}c_{44} + c_{23}^2 - c_{22}c_{33} - c_{22}c_{55} \\ & - c_{66}c_{44} - c_{55}c_{44} - c_{33}c_{66})_2^2]\zeta^2 \ + (c_{33} + c_{44} + c_{55})\zeta^4 \end{aligned}$$

$$\begin{aligned} P_3 = {} & (c_{55}l_1^2 + c_{44}l_2^2 - \zeta^2)\big\{[-\zeta^4 + (1 + c_{66})l_1^2 + (c_{22} + c_{66})l_2^2]\zeta^2 \\ & + [2c_{22}c_{66} + c_{12}^2 - c_{22})c_{55}]l_1^2l_2^2 \ - c_{2266}l_2^4 - c_{66}l_1^4\big\} \end{aligned}$$

$$B_1 = -c_{44}c_{55}\bar{\beta}_3^2$$

$$\begin{aligned} B_2 = {} & (c_{55} + c_{44})\bar{\beta}_3^2\zeta^2 + \big[2(c_{13}c_{44} + c_{44}c_{55})\bar{\phi}_3 - c_{33}c_{44} \text{-} (\ c_{44} + c_{66}c_{55})\bar{\beta}_3^2\big]l_1^2 \\ & + \big[2(c_{23}c_{55} + c_{44}c_{55})\bar{\beta}_2\bar{\beta}_3 - c_{33}c_{55})\bar{\beta}_2^2 - (c_{22}c_{55} + c_{44}c_{66})\big]l_2^2 \end{aligned}$$

$$\begin{aligned} B_3 = {} & [-\bar{\beta}_3^2 + 2(c_{55} + c_{13})\bar{\beta}_3 - c_{33})c_{66} - c_{44}c_{55}]l_1^4 + \big\{\big[2c_{23}c_{44} + (c_{13}^2 - c_{33} + 2c_{13}c_{55})\bar{\beta}_2^2 - c_{33}c_{22} \\ & + (c_{12}^2 - c_{22} + 2c_{12}c_{66})\bar{\beta}_3^2 + [2(c_{23} - c_{13}c_{66} + c_{44} - c_{12}c_{55} - c_{66}c_{55} - c_{12}c_{13})\bar{\beta}_3 \\ & + 2(c_{12}c_{13} + c_{33}c_{66} - c_{44}c_{55} - c_{13}c_{23} - c_{23}c_{55} - c_{23}c_{55}]\bar{\beta}_2 \\ & + 2(c_{22}c_{55} + c_{13}c_{22} - c_{12}c_{44} - c_{44}c_{66} - c_{12}c_{23} - c_{23}c_{66})\bar{\beta}_3 + c_{23}^2\big]l_2^2 \\ & + (1 + c_{66})\bar{\beta}_3^2\zeta^2 + (c_{44} + c_{33})\zeta^2 - 2(c_{55} + c_{13})\bar{\beta}_3\zeta^2\big\}l_1^2\zeta^2 + \big[-c_{22}c_{66}\bar{\beta}_3^2 - (c_{33}c_{66} + c_{44}c_{55})\bar{\beta}_2^2 \\ & + 2(\ c_{44}c_{66} + c_{66}c_{23})\bar{\beta}_2\bar{\beta}_3\big]l_2^4 + \big[(c_{22} + c_{66})\bar{\beta}_3^2 + 2(c_{33} + c_{55})\bar{\beta}_2^2 - 2(c_{23} + c_{44})\bar{\beta}_2\bar{\beta}_3\big]l_2^2\zeta^2 - \bar{\beta}_3\zeta^4 \end{aligned}$$

$$\begin{aligned} B_4 = {} & (c_{55}l_1^2 + c_{44}l_2^2 - \zeta^2)\big\{[-\zeta^4 + (1 + c_{66})l_1^2 + (c_{22} + c_{66})l_2^2]\zeta^2 \\ & + [2c_{22}c_{66} + c_{12}^2 - c_{22})c_{55}]l_1^2l_2^2 - c_{22}l_2^4 - c_{66}l_1^4\big\} \end{aligned}$$

Appendix-II

$$D_1(\alpha_q) = M_{23}(\alpha_q)M_{34}(\alpha_q)M_{41}(\alpha_q) + M_{24}(\alpha_q)M_{33}(\alpha_q)M_{41}(\alpha_q) - M_{13}(\alpha_q)M_{24}(\alpha_q)M_{43}(\alpha_q)$$
$$+M_{12}(\alpha_q)M_{34}(\alpha_q)M_{43}(\alpha_q) + M_{13}(\alpha_q)M_{23}(\alpha_q)M_{44}(\alpha_q) - M_{12}(\alpha_q)M_{33}(\alpha_q)M_{44}(\alpha_q)$$
$$D_2(\alpha_q) = M_{23}(\alpha_q)M_{24}(\alpha_q)M_{41}(\alpha_q) + M_{12}(\alpha_q)M_{23}(\alpha_q)M_{44}(\alpha_q) + M_{13}(\alpha_q)M_{24}(\alpha_q)M_{42}(\alpha_q)$$
$$+M_{22}(\alpha_q)M_{34}(\alpha_q)M_{41}(\alpha_q) - M_{13}(\alpha_q)M_{22}(\alpha_q)M_{44}(\alpha_q) - M_{12}(\alpha_q)M_{34}(\alpha_q)M_{42}(\alpha_q)$$
$$D_3(\alpha_q) = M^2{}_{23}(\alpha_q)M_{41}(\alpha_q) - M_{22}(\alpha_q)M_{33}(\alpha_q)M_{41}(\alpha_q) - M_{12}(\alpha_q)M_{23}(\alpha_q)M_{43}(\alpha_q)$$
$$+M_{13}(\alpha_q)M_{22}(\alpha_q)M_{43}(\alpha_q) + M_{12}(\alpha_q)M_{33}(\alpha_q)M_{42}(\alpha_q) - M_{13}(\alpha_q)M_{23}(\alpha_q)M_{42}(\alpha_q)$$
$$D(\alpha_q) = M_{23}(q_j)M_{34}(q_j)M_{42}(q_j) - M_{24}(q_j)M_{33}(q_j)M_{42}(q_j) - M_{22}(q_j)M_{34}(q_j)M_{43}(q_j)$$
$$+M_{22}(q_j)M_{33}(q_j)M_{44}(q_j) - M^2{}_{23}(q_j)M_{44}(q_j) + M_{23}(q_j)M_{24}(\alpha_q)M_{43}(q_j)$$

Appendix-III

$A_{1j} = 1$	$\overline{A}_{1j} = 1,$
$A_{2j} = \gamma_j^{(1)}$	$\overline{A}_{2j} = \gamma_j^{(2)},$
$A_{3j} = \delta_j^{(1)}$	$\overline{A}_{3j} = \delta_j^{(2)},$
$A_{4j} = \theta_j^{(1)}$	$\overline{A}_{4j} = \theta_j^{(2)}$
$A_{5j} = b_{1j}^{(1)} c_{55}^{(1)}$	$\overline{A}_{5j} = \eta b_{1j}^{(2)} c_{55}^{(2)}$,
$A_{6j} = b_{2j}^{(1)} c_{44}^{(1)}$	$\overline{A}_{6j} = \eta b_{2j}^{(2)} c_{44}^{(2)},$
$A_{7j} = b_{3j}^{(1)}$	$\overline{A}_{7j} = \eta b_{3j}^{(2)},$
$A_{8j} = -(l_3 + \alpha_j^{(1)})\theta_j^{(1)}$	$\overline{A}_{8j} = \eta(l_3 + \alpha_j^{(2)})\theta_j^{(2)},$
$B_{jk} = A_{jk} E_k^-$	$\overline{B}_{jk} = A_{jk} E_k^+,$
$E_j^- = e^{-iQ\left(l_3 + \alpha_j^{(1)}\right)h_1/h}$	$Q = \xi(h_1 + h_2),$
$E_j^+ = e^{-iQ(l_3 + \alpha_j^{(2)})h_2/h}$	$\eta = c_{11}^{(2)} \big/ c_{11}^{(1)},$
$b_{1j}^{(m)} = l_1 \delta_j^{(m)} - \alpha_j^{(m)}$	$b_{2j}^{(m)} = l_2 \delta_j^{(m)} - \alpha_j^{(m)} \gamma_j^{(m)},$

$$b_{3j}^{(m)} = \overline{c}_{13}^{(m)} l_1 + \overline{c}_{23}^{(m)} l_2 \gamma_j^{(m)} - \overline{c}_{33}^{(m)} \alpha_j^{(m)} \delta_j^{(m)} - \beta_3 \theta_j^{(m)} \quad ,$$
$$\overline{c}_{jk}^{(m)} = c_{jk}^{(m)} \big/ c_{11}^{(m)}, \quad m = 1,2$$

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K.L. Verma

Department of Mathematics

Government Post Graduate College Hamirpur, (H.P.) 177005, INDIA

klverma@netscape.net, kl.verma@rediffmail.com