Principle of virtual work and the discontinuous Galerkin method

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Summary. The article describes the application of the so-called discontinuous Galerkin method to bar and beam trusses. The formulation is based on the principle of virtual work modified by additional interface terms. This extended principle of virtual work and discontinuous approximation give the usual discontinuous Galerkin method as a particular case and allows one to proceed in the same manner as with the standard principle of virtual work and continuous approximation. The key features are a two-field formulation suitable for structures combining different engineering models, and a computationally efficient one–field implementation. Particular emphasis is placed on explaining the logic of the terms of the extended principle without too many technical details that may shadow the underlying idea.

Key words: principle of virtual work, discontinuous Galerkin method, beam, truss

Introduction

Main contents

The so-called discontinuous Galerkin finite element method has obtained more and more attention in recent years. In it the finite element representations for certain quantities are allowed on purpose to have the possibility of discontinuities. This gives flexibility to the approximation. For example, the degree of approximation can vary from element to element as may be locally needed without problems since strict continuity at the element interfaces is not required. On the other hand, for a fixed mesh and degree of polynomials in the approximation, the degrees of freedom of a discontinuous Galerkin method often exceed considerably that of the standard Galerkin method, although the two methods converge roughly at the same rate. Implementation on an existing software designed for continuous approximations may not also be straightforward.

The purpose of this article is to try to explain to a reader not familiar with the discontinuous Galerkin method how the maybe initially mysteriously looking terms in the various equations can be motivated without going deep into analysis of the solution method properties and mathematics needed therein. We further concentrate on structural applications and especially on truss applications, as the writers are not aware of attempts to use the method in complex truss geometry. The two-field formulation presented

allows one to treat structures combining different engineering models straightforwardly much in the same manner as when using the standard Galerkin method.

The other purpose is to discuss the ways to make a discontinuous Galerkin method competitive with the standard Galerkin method in computational simplicity. Although introduction of an additional unknown may seem initially to be a bad idea in this respect, it turns out to be the key feature. The first option is based on an additional algebraic relationship between the fields and leads to what is called here as the standard discontinuous Galerkin method. An alternative option, based on static condensation on the element level, is a novelty in connection with the discontinuous Galerkin method, as far as the writers are aware of.

As a first step, the method is explained starting from a simple setting and detailed explanations. Second, the outcome is generalized to an extended principle of virtual work. The principle corresponds to local forms of the balance laws of mechanics written under less severe continuity assumptions than usually. To be more specific, differential equations are replaced by physically correct jump conditions on lines or surfaces of discontinuities. In the third and final step, the generic extended principle of virtual work is applied to bar- and beam-trusses. These applications indicate that the extended principle of virtual work and discontinuous approximation can be used much in the same manner as the standard principle of virtual work and continuous approximation.

Nitsche's formulation

Some references consider an article by Nitsche [1] as containing the main ingredients on which the discontinuous Galerkin method can be based. We therefore start here with an analogous setting consisting of the equation set

$$\nabla \cdot \vec{s} + b^\circ = 0 \quad \text{in } \Omega, \tag{1}$$

$$t - t^\circ = 0 \quad \text{on } \Gamma_{\text{t}}, \tag{2}$$

$$u - u^\circ = 0 \quad \text{on } \Gamma_{\text{u}}, \tag{3}$$

in which $t = \vec{n} \cdot \vec{s}$, $\vec{n}$ is the unit outward normal vector to Γ and the constitutive equation is taken as

$$\vec{s} = S\nabla u. \tag{4}$$

The equations may be considered to represent for instance the small transverse deflection $u = u(x, y)$ of a stretched membrane in a plane domain Ω with a given uniform tension force S per unit length. The traction t is given on the traction boundary Γ_{t}. Similarly, the deflection u is given on the displacement boundary Γ_{u}. The disjoint boundary parts Γ_{t} and Γ_{u} form together the whole boundary Γ. The generic symbols Ω and Γ are commonly in use in the finite element literature although if wanted they could be replaced here say with A and s. The meaning of the rest of the notations in (1) to (4) is obvious. In what to follow, we should remember that $\vec{s} = \vec{s}(u)$ and $t = t(u)$, that is, these quantities depend on the deflection. However, to shorten the formulas, these dependencies are usually not shown.

One well-known conventional starting point towards the finite element treatment of the problem described by equations (1) to (4) is to write down the functional (potential energy of the system corresponding to the standard Galerkin method when minimized in a finite dimensional space)

$$V^{\mathrm{G}} = \frac{1}{2} \int_{\Omega} \vec{s} \cdot \nabla u \, \mathrm{d}\Omega - \int_{\Omega} b^{\circ} u \, \mathrm{d}\Omega - \int_{\Gamma_{\mathrm{t}}} t^{\circ} u \, \mathrm{d}\Gamma \,. \qquad (5)$$

Normally u in (5) is demanded to satisfy in advance (3). The variation of (5) gives first

$$\delta V^{\mathrm{G}} = \int_{\Omega} \vec{s} \cdot \nabla \delta u \, \mathrm{d}\Omega - \int_{\Omega} b^{\circ} \delta u \, \mathrm{d}\Omega - \int_{\Gamma_{\mathrm{t}}} t^{\circ} \delta u \, \mathrm{d}\Gamma \,. \qquad (6)$$

Integration by parts in the first term of the expression gives further the identity

$$\int_{\Omega} \vec{s} \cdot \nabla \delta u \, \mathrm{d}\Omega = \int_{\Gamma_{\mathrm{t}}} t \, \delta u \, \mathrm{d}\Gamma - \int_{\Omega} (\nabla \cdot \vec{s}) \, \delta u \, \mathrm{d}\Omega \,. \qquad (7)$$

It is to be noted that the boundary integral is here only over Γ_{t} since due to (3), $\delta u = 0$ on Γ_{u}. Collecting the terms above, the variation is found to become

$$\delta V^{\mathrm{G}} = -\int_{\Omega} (\nabla \cdot \vec{s} + b^{\circ}) \, \delta u \, \mathrm{d}\Omega + \int_{\Gamma_{\mathrm{t}}} (t - t^{\circ}) \, \delta u \, \mathrm{d}\Gamma \,. \qquad (8)$$

Setting the requirement $\delta V^{\mathrm{G}} = 0$ for arbitrary δu gives the field equation (1) and the traction boundary condition (the so-called natural condition) (2). On the basis of these results, the standard steps applied in the conventional finite element method with continuous approximation making use of (5) can be understood.

Nitsche changed the setting described above by not demanding u to satisfy in advance equation (3) in the variational principle. The change can be performed as follows. Let us alter the situation by first using the well-known Lagrange multiplier method. Thus we consider (3) as a constraint on functional (5) and use a modified functional

$$V^{\mathrm{L}} = V^{\mathrm{G}} + \int_{\Gamma_{\mathrm{u}}} \lambda (u - u^{\circ}) \mathrm{d}\Gamma \,, \qquad (9)$$

where λ is the Lagrange multiplier. When we now take the variation, we obtain with respect of the right-hand side of (6) the extra terms

$$\int_{\Gamma_{\mathrm{u}}} \lambda \delta u \, \mathrm{d}\Gamma + \int_{\Gamma_{\mathrm{u}}} \delta\lambda (u - u^{\circ}) \mathrm{d}\Gamma \,. \qquad (10)$$

Further, in the integration by parts the boundary term in the manipulation (7) is now over the whole Γ as constraint (3) is not put in advance on u and δu is thus arbitrary also on Γ_{u}. We obtain

$$\delta V^{\mathrm{L}} = -\int_{\Omega} (\nabla \cdot \vec{s} + b^{\circ}) \, \delta u \, \mathrm{d}\Omega$$

$$+ \int_{\Gamma_{\mathrm{t}}} (t - t^{\circ}) \, \delta u \mathrm{d}\Gamma + \int_{\Gamma_{\mathrm{u}}} [(t + \lambda)\delta u + \delta\lambda (u - u^{\circ})] \mathrm{d}\Gamma \,. \qquad (11)$$

Here δu and $\delta \lambda$ and are arbitrary. We can deduce now equations (1), (2) and (3) and further, the interpretation

$$\lambda = -t \, . \tag{12}$$

This gives the possibility to substitute result (12) back into (9) to arrive at the functional

$$V^{\mathrm{N}} = V^{\mathrm{G}} - \int_{\Gamma_{\mathrm{u}}} t(u - u^{\circ}) \mathrm{d}\Gamma + \frac{1}{2} \int_{\Gamma_{\mathrm{u}}} \tau(u - u^{\circ})^2 \mathrm{d}\Gamma \, . \tag{13}$$

The additional last term which has emerged on the right-hand side is explained as follows. Deeper mathematical analysis shows that for the formulation to work properly in finite dimensional cases (in the finite element method) some additional weighting should be put on the satisfaction of constraint (3). This is achieved by adding on the right-hand side of (9) the least-squares (boundary) functional (the so-called penalty term)

$$V^{\mathrm{LSB}} = \frac{1}{2} \int_{\Gamma_{\mathrm{u}}} \tau(u - u^{\circ})^2 \mathrm{d}\Gamma \, , \tag{14}$$

where τ is a non-negative parameter which may vary on Γ_{u}. It is seen that demanding the variation of (14) to vanish gives immediately equation (3). It may be noted that the factor $1/2$ in (14) is used here just for aesthetic reasons and is not essential as the magnitude of the term depends ultimately on the selection of τ. Expression (14) is in detail

$$V^{\mathrm{LSB}} = \frac{1}{2} \int_{\Gamma_{\mathrm{u}}} \tau u^2 \mathrm{d}\Gamma - \int_{\Gamma_{\mathrm{u}}} \tau u^{\circ} u \mathrm{d}\Gamma + \frac{1}{2} \int_{\Gamma_{\mathrm{u}}} \tau(u^{\circ})^2 \mathrm{d}\Gamma \, . \tag{15}$$

As the last term is a constant, it will disappear when the variation is taken and, if wanted, it can thus be neglected. In [1], the corresponding term has been dropped. We prefer the more compact form (14).

The formulation by Nitsche in [1] has now produced the two last extra integrals in (13). Factors like τ are called in the literature usually "stabilization parameters" or "tuning parameters" or "weightings" and much research has been devoted on the selection of appropriate values for them. The selection of the parameter should make the expression dimensionally homogeneous and ensure that the boundary terms imply that the essential boundary condition is satisfied also in a finite dimensional space. In fact, from the structural point of view, expression (14) can be interpreted here as the strain energy contribution to the total potential energy from an imaginary spring with a spring constant τ attaching the structure to the surroundings.

Discontinuities in Ω

A line loading in the membrane problem or a point force in a string problem may induce a discontinuity in internal force. The discontinuity in force or even in transverse deflection may also be due to function set wherein the solution is sought and therefore the cause can be called as numerical. In the discontinuous Galerkin method,

discontinuities appear inside the computational domain along element interfaces as the finite dimensional set of function does not have any built-in continuity restrictions at the element interfaces. Here, we do not make difference between a physical or numerical source but rather try to treat both possible sources on the same footing.
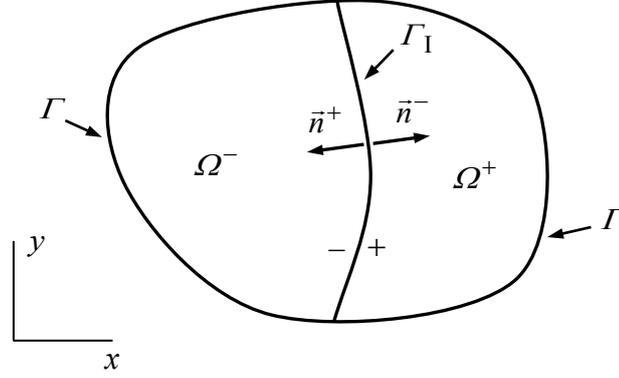

Figure 1 Domain consisting of two sub-domains.

To explain the notation to follow, we consider next the domain of Figure 1. Just one line of discontinuity crosses the domain dividing it into two sub-domains Ω^- and Ω^+ or with a more compact notation Ω^e $e \in E = \{+,-\}$. With the finite element method in mind, the sub-domains could be considered as two large elements and Γ_I the common interface of the two elements.

The unknown in the membrane or string problem is the transverse deflection u, which is considered to be continuous by its nature. This is represented mathematically

$$[\![u]\!] \equiv u^+ - u^- = 0 \text{ on } \Gamma_I, \tag{16}$$

where $u^\pm = u\,|\,\Omega^\pm$ (with the restriction notation) denote deflections inside sub-domains $\Omega^\pm$. To be precise, $u^\pm$ in (16) mean limit values at Γ_I taken from $\Omega^\pm$ but we omit the detail as technical one. The continuity condition (16) can alternatively be written in the form

$$\begin{cases} u^+ - u^* = 0 \\ u^- - u^* = 0 \end{cases} \text{ on } \Gamma_I, \tag{17}$$

and the displacement condition $u - u^\circ = 0$ on Γ_u can be expressed

$$\begin{cases} u - u^* = 0 \\ u^* - u^\circ = 0 \end{cases} \text{ on } \Gamma_u. \tag{18}$$

Finally, we may express (17) and the first equation of (18) in a concise form

$$u^e - u^* = 0 \text{ on } \Gamma, \tag{19}$$

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in which the superscript e takes as its values all the indices of the sub-domains $E = \{+, -\}$ having a point $p \in \Gamma$ in common. In the example case of Figure 1, $E_p = \{+, -\}$ $p \in \Gamma_I$, $E_p = \{+\}$ on Γ_u of Ω^+, and $E_p = \{-\}$ on Γ_u of Ω^-. Now we consider Γ as the entire boundary consisting of the interior and exterior boundaries. This is consistent with the usage of Γ in the previous chapter as there the interior boundary did not exist. Above we have introduced an interface displacement u^* defined everywhere on the sub-domain boundaries. The use of an additional unknown of this type seems to be a novelty in connection with a discontinuous Galerkin method although the idea of using a frame approximation has been discussed earlier e.g. in [2] and [3].

We try now to extend the Nitsche's idea of treating the displacement boundary condition in [1] to discontinuities inside the domain with the interface displacement concept. Instead of equations (1) to (3) , we assume the following set

$$\nabla \cdot \vec{s} + b^\circ = 0 \quad \text{in } \Omega \setminus \Gamma_I, \tag{20}$$

$$-\sum_e t^e + t^\circ = 0 \quad \text{on } \Gamma \setminus \Gamma_u, \tag{21}$$

$$u^e - u^* = 0 \quad \text{on } \Gamma, \tag{22}$$

$$u^* - u^\circ = 0 \quad \text{on } \Gamma_u, \tag{23}$$

in which $\Gamma = \Gamma_u \cup \Gamma_t \cup \Gamma_I$. The representation is implied by the generic balance laws of mechanics: differential equation (20) is obtained only under certain continuity assumptions on $\vec{s}$ if the material element is chosen inside the membrane and therefore it is not valid on Γ. If a highly localized force b° is idealized as line loading t° acting on Γ_I or Γ_t, the outcome is (21) where the sum is over set $E_p \subseteq E$. As discussed in connection to the simplified setting of Figure 1, the set contains the indices of the sub-domains having point $p \in \Gamma$ on their boundaries. In (22), the index takes all the values in E_p. In string and membrane applications, E_p contains one or two elements but later –in truss applications– any number of sub-domains may have point $p \in \Gamma$ on their boundaries.

We start with the functional

$$V^L = \frac{1}{2} \int_{\Omega \setminus \Gamma_I} \vec{s} \cdot \nabla u \, d\Omega - \int_{\Omega \setminus \Gamma_I} b^\circ u \, d\Omega - \int_{\Gamma \setminus \Gamma_u} t^\circ u^* \, d\Gamma$$

$$+ \int_\Gamma \sum_e \lambda^e (u^e - u^*) d\Gamma + \int_{\Gamma_u} \lambda (u^* - u^\circ) d\Gamma . \tag{24}$$

External potential energy expression has now a contribution from t° also on Γ_I. The constraints (22) and (23) have been taken into account with the Lagrange multipliers λ^e $e \in E$ and λ. Above we make difference between a sum inside integral and outside of it. In the latter case, the interface points are excluded. This will be indicated by notation

$$\int_{\Omega \setminus \Gamma_I} f \, d\Omega \equiv \sum_{e \in E} \int_{\Omega^e} f^e \, d\Omega . \tag{25}$$

If a sum is placed inside integral, only the sub-domains having a point in common are accounted for. This will be indicated by notation

$$\int_{\Gamma} \sum_{e} f \, \mathrm{d}\Gamma \equiv \int_{\Gamma} \left(\sum_{e \in E_{\mathrm{p}}} f^{e} \right) \mathrm{d}\Gamma . \tag{26}$$

The somewhat unconventional definition (26) becomes understandable later, when more than two sub-domains may share an interface point.

The variation of (24) is

$$\delta V^{\mathrm{L}} = \int_{\Omega \setminus \Gamma_{\mathrm{I}}} \vec{s} \cdot \nabla \delta u \, \mathrm{d}\Omega - \int_{\Omega \setminus \Gamma_{\mathrm{I}}} b^{\circ} \delta u^{e} \, \mathrm{d}\Omega - \int_{\Gamma \setminus \Gamma_{\mathrm{u}}} t^{\circ} \delta u^{*} \mathrm{d}\Gamma$$

$$+ \int_{\Gamma} \sum_{e} [\delta \lambda^{e} (u^{e} - u^{*}) + \lambda^{e} (\delta u^{e} - \delta u^{*})] \mathrm{d}\Gamma + \int_{\Gamma_{\mathrm{u}}} [\delta \lambda (u^{*} - u^{\circ}) + \lambda \delta u^{*}] \mathrm{d}\Gamma . \tag{27}$$

The steps needed are rather tedious but to gain confidence in the emerging formulas we obviously should go through them. Integration by parts over the sub-domains gives the identity

$$\int_{\Omega \setminus \Gamma_{\mathrm{I}}} \vec{s} \cdot \nabla \delta u \, \mathrm{d}\Omega = \int_{\Gamma} \sum_{e} t^{e} \delta u^{e} \mathrm{d}\Gamma - \int_{\Omega \setminus \Gamma_{\mathrm{I}}} (\nabla \cdot \vec{s}) \delta u \, \mathrm{d}\Omega \tag{28}$$

and the variation is found to become

$$\delta V^{\mathrm{L}} = - \int_{\Omega \setminus \Gamma_{\mathrm{I}}} (\nabla \cdot \vec{s} + b^{\circ}) \delta u \, \mathrm{d}\Omega - \int_{\Gamma \setminus \Gamma_{\mathrm{u}}} \left(\sum_{e} \lambda^{e} + t^{\circ} \right) \delta u^{*} \mathrm{d}\Gamma$$

$$+ \int_{\Gamma} \sum_{e} (t^{e} + \lambda^{e}) \delta u^{e} \, \mathrm{d}\Gamma + \int_{\Gamma} \sum_{e} \delta \lambda^{e} (u^{e} - u^{*}) \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{\mathrm{u}}} \delta \lambda (u^{*} - u^{\circ}) \mathrm{d}\Gamma + \int_{\Gamma_{\mathrm{u}}} \left(\lambda - \sum_{e} \lambda^{e} \right) \delta u^{*} \mathrm{d}\Gamma . \tag{29}$$

We obtain thus from the requirement $\delta V^{\mathrm{L}} = 0$ the correct field equations and the displacement continuity equations (22) and (23). Further, we have the interpretations

$$\lambda^{e} = -t^{e} \quad \text{on } \Gamma , \tag{30}$$

$$\lambda = \sum_{e} \lambda^{e} \quad \text{on } \Gamma_{\mathrm{u}} \tag{31}$$

and when these are applied in (29) we finally obtain the satisfaction of (21). The next step is to substitute (30) and (31) back to (24) to obtain the functional

$$V^{\mathrm{L}} = \frac{1}{2} \int_{\Omega \setminus \Gamma_{\mathrm{I}}} \vec{s} \cdot \nabla u \, \mathrm{d}\Omega - \int_{\Omega \setminus \Gamma_{\mathrm{I}}} b^{\circ} u \, \mathrm{d}\Omega - \int_{\Gamma \setminus \Gamma_{\mathrm{u}}} t^{\circ} u^{*} \mathrm{d}\Gamma$$

$$- \int_{\Gamma} \sum_{e} t^{e} (u^{e} - u^{*}) \mathrm{d}\Gamma - \int_{\Gamma_{\mathrm{u}}} \sum_{e} t^{e} (u^{*} - u^{\circ}) \mathrm{d}\Gamma . \tag{32}$$

Summarizing, the relevant functional is thus

$$V = V^{\mathrm{G}} + V^{\mathrm{JI}} + V^{\mathrm{JB}} + V^{\mathrm{LSI}} + V^{\mathrm{LSB}} , \tag{33}$$

where the standard contribution, taking into account equation (20) and partly equation (21), is

$$V^{\mathrm{G}} = \frac{1}{2}\int_{\Omega\backslash\Gamma_{\mathrm{I}}} \vec{s}\cdot\nabla u \,\mathrm{d}\Omega - \int_{\Omega\backslash\Gamma_{\mathrm{I}}} b^{\circ}u \,\mathrm{d}\Omega - \int_{\Gamma\backslash\Gamma_{\mathrm{u}}} t^{\circ}u^{*}\mathrm{d}\Gamma . \tag{34}$$

Points where the integrand may be discontinuous have explicitly been excluded as indicated by notation $\Omega\backslash\Gamma_{\mathrm{I}}$. The jump contributions taking into account partly equation (21) and equations (22) and (23) are

$$V^{\mathrm{JI}} = -\int_{\Gamma}\sum_{e}t^{e}(u^{e}-u^{*})\mathrm{d}\Gamma , \tag{35}$$

$$V^{\mathrm{JB}} = -\int_{\Gamma_{\mathrm{u}}} t(u^{*}-u^{\circ})\mathrm{d}\Gamma . \tag{36}$$

We have not yet performed a similar step as in equation (13) of putting extra weight on the constraints. Thus we can speculate on using additionally the least-squares interface and boundary functionals

$$V^{\mathrm{LSI}} = \frac{1}{2}\int_{\Gamma}\sum_{e}\tau^{e}(u^{e}-u^{*})^{2}\mathrm{d}\Gamma , \tag{37}$$

$$V^{\mathrm{LSB}} = \frac{1}{2}\int_{\Gamma_{\mathrm{u}}}\tau(u^{*}-u^{\circ})^{2}\mathrm{d}\Gamma , \tag{38}$$

where parameters τ^{e} and τ are dimensionally correct but otherwise arbitrary. Now, expressions (37) could also be interpreted as the strain as the strain energy contributions from imaginary springs with spring constants τ^{e} attaching the sub-domains to the frame.

We notice that the exact solution to equations (20) to (23) makes all the terms in excess to V^{G} vanish. To simplify the expressions, one may impose additional assumptions on the function set wherein the stationary value of the functional is sought in the same manner as with the standard formulation. For example, the displacement boundary condition $u^{*}-u^{\circ}=0$ on Γ_{u} can be satisfied in the strong sense so that terms (36) and (38) vanish. The selection is particularly useful in the one-dimensional case when Γ consists of points as no interpolations are involved.

Principle of virtual work for the membrane

Extended principle of virtual work

For comparison with the presentation to follow, we may represent the formulation (33) also using the principle of virtual work. Then the goal is to find u, u^{*} such that

$$\delta W = 0 \quad \forall \delta u, \delta u^{*} . \tag{39}$$

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The detailed specification of the function set wherein the solution is sought is a technical detail in this connection. Roughly, the function set is assumed to be consistent with the assumptions used in the strong form. The virtual work expression consists of the standard, jump, and least-squares contributions each giving two terms to

$$\delta W = \delta W^{\text{INT}} + \delta W^{\text{EXT}} + \delta W^{\text{JI}} + \delta W^{\text{JB}} + \delta W^{\text{LSI}} + \delta W^{\text{LSB}} . \qquad (40)$$

This will be called here as extended principle of virtual work to discern it from the standard form.

The standard contributions of internal and external forces, taking into account equation (20) to and partly equation (21) are

$$\delta W^{\text{INT}} = -\int_{\Omega \backslash \Gamma} \vec{s} \cdot \nabla \delta u \, \mathrm{d}\Omega , \qquad (41)$$

$$\delta W^{\text{EXT}} = \int_{\Omega \backslash \Gamma} b^{\circ} \delta u \, \mathrm{d}\Omega + \int_{\Gamma \backslash \Gamma_{\text{u}}} t^{\circ} \delta u^{*} \, \mathrm{d}\Gamma . \qquad (42)$$

The jump contributions taking into account partly equation (21) and equations (22) and (23) are

$$\delta W^{\text{JI}} = \int_{\Gamma} \sum_{e} \delta [t^{e}(u^{e} - u^{*})] \, \mathrm{d}\Gamma , \qquad (43)$$

$$\delta W^{\text{JB}} = \int_{\Gamma_{\text{u}}} \delta [t(u^{*} - u^{\circ})] \, \mathrm{d}\Gamma , \qquad (44)$$

and the least-squares contributions are

$$\delta W^{\text{LSI}} = -\frac{1}{2} \int_{\Gamma} \sum_{e} \delta [\tau^{e}(u^{e} - u^{*})^{2}] \, \mathrm{d}\Gamma , \qquad (45)$$

$$\delta W^{\text{LSB}} = -\frac{1}{2} \int_{\Gamma_{\text{u}}} \delta [\tau(u^{*} - u^{\circ})^{2}] \, \mathrm{d}\Gamma . \qquad (46)$$

The principle of virtual work can be taken as the stationarity condition of functional (33), with a slightly modified notation and, therefore, the two formulations are equivalent in the present case. However, the principle of virtual work makes sense also when a stationary principle of a functional does not exist and therefore it can be taken as more general. When expanded with respect to the δ − operator, the jump contributions become

$$\delta W^{\text{JI}} = \int_{\Gamma} \sum_{e} [\beta \delta t^{e}(u^{e} - u^{*}) + t^{e}(\delta u^{e} - \delta u^{*})] \, \mathrm{d}\Gamma , \qquad (47)$$

$$\delta W^{\text{JB}} = \int_{\Gamma_{\text{u}}} [\beta \delta t(u^{*} - u^{\circ}) + t(\delta u^{*} - \delta u^{\circ})] \, \mathrm{d}\Gamma , \qquad (48)$$

in which $\beta = 1$ and $\delta u^{\circ} = 0$. If parameter β is given some other value, consistency with (20) to (23) is not affected, but the principle of virtual work does not represent a stationarity condition of any functional. The sign change modification $\beta = -1$ discussed e.g. in [4], [5] has a favorable effect on stability properties on finite dimensional spaces,

and one may speculate that parameter β can be used to balance the accuracy and stability of the discrete method.

The virtual work equation (39) is satisfied by the solution to equations (20) to (23) and therefore the formulation is consistent. Standard manipulation using integration by parts and fundamental lemma of the calculus of variations show that the principle implies equations (20) to (23) and therefore, the two representations are equivalent. Besides that, one would like to show that the extended principle of virtual work gives an unique solution with, say, piecewise continuous polynomial approximations to the unknown u and u^*. This theme is, however, out of the scope of the present article.

One-field formulation

The extended principle of virtual work discussed in the previous section does not require inter-element continuity of the approximation in a finite element application, but as a drawback involves separate approximations for elements and their interfaces. From the practical viewpoint, this makes the method less attractive than a one-field formulation.

The usual Galerkin method would be obtained by selecting the interface displacement to be the restriction of element displacement to Γ i.e. by using condition

$$u^* = u^e \quad \text{on } \Gamma \tag{49}$$

in the virtual work expression. Then also the approximation should be continuous to be consistent with the assumptions of the formulation. A slightly less restrictive algebraic relationship, along the same lines, follows from

$$\sum_e (u^e - u^*) = 0 \quad \text{on } \Gamma . \tag{50}$$

The selection can be motivated e.g. as follows. As τ is a free parameter of the formulation, one may consider term (37) as constraint or penalty term and let $\tau \to \infty$ so that the minimizer of (33) essentially satisfies (50). If, as in most cases, the number of terms in the sum is two on Γ_I and one on $\Gamma \setminus \Gamma_\mathrm{I}$, the equation gives the mean value $u^* = \bar{u}$ on Γ_I and $u^* = u$ on $\Gamma \setminus \Gamma_\mathrm{I}$. With these relationships the virtual work expressions (43) and (44) become the usual terms of the standard discontinuous Galerkin method

$$\delta W^{\mathrm{JI}} = \int_\Gamma \frac{1}{2} \delta[\![t]\!][\![u]\!] \mathrm{d}\Gamma , \tag{51}$$

$$\delta W^{\mathrm{JB}} = \int_{\Gamma_\mathrm{u}} \delta[t(u - u^\circ)] \mathrm{d}\Gamma . \tag{52}$$

The least-squares contributions associated with jump contributions take also the familiar forms

$$\delta W^{\mathrm{LSI}} = -\int_\Gamma \frac{1}{4} \delta[\![u]\!]\tau[\![u]\!] \mathrm{d}\Gamma , \tag{53}$$

$$\delta W^{\text{LSB}} = -\int_{\Gamma_u} \frac{1}{2} \delta[(u - u^\circ)\tau(u - u^\circ)] \mathrm{d}\Gamma , \tag{54}$$

assuming that $\tau^e = \tau$. A discontinuous Galerkin method of this type is known to share the convergence and stability properties of the standard method based on a continuous approximation [5], [6]. Although an algebraic relationship like (50) is simple in the elastic membrane or string problem, the outcome can be quite complex when a structural model combines beams, plates, etc. Also the improvement in computational simplicity is not enough to make the method competitive with the standard Galerkin method.

We suggest here a quite different approach based on the elimination of element approximation u^e without additional restrictions like (50). Then manipulations are not possible on the virtual work expression level, but they are rather performed in discrete form of element contributions

$$\delta W^e = -\begin{Bmatrix} \delta \mathbf{a}^e \\ \delta \mathbf{a}^* \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \mathbf{K}^{ee} & \mathbf{K}^{e*} \\ \mathbf{K}^{*e} & \mathbf{K}^{**} \end{bmatrix} \begin{Bmatrix} \mathbf{a}^e \\ \mathbf{a}^* \end{Bmatrix} - \begin{Bmatrix} \mathbf{F}^e \\ \mathbf{F}^* \end{Bmatrix} \right) \tag{55}$$

in which $\mathbf{a}^e$ and $\mathbf{a}^*$ are the parameters of the element and interface approximations, respectively. Expression (55) can be manipulated into

$$\mathbf{a}^e = -(\mathbf{K}^{ee})^{-1}(\mathbf{K}^{e*}\mathbf{a}^* - \mathbf{F}^e) , \tag{56}$$

$$\delta W^e = -(\delta \mathbf{a}^*)^{\mathrm{T}}[\{\mathbf{K}^{**} - \mathbf{K}^{*e}(\mathbf{K}^{ee})^{-1}\mathbf{K}^{e*}\}\mathbf{a}^* - \{\mathbf{F}^* - \mathbf{K}^{*e}(\mathbf{K}^{ee})^{-1}\mathbf{F}^e\}] \tag{57}$$

so that the final virtual work expression is obtained by summing the element contributions of expressions (57) containing only the parameters of interface approximations. After solving these, the parameters of element approximations follow from (56). Then, the computational work is dominated by the number of parameters of interface approximations which roughly coincides with that of the standard method. In principle, one may combine a low order interface approximation with a high order element approximation and use a discontinuous Galerkin method without much computational overhead, although accuracy may be restricted by the interface approximation. As far as the writers know this type of elimination procedure has not been put into practice although the possibility has been mentioned already in [2].

The implementation used in the application examples to be discussed later, makes use of the elimination method, in which the extended principle of virtual work is used as it is. As an attractive feature, the parameters of element approximations are treated as internal variables not necessarily visible to the user of software at all. From the user viewpoint, the implementation looks exactly like the one based on a continuous approximation with additional parameters defining the polynomial degree of element approximations.
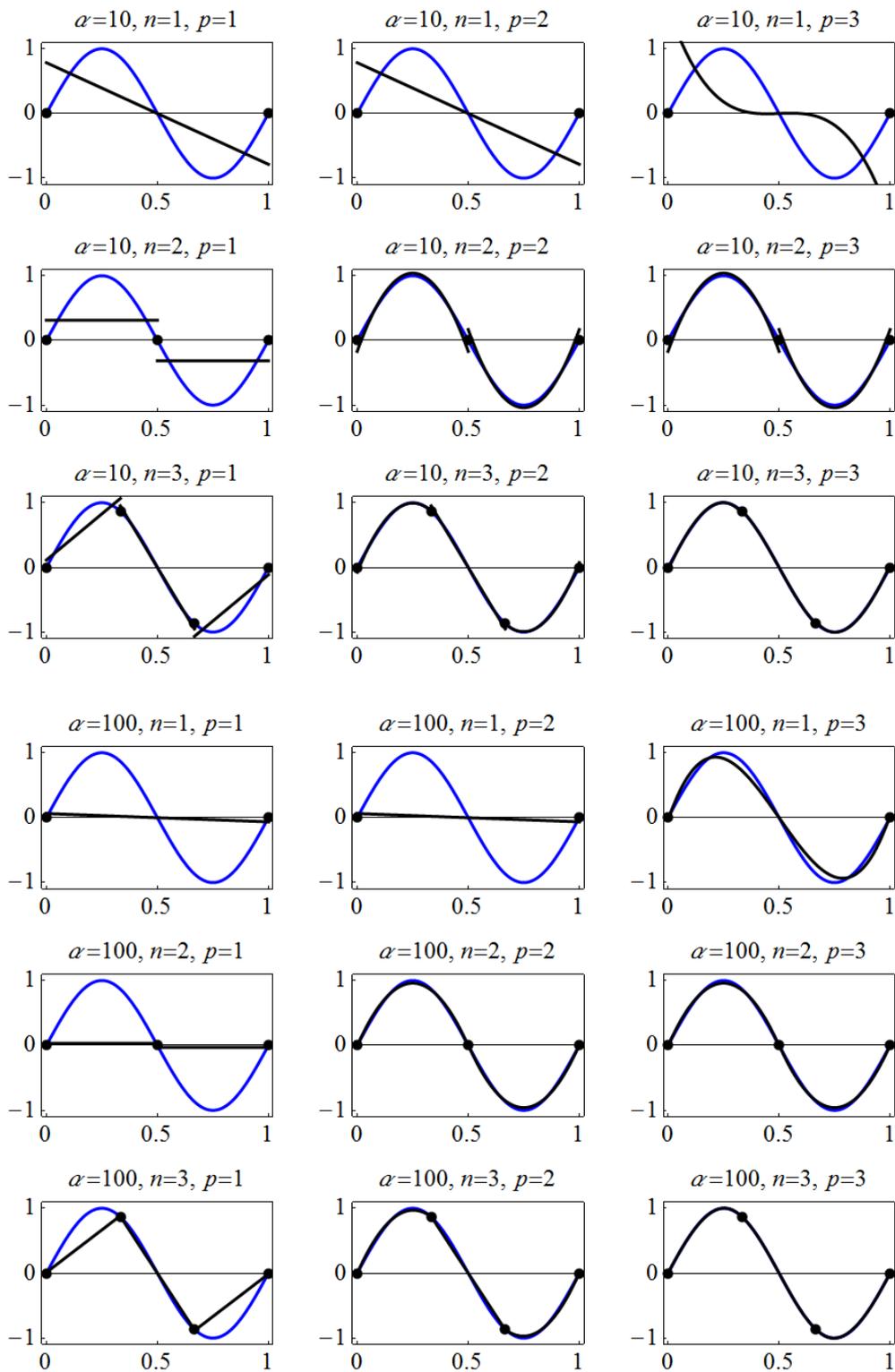
Figure 2 Transverse displacement u / U of a string as function of x / L. Exact solution in blue line, element solution in black line, and interface solution in black markers.

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String application example

As a preliminary example, the method is applied to the one-dimensional string problem on $\Omega =]0, L[$ with a constant S, sinusoidal b° and a specified value of u on the boundary $\Gamma_{\mathrm{u}} = \{0, L\}$. Exact solution to the problem is given by

$$\frac{u}{U} = \sin(2\pi \frac{x}{L}), \tag{58}$$

where U is the maximum transverse deflection. The discrete solution method is based on the extended principle of virtual work and piecewise polynomial approximation of degree p inside the elements without any continuity restrictions on Γ_{I}. The stabilization parameter expression was chosen as

$$\tau = \alpha \frac{S}{h}, \tag{59}$$

in which h is the element size and $\alpha > 0$.

Discrete solutions for some representative selections of stabilization parameter α, symmetry parameter β, number of elements n, and polynomial degree of approximation p are compared to the exact solution in Figure 2. The interface solution with $\beta = 1$ is exact and therefore also independent of α, p and n, which was verified by solving a set of discrete problems with the exact arithmetic of Mathematica environment. Therefore, large α does not mean numerical troubles when $\beta = 1$. For large values of α, the element solution is practically continuous no matter the polynomial degree $p \geq 1$ and independent of β. The numerical solution by non-symmetric formulation $\beta = -1$ behaves much in the same manner, although the interface solution is not exact as with the symmetric formulation.

The precise roles of parameters α and β were studied from analytical solutions to the discrete problem. As the outcome, the minimum value of α for stability increases in p according to (save a positive multiplier)

$$\alpha_{\mathrm{min}} = p(p+1)\beta. \tag{60}$$

Therefore, if β is chosen to be negative, any positive value of α suffices for stability (but not necessarily for accuracy). The same conclusion follows also from a more careful stability analysis [5]. The selection

$$\alpha(p) = 10^4 \cdot \max\{p(p+1)\beta, 1\} \tag{61}$$

based on (60) will be used later in connection with examples where the main interest is not in the effect of α.

Principle of virtual work

Some notations

We now try to generalize the approach described in the previous chapter to linear elasticity. The goal is to use the extended principle of virtual work and discontinuous

approximation in the same manner as the standard principle of virtual work and continuous approximation. A coordinate system invariant representation based on vectors and dyads, denoted $\vec{a}$ and $\ddot{a}$, will be used to keep the expressions concise. There, double dots in expression $\ddot{a} : \ddot{c}$ means double inner product and conjugate dyad $\ddot{a}_c$, defined by $\ddot{a} \cdot \vec{c} = \vec{c} \cdot \ddot{a}_c$ $\forall \vec{c}$, represents transpose. If $\ddot{a} = \ddot{a}_c$, the dyad is taken symmetric and, if $\ddot{a} = -\ddot{a}_c$ anti-symmetric. The double inner product of a symmetric and anti-symmetric dyad $\ddot{a}$ and $\ddot{c}$ satisfies $\ddot{a} : \ddot{c} = 0$. The usual rules of vector algebra apply and e.g. the double inner product can be interpreted as just two inner products. In a Cartesian coordinate system, where $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$ (Kronecker delta), the detailed component representations of conjugate dyad and double inner product are

$$\ddot{a}_c = \sum_{i,j} \vec{e}_j \vec{e}_i a_{ij} , \tag{62}$$

$$\ddot{a} : \ddot{c}_c = (\sum_{i,j} \vec{e}_i \vec{e}_j a_{ij}) : (\sum_{k,l} \vec{e}_l \vec{e}_k c_{kl}) = \sum_{i,j} a_{ij} c_{ij} . \tag{63}$$

The integration by parts formula, needed in connection with a linear elasticity problem,

$$\int_V (\nabla \cdot \ddot{a}) \cdot \vec{c} \, dV = \int_A \vec{n} \cdot \ddot{a} \cdot \vec{c} \, dA - \int_V \ddot{a} : (\nabla \vec{c})_c \, dV , \tag{64}$$

assumes that $\ddot{a}$ and $\vec{c}$ are continuous in V (or have even continuous first derivatives in connection with the fundamental theorem of calculus). In the present context, continuity assumption of (64) does not hold as it is, but V consists of disjoint sub-domains wherein (64) is valid. Then the integration by parts formula can be written as

$$\int_{V \setminus A_I} (\nabla \cdot \ddot{a}) \cdot \vec{c} \, dV = \int_A \sum_e \vec{n} \cdot \ddot{a} \cdot \vec{c} \, dA - \int_{V \setminus A_I} \ddot{a} : (\nabla \vec{c})_c \, dV , \tag{65}$$

where $V \setminus A_I$ is the union of the sub-domains and A consist of the interior boundary A_I and of the exterior boundary ∂V. If $A_I = \varnothing$, as in connection with (64), $A = \partial V$. The sum is over the sub-domains having a point $p \in A$ on their boundaries. On physical sub-domain interfaces or interior boundary the number of those is two and on the exterior boundary one. In connection with engineering models and truss-like structures, the number of terms may exceed two.

Standard form

The principle of virtual work in its standard form states that the sum of the virtual works of internal forces and external forces acting on a body is zero for any kinematically admissible virtual displacement. In mathematical notation

$$\delta W = \delta W^{INT} + \delta W^{EXT} = 0 \quad \forall \delta \vec{u} \tag{66}$$

in which the virtual work of the internal forces is given by (small displacement theory is assumed)

$$\delta W^{INT} = -\int_V \ddot{\sigma} : \delta \ddot{\varepsilon}_c \, dV \tag{67}$$

and that of the external forces is

$$\delta W^{\text{EXT}} = \int_V \vec{b}^{\,\circ} \cdot \delta \vec{u} \, \mathrm{d}V + \int_{A_{\mathrm{t}}} \vec{t}^{\,\circ} \cdot \delta \vec{u} \, \mathrm{d}A \, . \tag{68}$$

The meaning of the notations should be rather obvious but some comments are in place. First, no variations of quantities called internal work or external work are involved. The left-hand sides are just short-hand notations for the expressions appearing on the right-hand sides. Second, quite often in the literature, the minus sign is not used in the definition of the internal virtual work. Then the corresponding term appears on the other side of virtual work equation. In what to follow, some expression can be taken as variations of some other expression and some should be considered independent. To make difference with the two meanings of the delta-symbol, we use the operator notation $\delta[V]$ with brackets in the former case and omit the brackets when we mean an independent quantity like δW.

We will recall shortly how the above expressions are arrived at. The well-known local forms of the balance laws of momentum, moment of momentum and the additional displacement restriction on the boundary

$$\nabla \cdot \vec{\vec{\sigma}} + \vec{b}^{\,\circ} = 0 \quad \text{in } V \, , \tag{69}$$

$$\vec{\vec{\sigma}} - \vec{\vec{\sigma}}_{\mathrm{c}} = 0 \quad \text{in } V \, , \tag{70}$$

$$\vec{t} - \vec{t}^{\,\circ} = 0 \quad \text{on } A_{\mathrm{t}} \, . \tag{71}$$

$$\vec{u} - \vec{u}^{\,\circ} = 0 \quad \text{on } A_{\mathrm{u}} \, , \tag{72}$$

in which $\vec{t} = \vec{n} \cdot \vec{\vec{\sigma}}$ and $\vec{n}$ is the unit outward normal vector to A, are taken as the governing equations. The balance law of moment of momentum excludes volume moments in its present form (70).

We remember the important point that the principle of virtual work is valid for a body irrespective of the constitutive law of the material of the body. This is reflected in equations (69) to (71) as no material law appears there. Naturally, to solve an actual problem, we must introduce at some level information about the material behavior, as we did in equation (4). Preferably, this should be done as late as possible to keep the formulation as general as possible. Here we proceed thus assuming that the stress $\vec{\vec{\sigma}}$ finally depends in some way through the constitutive law on the displacement $\vec{u}$. However, in most to follow, we do not show directly this dependence. Also, the constitutive equation is assumed to satisfy equation (70) 'a priori' and therefore (70) does not enter into the discussion. We would like to emphasize, however, that we could well treat (70) on the same footing as the other conditions implied by the basic laws of mechanics.

The steps are as follows. Equation (69) is multiplied by an arbitrary (weighting) function $\delta \vec{u}$ of position, and integrated over the domain. A scalar equation is obtained:

$$\int_V (\nabla \cdot \vec{\vec{\sigma}}) \cdot \delta \vec{u} \, \mathrm{d}V + \int_V \vec{b}^{\,\circ} \cdot \delta \vec{u} \, \mathrm{d}V = 0 \, . \tag{73}$$

Integration by parts with (64) gives

$$\int_V (\nabla \cdot \vec{\vec{\sigma}}) \cdot \delta \vec{u} \, dV = \int_A \vec{n} \cdot \vec{\vec{\sigma}} \cdot \delta \vec{u} \, dA - \int_V \vec{\vec{\sigma}} : (\nabla \delta \vec{u})_c \, dV \, . \tag{74}$$

The well-known manipulations based on the assumed symmetry of $\vec{\vec{\sigma}}$, division of displacement gradient into the symmetric and anti-symmetric parts according to

$$\nabla \delta \vec{u} = \frac{1}{2}[\nabla \delta \vec{u} + (\nabla \delta \vec{u})_c] + \frac{1}{2}[\nabla \delta \vec{u} - (\nabla \delta \vec{u})_c] \equiv \delta \vec{\vec{\varepsilon}} + \delta \vec{\vec{\omega}} \, , \tag{75}$$

identity $\vec{\vec{\sigma}} : \delta \vec{\vec{\omega}} = 0$, and definition of traction $\vec{t} = \vec{n} \cdot \vec{\vec{\sigma}}$ in (74) give

$$- \int_V \vec{\vec{\sigma}} : \delta \vec{\vec{\varepsilon}}_c \, dV + \int_V \vec{b}^\circ \cdot \delta \vec{u} \, dV + \int_A \vec{t} \cdot \delta \vec{u} \, dA = 0 \, . \tag{76}$$

Now (76) is seen to be nearly the virtual work equation (66). In the conventional application form, one additionally takes the interpretation $\delta \vec{u} = \delta[\vec{u}]$, so that the additional restriction (72) imposed on the boundary implies

$$\delta \vec{u} = 0 \ \text{on} \ A_u \, , \tag{77}$$

and uses the condition of (71)

$$\vec{t} = \vec{t}^\circ \ \text{on} \ A_t \tag{78}$$

to obtain the standard form. The purpose of the selection (77) in (76) is to prevent the unknown traction on boundary A_u to appear in the formulation.

We see many analogues here with the manipulations performed in the earlier chapters. However, there is one important difference. The principle on virtual work in its general form is not a variational principle in the sense that the first variation of a functional is set to zero and we thus cannot make use, say of a principle $\delta[W] = 0$.

Extended form

The principle of virtual work will be modified next to take into account a possible discontinuity in stress $\vec{\vec{\sigma}}$ and displacement $\vec{u}$ inside the domain. We assume that the domain can be divided into disjoint sub-domains so that conditions for (69) to (72) are still valid everywhere except on the sub-domain interfaces. Displacement $\vec{u}^*$ on the sub-domain boundaries becomes an additional unknown of the problem, displacement inside the sub-domains is denoted by $\vec{u}$ and a superscript $\vec{u}^e$ will be used to denote displacement in a typical sub-domain $e \in E$.

The local forms of the balance laws of momentum, moment of momentum and the additional displacement restrictions on the boundary become now

$$\nabla \cdot \vec{\vec{\sigma}} + \vec{b}^\circ = 0 \ \text{in} \ V \setminus A_I \, , \tag{79}$$

$$\vec{\vec{\sigma}} - \vec{\vec{\sigma}}_c = 0 \ \text{in} \ V \setminus A_I \, , \tag{80}$$

$$-\sum_e \vec{t}^e + \vec{t}^\circ = 0 \quad \text{on } A \setminus A_{\mathrm{u}}, \tag{81}$$

$$\vec{u}^e - \vec{u}^* = 0 \quad \text{on } A, \tag{82}$$

$$\vec{u}^* - \vec{u}^\circ = 0 \quad \text{on } A_{\mathrm{u}}, \tag{83}$$

where $A = A_{\mathrm{u}} \cup A_{\mathrm{t}} \cup A_{\mathrm{I}}$. As the derivation of the balance law of momentum (79) assumes certain continuity on stress and require that a material is inside the domain, the equation is not valid e.g. at points where an external surface load $\vec{t}^\circ$ is acting. These points, inside the domain or at the traction boundary, have been excluded from (79) and taken into account in (81) (implied by the very same law) instead. Continuity of the displacement is represented by (82) and the displacement boundary condition by (83).

Instead of going through in detail the straightforward but somewhat lengthy manipulations of the previous sections, we start directly analogously from the extended principle of virtual work: find $\vec{u}, \vec{u}^*$ such that

$$\delta W = 0 \quad \forall \delta \vec{u}, \delta \vec{u}^*. \tag{84}$$

Again, the virtual work expression consists of the standard, jump, and least-squares contributions each giving two terms to

$$\delta W = \delta W^{\mathrm{INT}} + \delta W^{\mathrm{EXT}} + \delta W^{\mathrm{JI}} + \delta W^{\mathrm{JB}} + \delta W^{\mathrm{LSI}} + \delta W^{\mathrm{LSB}}, \tag{85}$$

where the detailed expressions are now chosen to match (79) to (83). The standard contributions of internal and external forces, taking into account equation (79) and partly equation (81)

$$\delta W^{\mathrm{INT}} = -\int_{V \setminus A_{\mathrm{I}}} \vec{\vec{\sigma}} : \delta \vec{\vec{\varepsilon}}_{\mathrm{c}} \, \mathrm{d}V, \tag{86}$$

$$\delta W^{\mathrm{EXT}} = \int_{V \setminus A_{\mathrm{I}}} \vec{b}^\circ \cdot \delta \vec{u} \, \mathrm{d}V + \int_{A \setminus A_u} \vec{t}^\circ \cdot \delta \vec{u}^* \mathrm{d}A \tag{87}$$

should have rather clear physical meanings. Notation $V \setminus A_{\mathrm{I}}$ indicates that points where the integrand may be discontinuous have been excluded. The symmetric jump contributions taking into account partly equation (81) and equations (82) and (83) are

$$\delta W^{\mathrm{JI}} = \int_A \sum_e \delta[\vec{t}^e \cdot (\vec{u}^e - \vec{u}^*)] \mathrm{d}A, \tag{88}$$

$$\delta W^{\mathrm{JB}} = \int_{A_{\mathrm{u}}} \delta[\vec{t} \cdot (\vec{u}^* - \vec{u}^\circ)] \mathrm{d}A. \tag{89}$$

The generalizations of (88) and (89) containing the symmetry parameter β and corresponding to (47) and (48) should be obvious. Finally, the least-squares contributions associated with (88) and (89) are

$$\delta W^{\mathrm{LSI}} = -\frac{1}{2} \int_A \sum_e \delta[(\vec{u}^e - \vec{u}^*) \cdot \vec{\vec{\tau}} \cdot (\vec{u}^e - \vec{u}^*)] \mathrm{d}A, \tag{90}$$

$$\delta W^{\text{LSB}} = -\frac{1}{2} \int_{A_{\text{u}}} \delta[(\vec{u}^* - \vec{u}^\circ) \cdot \vec{\tau} \cdot (\vec{u}^* - \vec{u}^\circ)] \mathrm{d}A , \tag{91}$$

in which one may assume $\vec{\tau} = \vec{\tau}_{\text{c}}$ without loss of generality as only the symmetric part of $\vec{\tau}$ has effect on the setting.

We apply next the generic equations to derive the extended virtual work expressions of bar-truss and beam-truss models in the same manner as is done with the standard principle and more severe continuity assumptions. We are not aware of a similar consistent development in connection with discontinuous Galerkin methods.

Bar- truss application

Some notations

The first application is a truss consisting of bars connected by frictionless spherical or cylindrical joints. Displacements at the joints are continuous but rotations of bars may show jumps there. Joints are also able to transmit forces but not moments. A bar truss can be loaded by distributed forces in the directions of the bars and point forces acting on the joints. Although the bar-joint system is an elastic body having just a complex network geometry, the joints are idealized as small rigid bodies so that the geometrical axes of bars extend to the same point at a joint. With this idealization, detailed geometry of a joint becomes immaterial.

Geometrically, bars are slender cylindrical bodies so that $V = A \times \Omega$ in which Ω is the (mathematical) solution domain. Material xyz – systems are used to identify the particles of bars. Although not necessary, the x – axes are chosen to coincide with the geometrical axes. A structural XYZ – system is used as common reference frame. The unit vectors of the material and structural coordinates are denoted by $\vec{i}, \vec{j}, \vec{k}$ and $\vec{I}, \vec{J}, \vec{K}$, respectively. The well-known kinematic and kinetic assumptions of the bar model are

$$\vec{u} = u(x)\vec{i} , \tag{92}$$

$$\vec{\sigma} = \sigma \vec{i}\vec{i} \tag{93}$$

and interface (joint) displacement

$$\vec{u}^* = u_X^* \vec{I} + u_Y^* \vec{J} + u_Z^* \vec{K} \tag{94}$$

may have one, two or three components depending on the truss type. It is noteworthy that the interface displacement (94) is represented in the basis of the structural coordinate system whereas the relative displacement (92) is represented in the basis of material coordinate system.

It is noteworthy that the displacements of bar particles are defined by (92) and (94) together. By assumption, cross-sections move as rigid bodies so that one may attach a material coordinate system to any of them. The positions and orientations of material coordinate systems are determined by the interface points. Only the rigid body rotation around the x – axis remains non-specified, but that does not matter in the bar model.

Bar-truss equations

The differential equation of the bar-truss problem is given by

$$(\vec{F}' + \vec{B}^\circ) \cdot \vec{i} = 0 \quad \text{in } \Omega \setminus \Gamma_\text{I}, \tag{95}$$

in which $\vec{F}$ is the stress resultant, $\vec{B}^\circ$ is the resultant of the external volume forces, and the superimposed comma means derivative with respect to x. Joints located on Γ_I are excluded as the derivation of (95) assumes certain continuity of $\vec{F}$. On points of discontinuities, the balance law of momentum gives instead

$$-\sum_e \vec{T}^e + \vec{T}^\circ = 0 \quad \text{on } \Gamma \setminus \Gamma_\text{u}, \tag{96}$$

in which $\vec{T}^e \equiv n^e \vec{F}^e$ and $n^e = \pm 1$ is the unit outward normal to Ω^e. In the bar-truss model, the number of bars connected to a joint is not restricted to two and therefore the sum may have more than two terms, too. The given external point force $\vec{T}^\circ$ is assumed to act on joints where displacement is not specified as indicated by $\Gamma \setminus \Gamma_\text{u}$. Continuity of displacement at joints is expressed as

$$(\vec{u}^e - \vec{u}^*) \cdot \vec{i}^e = 0 \quad \text{on } \Gamma, \tag{97}$$

$$(\vec{u}^* - \vec{u}^\circ) \cdot \vec{i}^e = 0 \quad \text{on } \Gamma_\text{u}, \tag{98}$$

which stand for a set of equations written for all the bars connected to a joint. The domain notation correspond to that used earlier and $\Gamma = \Gamma_\text{u} \cup \Gamma_\text{t} \cup \Gamma_\text{I}$.

Assuming that the given traction $\vec{t}^\circ$ is acting only on the end planes of bars, the resultants of external surface and volume forces become

$$\vec{T}^\circ = \int \vec{t}^\circ \mathrm{d}A, \tag{99}$$

$$\vec{B}^\circ = \int \vec{b}^\circ \mathrm{d}A, \tag{100}$$

in which $\vec{b}^\circ$ is the given volume force, and the integrals are over cross-sections. Finally, the resultant of internal forces acting on the cross-sections is

$$\vec{F} = \int \vec{i} \cdot \vec{\sigma} \, \mathrm{d}A. \tag{101}$$

Equations above are valid no matter the material model.

Virtual work expression

The extended virtual work expressions of the bar-truss problem follow from generic expressions (86) to (90) when the kinematic and kinetic assumptions of the model are applied there. Hence, we proceed exactly in the same manner as with the (standard) principle of virtual work.
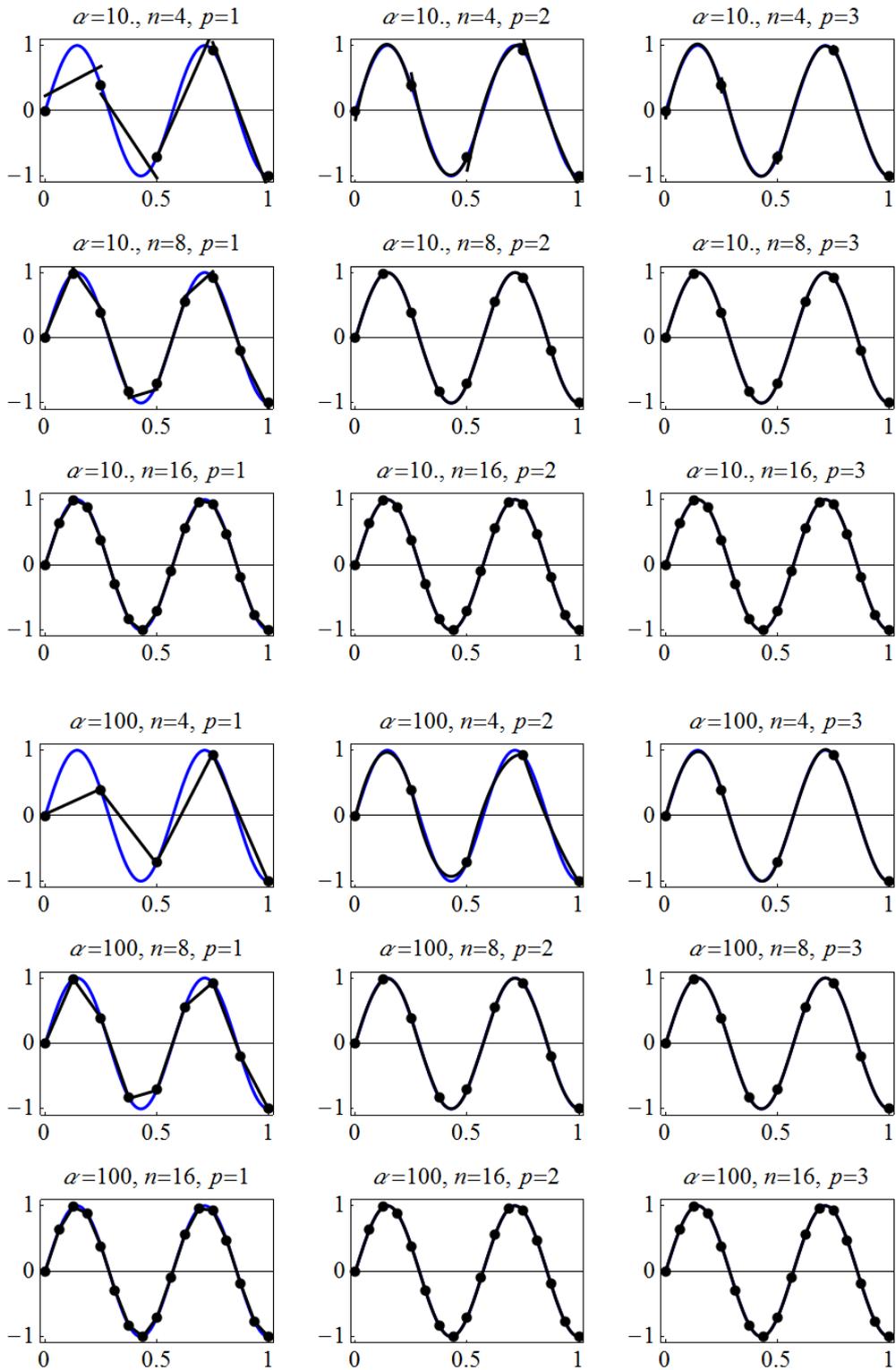
Figure 3 Axial displacement u/U of bar as function of x/L. Exact solution in blue line, element solution in black line, and interface solution in black markers.

The standard contributions of internal and external forces, taking into account equation (95) to and partly equation (96)

$$\delta W^{\mathrm{INT}} = -\int_{\Omega \backslash \Gamma} \vec{F} \cdot \delta \vec{u}' \mathrm{d}\Omega, \qquad (102)$$

$$\delta W^{\mathrm{EXT}} = \int_{\Omega \backslash \Gamma} \vec{B}^{\circ} \cdot \delta \vec{u} \mathrm{d}\Omega + \sum_{\Gamma \backslash \Gamma_u} (\vec{T}^{\circ} \cdot \delta \vec{u}^*) \qquad (103)$$

should be obvious. The jump contributions taking into account partly equation (96) and equations (97) and (98) are

$$\delta W^{\mathrm{JI}} = \sum_{\Gamma} \sum_e \delta[\vec{T}^e \cdot (\vec{u}^e - \vec{u}^*)], \qquad (104)$$

$$\delta W^{\mathrm{JB}} = \sum_{\Gamma_u} \sum_e \delta[\vec{T}^e \cdot (\vec{u}^* - \vec{u}^{\circ})], \qquad (105)$$

and the least-squares contributions associated with the jump terms become

$$\delta W^{\mathrm{LSI}} = -\frac{1}{2} \sum_{\Gamma} \sum_e \delta[(\vec{u}^e - \vec{u}^*) \cdot \vec{\tau}^e \cdot (\vec{u}^e - \vec{u}^*)], \qquad (106)$$

$$\delta W^{\mathrm{LSB}} = -\frac{1}{2} \sum_{\Gamma_u} \sum_e \delta[(\vec{u}^* - \vec{u}^{\circ}) \cdot \vec{\tau}^e \cdot (\vec{u}^* - \vec{u}^{\circ})]. \qquad (107)$$

More details of the rather straightforward derivation of these expressions are given in Appendix A.

Linearly elastic material model

In linearly elastic, homogeneous and isotropic case when the stress-strain relationship is given by the generalized Hooke's law and the linear strain-displacement relationship suffices, constitutive equation (101) becomes

$$\vec{F} = \int \vec{i} \cdot \vec{\sigma} \mathrm{d}A = \vec{i}EAu', \qquad (108)$$

in which E is the Young's modulus of material. One parameter family of weightings

$$\vec{\tau} = \alpha \frac{EA}{h} \vec{i}\vec{i}, \qquad (109)$$

of the least-squares terms (107) and (106) suffices here, as the main emphasis is on the explanation of the two-field formulation.

Bar application example

The first application example is a bar under a sinusoidal distributed load. Left end of the bar is fixed and the right end is free meaning that no external force is acting there. The exact solution to the problem is

$$\frac{u}{U} = \sin(7\pi \frac{x}{2L}),\qquad(110)$$

in which U is the maximal displacement. The discrete solution method is based on the extended principle of virtual work and piecewise polynomial approximations of degree p without any continuity restrictions at joints or at the element interfaces. The displacement boundary condition is satisfied in the strong sense.

The discrete solution with the symmetry parameter $\beta = 1$ and some representative values of stabilization parameter α, number of elements n, and polynomial degree p are compared to the exact solution in Figure 3. From the numerical viewpoint, an elastic bar problem does not differ from the string problem and the discussion and conclusion therein applies also here.
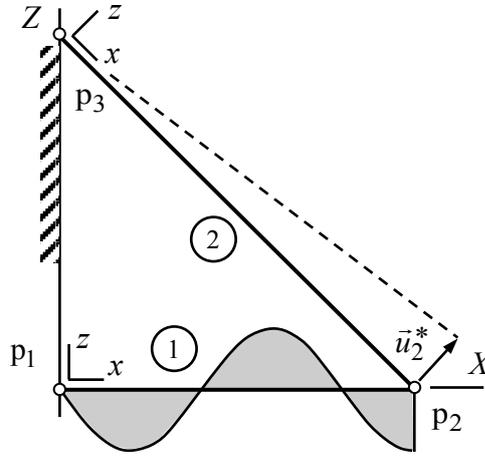
Figure 4 Truss consisting of two bars.

Bar-truss application example

The second application example is a bar-truss consisting of two elastic bars connected by frictionless spherical joints shown in Figure 4. The joints of the initial geometry are at points $p_1 = (0,0,0)$, $p_2 = (L,0,0)$, $p_3 = (0,0,L)$ on the $X-$ and $Z-$axes of the structural coordinate system. Bars 1 and 2 of the truss have end points (p_1,p_2) and (p_3,p_2), respectively, in which the order of points indicates the positive direction of the $x-$axis. Joint at p_2 is free to move and the remaining two are fixed. The cross-sectional area A and Young's modulus E are constants. Bar 1 is subjected to a distributed sinusoidal loading acting in the $x-$direction so that the exact solution is

$$\frac{\vec{u}_2^*}{U} = \sin(5\pi \frac{1}{2})(\vec{I} + \vec{K}),\qquad(111)$$

$$\frac{u^1}{U} = \sin(5\pi \frac{x}{2L}).\qquad(112)$$

The reference value U controls the maximum displacement. Although bar 2 does not undergo a length change and $u^2 = 0$, its orientation changes, as both bars are connected to the joint at p_2.

Again, the discrete solution method is based on the extended principle of virtual work and piecewise polynomial approximations of degree p without any continuity restrictions at joints or element interfaces. The displacement boundary condition is satisfied in the strong sense. Stabilization parameter expression (61) is used to obtain a near-continuous discrete solution. The solution obtained with the representative values of symmetry parameter β and polynomial degree p are compared to the exact solution in Figure 5. The discrete solution behaves in the same manner as in the bar application.
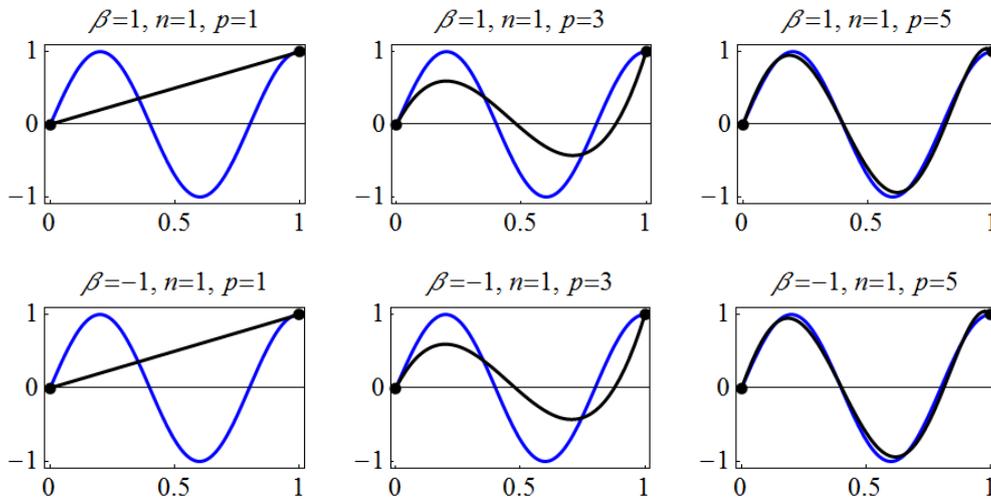
Figure 5 Axial displacement u / U of bar 1 as function of x / L. Exact solution in blue line, element solution in black line, and interface solution in black markers.

Beam- truss application

Some notations

The second engineering model application is a beam-truss. In the same manner as with the bar model, the solution domain is a 2D or 3D network of beams connected by spherical, cylindrical, or etc. joints transmitting force and moment components in some combination. The truss can be loaded by distributed external forces and moments acting on the beams and point forces and moments acting on the joints. The coordinate systems coincide with those of the bar-truss problem. The kinematic and kinetic beam assumptions are

$$\vec{u} = \vec{u}_0(x) + \vec{\theta}(x) \times \vec{\rho}, \tag{113}$$

$$\vec{\sigma} = \sigma_{xx}\vec{i}\vec{i} + \sigma_{xy}\vec{i}\vec{j} + \sigma_{yx}\vec{j}\vec{i} + \sigma_{xz}\vec{i}\vec{k} + \sigma_{zx}\vec{k}\vec{i}, \tag{114}$$

in which $\vec{u}_0(x)$ and $\vec{\theta}(x)$ denote the translation and (small) rotation of cross-sections and $\vec{\rho}(y,z) = \vec{j}y + \vec{k}z$ is the position vector relative to the intersection of cross-section and the $x-$axis. Interface displacement and rotation

$$\vec{u}^* = u_X^* \vec{I} + u_Y^* \vec{J} + u_Z^* \vec{K} , \qquad (115)$$

$$\vec{\theta}^* = \theta_X^* \vec{I} + \theta_Y^* \vec{J} + \theta_Z^* \vec{K} \qquad (116)$$

may have one, two or three components depending on the truss type. In the present case, the unknowns of the problem are the displacement and rotation components which should be accounted for in the interpretation $\delta \vec{u} = \delta[\vec{u}]$. Also now, the displacements of beam particles are defined by interface and element displacements together so that the positions and orientations of material coordinate systems are determined by (115), and (116) and relative displacements by (113).

Timoshenko model

The strong form of a Timoshenko beam truss problem consists of differential equations

$$\vec{F}' + \vec{B}^\circ = 0 \quad \text{in } \Omega \setminus \Gamma_\mathrm{I}, \qquad (117)$$

$$\vec{M}' + \vec{i} \times \vec{F} + \vec{C}^\circ = 0 \quad \text{in } \Omega \setminus \Gamma_\mathrm{I}, \qquad (118)$$

in which $\vec{F}$ and $\vec{M}$ are the stress resultants, $\vec{B}^\circ$ and $\vec{C}^\circ$ are distributed forces and moments. When written for joints and other possible points of force and moment resultant discontinuities, the balance laws take the forms

$$-\sum_e \vec{T}^e + \vec{T}^\circ = 0 \quad \text{on } \Gamma \setminus \Gamma_\mathrm{u} , \qquad (119)$$

$$-\sum_e \vec{S}^e + \vec{S}^\circ = 0 \quad \text{on } \Gamma \setminus \Gamma_\theta , \qquad (120)$$

in which $\vec{T}^e \equiv n^e \vec{F}^e$, $\vec{S}^e \equiv n^e \vec{M}^e$, and $n^e = \pm 1$ is the unit outward normal to domain Ω^e. The given external forces and moment $\vec{T}^\circ$ and $\vec{S}^\circ$ are assumed to act on joints where displacement or rotation are not given. Continuity of displacement and rotation at joints can be expressed as

$$\vec{u}^* - \vec{u}^\circ = 0 \quad \text{on } \Gamma_\mathrm{u} , \qquad (121)$$

$$\vec{\theta}^* - \vec{\theta}^\circ = 0 \quad \text{on } \Gamma_\theta , \qquad (122)$$

$$\vec{\theta}^e - \vec{\theta}^* = 0 \quad \text{on } \Gamma , \qquad (123)$$

$$\vec{u}^e - \vec{u}^* = 0 \quad \text{on } \Gamma . \qquad (124)$$

Assuming again that the given traction $\vec{t}^\circ$ is acting only on the end planes of beams, the resultants of external volume and surface forces become

$$\vec{B}^{\circ} = \int \vec{b}^{\circ} \mathrm{d}A, \tag{125}$$

$$\vec{C}^{\circ} = \int \vec{\rho} \times \vec{b}^{\circ} \mathrm{d}A, \tag{126}$$

$$\vec{T}^{\circ} = \int \vec{t}^{\circ} \mathrm{d}A, \tag{127}$$

$$\vec{S}^{\circ} = \int \vec{\rho} \times \vec{t}^{\circ} \mathrm{d}A, \tag{128}$$

in which $\vec{\rho} = y\vec{j} + z\vec{k}$ is the relative position vector, $\vec{b}^{\circ}$ is the given body force, and the integrals are over cross-sections. Therefore, the moment resultants $\vec{C}^{\circ}$ and $\vec{S}^{\circ}$ are seen to depend on the selection of material coordinate systems. Finally, the resultants of internal forces acting on the cross-sections are

$$\vec{F} = \int \vec{i} \cdot \vec{\sigma} \mathrm{d}A, \tag{129}$$

$$\vec{M} = -\int \vec{i} \cdot \vec{\sigma} \times \vec{\rho} \mathrm{d}A, \tag{130}$$

$$\vec{T} = n\vec{F}, \tag{131}$$

$$\vec{S} = n\vec{M}. \tag{132}$$

Equations above are valid no matter the material model which will again be introduced as late as possible to keep the formulation as general as possible.

Bernoulli model

In the Bernoulli beam-truss model, equations (117) and (118) are just rewritten as

$$\vec{i} \cdot (\vec{F}' + \vec{B}^{\circ}) = 0 \quad \text{in } \Omega \setminus \Gamma_{\mathrm{I}}, \tag{133}$$

$$\vec{M}'' - \vec{i} \times \vec{B}^{\circ} + (\vec{C}^{\circ})' = 0 \quad \text{in } \Omega \setminus \Gamma_{\mathrm{I}}. \tag{134}$$

$$\vec{M}' + \vec{i} \times \vec{F} + \vec{C}^{\circ} = 0 \quad \text{in } \Omega \setminus \Gamma_{\mathrm{I}}, \tag{135}$$

and the shear force components F_y and F_z are taken as constraint forces to be solved from (135). Also, the corresponding kinematical so-called Bernoulli constraints

$$\theta_y = -u_z', \tag{136}$$

$$\theta_z = u_y' \tag{137}$$

are used to eliminate the rotation components θ_y and θ_z from constitutive equations to be discussed later. In short, the main difference between the two models are the Bernoulli model expression

$$\delta\vec{\theta} = \delta\theta_x \vec{i} - \delta u_z' \vec{j} + \delta u_y' \vec{k}, \tag{138}$$

$$\vec{F} = F_x\vec{i} - M'_z\vec{j} + M'_y\vec{k} \tag{139}$$

valid when $C_y^{\mathrm{o}} = C_z^{\mathrm{o}} = 0$. The interface displacement $\vec{u}^*$ and rotation $\vec{\theta}^*$ are the same as with the Timoshenko model.

Virtual work expression

The (extended) principle of virtual work follow from the generic expressions and the Timoshenko beam assumptions (113) and (114). Hence, we proceed exactly in the same manner as with the (standard) principle of virtual work. The weak form of the Bernoulli beam model follows from that of the Timoshenko model, when expressions (138) and (139) are substituted there.

The standard contributions of internal and external forces, taking into account equations (117), (118) and partly (119), (120), become

$$\delta W^{\mathrm{INT}} = -\int_{\Omega\setminus\Gamma_\mathrm{I}} [\vec{F}\cdot\delta\vec{u}' + \vec{M}\cdot\delta\vec{\theta}' + (\vec{F}\times\vec{i})\cdot\delta\vec{\theta}]\mathrm{d}\Omega, \tag{140}$$

$$\delta W^{\mathrm{EXT}} = \int_{\Omega\setminus\Gamma_\mathrm{I}} (\vec{B}^{\mathrm{o}}\cdot\delta\vec{u} + \vec{C}^{\mathrm{o}}\cdot\delta\vec{\theta})\mathrm{d}\Omega$$

$$+\sum_{\Gamma\setminus\Gamma_{\mathrm{u}\theta}}(\vec{T}^{\mathrm{o}}\cdot\delta\vec{u}^*) + \sum_{\Gamma\setminus\Gamma}(\vec{S}^{\mathrm{o}}\cdot\delta\vec{\theta}^*) \tag{141}$$

The jump contributions taking into account equations (121) to (124), are

$$\delta W^{\mathrm{JI}} = \sum_{\Gamma}\sum_{e}\delta[\vec{T}^e\cdot(\vec{u}^e - \vec{u}^*)] + \sum_{\Gamma}\sum_{e}\delta[\vec{S}^e\cdot(\vec{\theta}^e - \vec{\theta}^*)], \tag{142}$$

$$\delta W^{\mathrm{JB}} = \sum_{\Gamma_{\mathrm{u}\theta}}\sum_{e}\delta[\vec{T}^e\cdot(\vec{u}^* - \vec{u}^{\mathrm{o}})] + \sum_{\Gamma}\sum_{e}\delta[\vec{S}^e\cdot(\vec{\theta}^* - \vec{\theta}^{\mathrm{o}})], \tag{143}$$

and the least-squares terms are

$$\delta W^{\mathrm{LSI}} = -\sum_{\Gamma}\sum_{e}\frac{1}{2}\delta[(\vec{u}^e - \vec{u}^*)\cdot\vec{\tau}_\mathrm{u}^e\cdot(\vec{u}^e - \vec{u}^*)]$$

$$-\sum_{\Gamma}\sum_{e}\frac{1}{2}\delta[(\vec{\theta}^e - \vec{\theta}^*)\cdot\vec{\tau}_\theta^e\cdot(\vec{\theta}^e - \vec{\theta}^*)], \tag{144}$$

$$\delta W^{\mathrm{LSB}} = -\sum_{\Gamma_\mathrm{u}}\sum_{e}\frac{1}{2}\delta[(\vec{u}^* - \vec{u}^{\mathrm{o}})\cdot\vec{\tau}_\mathrm{u}^e\cdot(\vec{u}^* - \vec{u}^{\mathrm{o}})]$$

$$-\sum_{\Gamma_\theta}\sum_{e}\frac{1}{2}\delta[(\vec{\theta}^* - \vec{\theta}^{\mathrm{o}})\cdot\vec{\tau}_\theta^e\cdot(\vec{\theta}^* - \vec{\theta}^{\mathrm{o}})]. \tag{145}$$

Some details of the derivation are given in Appendix B.

Linearly elastic material model

In the linearly elastic, homogeneous and isotropic case, when the stress-strain relationship is given by the generalized Hooke's law and the linear strain-displacement

relationship suffices, the Timoshenko beam constitutive equations (129) and (130) become

$$\vec{F} = \int \vec{i} \cdot \vec{\sigma} \mathrm{d}A = \vec{i}EAu'_x + \vec{j}GA(u'_y - \theta_z) + \vec{k}GA(u'_z + \theta_y),\qquad(146)$$

$$\vec{M} = -\int \vec{i} \cdot \vec{\sigma} \times \vec{\rho} \mathrm{d}A = \vec{i}G(I_{yy} + I_{zz})\theta'_x + \vec{j}EI_{zz}\theta'_y + \vec{k}EI_{yy}\theta'_z\qquad(147)$$

in which E and G are the Young's and shear module of material, respectively. In the beam model, the geometry of the cross-sections is described by the moments

$$A = \int \mathrm{d}A,\qquad(148)$$

$$S_r = \int r\mathrm{d}A,\qquad(149)$$

$$I_{rs} = \int rs\mathrm{d}A,\qquad(150)$$

where $r,s \in \{y,z\}$. The simple constitutive equations assume that the material coordinate system is chosen so that the first moments $S_y = S_z = 0$ and the second moments $I_{yz} = I_{zy} = 0$. The weightings

$$\vec{\tau}_{\mathrm{u}} = \alpha_{\mathrm{u}} \frac{1}{h}(\vec{i}\vec{i}EA + \vec{j}\vec{j}GA + \vec{k}\vec{k}GA),\qquad(151)$$

$$\vec{\tau}_{\theta} = \alpha_{\theta} \frac{1}{h}[\vec{i}\vec{i}G(I_{yy} + I_{zz}) + \vec{j}\vec{j}EI_{zz} + \vec{k}\vec{k}EI_{yy}]\qquad(152)$$

of the least-squares terms (145) and (144) suffice for demonstrative purposes. Expressions can be obtained by considering polynomial approximations and the cases of pure bending, torsion etc. separately and combining the results.

Bernoulli model constitutive equations follow from (146) and (147) when Bernoulli constraints (136) and (137) are used there (actually, the Bernoulli constraints follow from the constitutive equations (146) and (147), but this theme is somewhat out of scope). When combined with expression (139), the outcome is

$$\vec{F} = \vec{i}EAu'_x - \vec{j}EI_{yy}u'''_y - \vec{k}EI_{zz}u'''_z,\qquad(153)$$

$$\vec{M} = \vec{i}G(I_{yy} + I_{zz})\theta'_x - \vec{j}EI_{zz}u''_z + \vec{k}EI_{yy}u''_y.\qquad(154)$$

The weightings of the least-squares terms are obtained also here by considering first the cases of pure bending, torsion etc. separately and combining the results. The outcome is

$$\vec{\tau}_{\mathrm{u}} = \alpha_{\mathrm{u}} \frac{1}{h}(\vec{i}\vec{i}EA + \vec{j}\vec{j}\frac{1}{h^2}EI_{yy} + \vec{k}\vec{k}\frac{1}{h^2}EI_{zz}),\qquad(155)$$

$$\vec{\tau}_{\theta} = \alpha_{\theta} \frac{1}{h}[\vec{i}\vec{i}G(I_{yy} + I_{zz}) + \vec{j}\vec{j}EI_{zz} + \vec{k}\vec{k}EI_{yy}].\qquad(156)$$
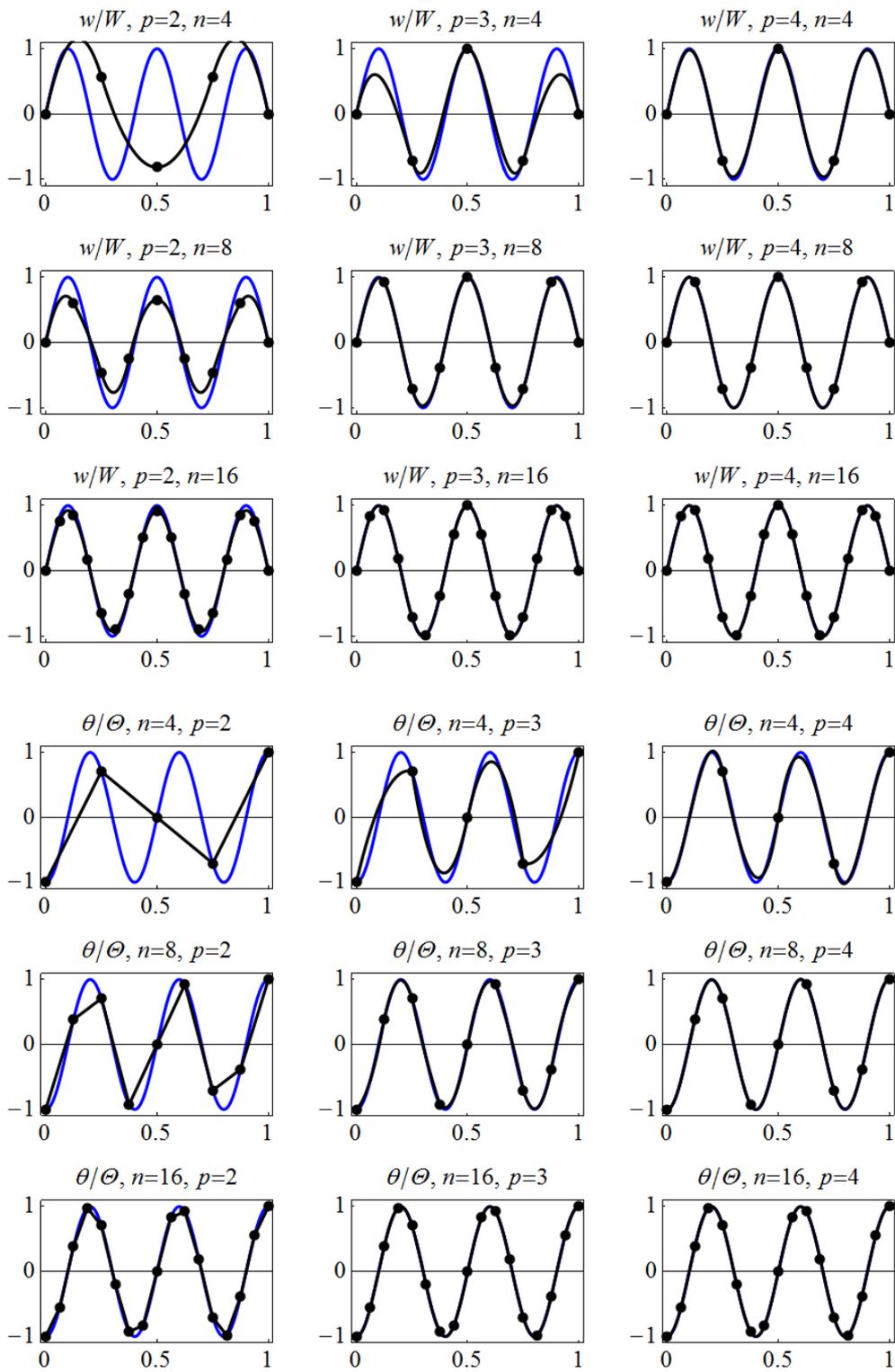
Figure 6 Deflection w/W and rotation θ/Θ as functions of x/L. Exact solution in blue line, element solution in black line, and interface solution in black markers.

Beam bending application example

The first application example is pure bending of a simply supported beam in the XZ – plane under a sinusoidal distributed load. Cross-sectional area A, second moment of area I, Young's modulus E, and shear modulus G are constants. The cross-sectional area is taken to be large so that

$$\frac{w}{W} = \sin(5\pi\frac{x}{L}),\qquad(157)$$

is the exact solution to the Bernoulli and Timoshenko models. Rotation follows from $\theta = -w'$. The discrete solution is based on expressions (140) to (145), and polynomial approximation of degree p for displacement in the Bernoulli model and p, $q = p - 1$ for displacement and rotation, respectively, in the Timoshenko model. Stabilization parameters $\alpha_u = \alpha(p)$ and $\alpha_\theta = \alpha(q)$ according (61) strive for a near continuous discrete solution.

Figure 6 shows the discrete solutions to w and θ with some representative values of the symmetry parameter β, number of elements n, and polynomial degree p. The discrete solutions by the two models practically coincide and β has a negligible effect due to the large α. Interface displacement and rotation are exact when $\beta = 1$, $p \geq 3$ and $q \geq p - 1$. A converging solution is obtained also when $p = 2$, but accuracy of the interface solution is lost and the values of α_u and α_θ become significant. When $p = q = 1$, the Timoshenko model solution shows severe shear locking.
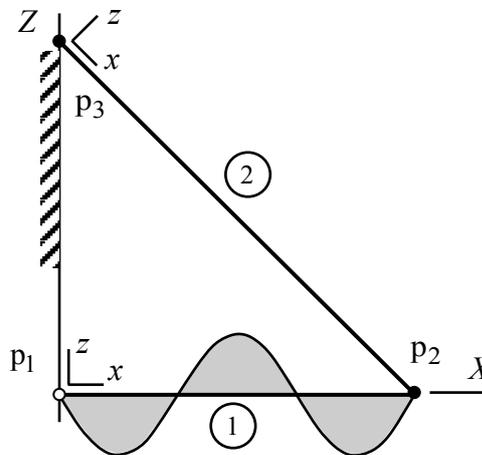

Figure 7 Truss consisting of two beams.

Beam-truss application example

The second application example is a truss consisting of two elastic beams shown in Figure 7. The geometry is the same as with the bar-truss application example. The cross-sectional area A, second moment of area I, Young's modulus E, and shear modulus G are constants. Cross-sectional area is large so that the exact solutions to the Bernoulli and Timoshenko models practically coincide and axial displacements are

negligible. Beam 1 is subjected to a distributed transverse sinusoidal load, beam 2 is clamped at p_3, beam 1 rotates freely at p_1 and displacements and rotations are continuous at p_2. The exact solution to the Bernoulli beam model is

$$\frac{w^1}{W} = \frac{1}{2}\sin(3\pi\frac{x}{L}) + \frac{3\sqrt{2}\pi}{6+4\sqrt{2}}(\frac{x}{L})[(\frac{x}{L})^2 - 1],\tag{158}$$

$$\frac{w^2}{W} = \frac{9\pi}{12+8\sqrt{2}}(\frac{x}{L})^2(\frac{x}{L} - \sqrt{2}).\tag{159}$$

The reference value of deflection W controls the maximum displacement. As the cross-sectional area is considered very large, beams are inextensible in the axial directions and joint at p_2 may undergo a rotation but not displacement.

   In the discrete solution, the displacement and rotation boundary conditions are satisfied in the strong sense. Discrete solutions to the deflection and rotation of beam 1 are compared to the exact solution (158) in Figure 8. Stability parameter expression was according to (61) i.e. $\alpha_u = \alpha(p)$ and $\alpha_\theta = \alpha(q)$. The interface solutions by the symmetric formulation are exact and therefore independent of α and $p \geq 3$. The discrete solution by the Bernoulli and Timoshenko models practically coincide and Figure 8 serves for both cases and also $\beta = \pm 1$ due to the large α used. The discrete solutions to beam 2 of the truss coincide with the exact solution in all cases of Figure 8.
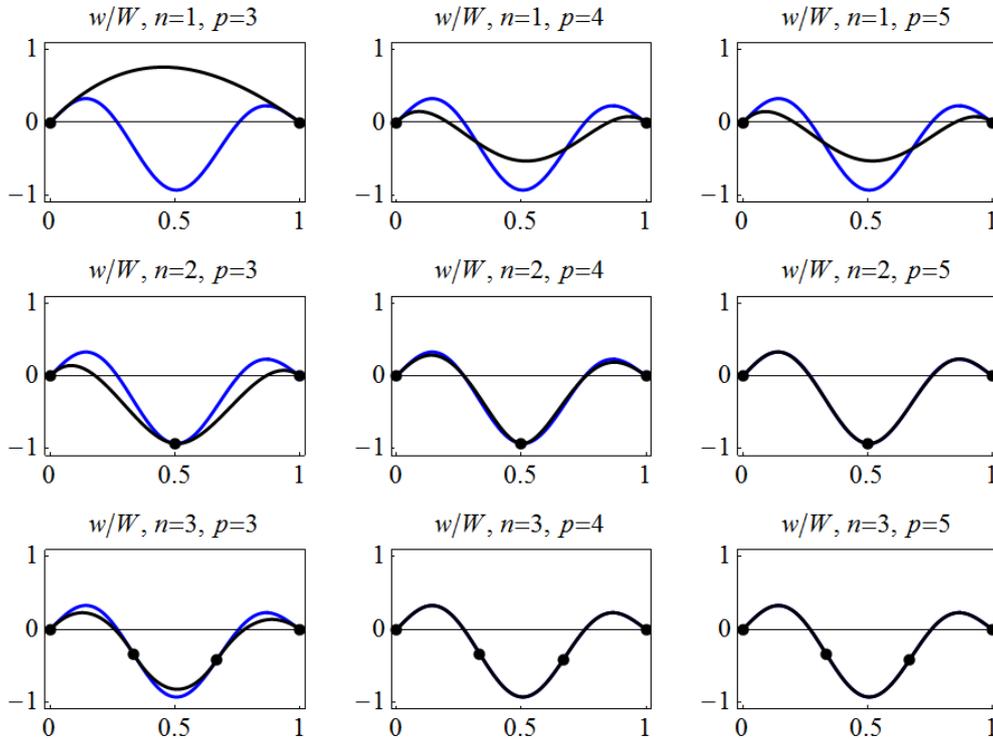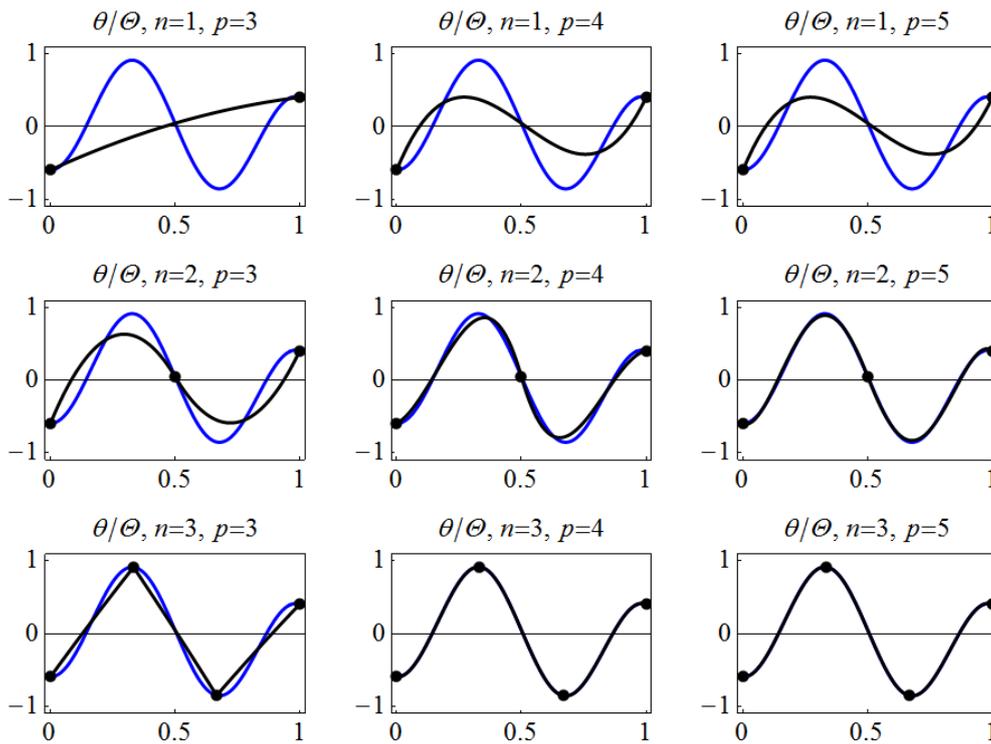
Figure 8 Deflection w/W and rotation θ/Θ of beam 1 as functions of x/L. Exact solution in blue line, element solution in black line, and interface solution in black markers.

Concluding remarks

In this work the standard principle of virtual work was extended to the case, where the usual continuity assumptions are not valid. The extended principle of virtual work was derived directly from the basic balance laws of mechanics taking into account discontinuities. As a novelty, a two-field representation was applied to treat the conditions for domains of smooth behavior and those for discontinuities.

The extended principle of virtual work for 3D continuum can be used in the same manner and for the same purposes as the standard one. First, the corresponding principle for an engineering model follows when kinematical and kinematic assumptions are substituted there as indicated by the bar- and beam-truss application examples. Although the extended principle of, say, for the elastic Timoshenko and Bernoulli beam models can be derived from the boundary value problem directly, the use of the generic form is more straightforward. Second, the differential equations and the correct jump conditions follow from the extended principle for an engineering model in the well-known manner. Third, the extended principle serves as a consistent and ready-to-use weak form of a discontinuous Galerkin method.

The principle of virtual work is attractive from an engineering viewpoint as the terms have clear physical meanings. The first two terms of the extended expression practically coincide with those of the standard expression. The jump terms combining the virtual work of internal forces in jumps and continuity constraint of displacement,

are unique up to the symmetry parameter β of the displacement constraint. Arbitrary multiplier α of least-squares terms brings another parameter that can be used to tune the method to match the approximation used. The application examples indicate that a symmetric formulation is exceptionally accurate. In connection with bar and beam models, the interface solutions or nodal values were exact, and therefore independent of α and polynomial degree p. Also, if α is chosen large, the discrete solution becomes insensitive to β.

Implementation and usability issues are important in practice. Implementation on a standard finite element framework with various conventions concerning concepts and algorithms should also be possible. One of the major advantages of standard Galerkin methods over discontinuous Galerkin methods has been computational simplicity. The ways to eliminate either the interface or element quantities presented here aim to make the method competitive with the standard one. The first option used e.g. in [5] seems not to help much in this respect. Another novelty of the present study is the use of the interface approximation as the primary unknown. Then the (asymptotical) computational complexity in arithmetic operations becomes the same as with the standard method no matter the element approximation, the method fits well in the existing implementations, and additional assumptions are not needed. The last item is important in truss applications or when different engineering models are used to model a structure. As far as the writers are aware, this type of elimination procedure has not been put into practice although the possibility has been mentioned in [2].

As an illustrative example, elimination method gives the element contribution

$$
\delta W = -\left\{ \begin{matrix} \delta u_{z1} \\ h\delta\theta_{y1} \\ \delta u_{z2} \\ h\delta\theta_{y2} \end{matrix} \right\}^{\mathrm{T}} \left(\frac{EI}{h^3} \frac{1}{12\varepsilon+1} \begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4(1+3\varepsilon) & 6 & 2(1-6\varepsilon) \\ -12 & 6 & 12 & 6 \\ -6 & 2(1-6\varepsilon) & 6 & 4(1+3\varepsilon) \end{bmatrix} \left\{ \begin{matrix} u_{z1} \\ h\theta_{y1} \\ u_{z2} \\ h\theta_{y2} \end{matrix} \right\} - \frac{B_z h}{12} \left\{ \begin{matrix} 6 \\ -1 \\ 6 \\ 1 \end{matrix} \right\} \right)
$$

for a Timoshenko beam in $xz-$plane bending and a distributed constant load, when $\beta = 1$, $p \geq 3$ and $q \geq p-1$. It is noteworthy that the weighting of the least squares terms or the degree of polynomial used in the element approximation do not affect the expression. Parameter $\varepsilon = EI/(h^2 GA)$ goes to zero in the Bernoulli limit and the element contribution is seen to coincide with the well-known expression of the Bernoulli beam. One may speculate that elimination along the same lines could be used to generate element contributions for models like Reissner-Mindlin plate or shells suffering from locking phenomena in the same manner as the Timoshenko beam model.

Exceptional accuracy of the interface solution is obtained as the Green function of the adjoint problem (self-adjoint in this particular case) is polynomial which belongs to the approximation space when p is chosen large enough. When the Green function is not polynomial as e.g. in connection with beam on an elastic foundation, using a large p is advantageous as the Green function can be approximated better and thereby also the interface solution becomes more accurate. The ability of the element approximation to be discontinuous at the interface points is essential in both cases. The prize of the

well-known remedy –use high degree polynomials– is no problem in the present context due to the elimination of most unknowns at the element level.

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Appendix A

We assume a Cartesian material coordinate system so that $\vec{i}$ is constant in a bar. The variations of the element and interface displacement and the gradient of the element displacement are

$$\delta \vec{u} = \delta[\vec{u}] = \delta u \vec{i} \, , \tag{A.1}$$

$$\delta \vec{u}^* = \delta[\vec{u}^*], \tag{A.2}$$

$$\nabla \delta \vec{u} = \vec{i} \, \delta \vec{u}' \, . \tag{A.3}$$

The gradient operator of the material coordinate system is $\nabla = \vec{i}\partial/\partial x + \vec{j}\partial/\partial y + \vec{k}\partial/\partial z$ and the derivative with respect to x is denoted by the usual comma notation. The generic virtual work densities (virtual work per unit volume or area indicated by a subscript) can be manipulated into the forms

$$\delta w_V^{\text{INT}} = -\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_c = -(\vec{i} \cdot \vec{\vec{\sigma}}) \cdot \delta \vec{u}' \, . \tag{A.4}$$

$$\delta w_V^{\text{EXT}} = \vec{b}^{\circ} \cdot \delta \vec{u} \, , \tag{A.5}$$

$$\delta w_A^{\text{EXT}} = \vec{t}^{\circ} \cdot \delta \vec{u}^* \, , \tag{A.6}$$

$$\delta w_A^{\text{JI}} = \sum_e \delta[\vec{t}^{\,e} \cdot (\vec{u}^{\,e} - \vec{u}^{\,*})], \tag{A.7}$$

$$\delta w_A^{\text{JB}} = \delta[\vec{t} \cdot (\vec{u}^{\,*} - \vec{u}^{\,\circ})]. \tag{A.8}$$

After integration over the cross-section, the virtual work expressions of internal and external forces become

$$\delta W^{\text{INT}} = -\int_{\Omega \backslash \Gamma_{\text{I}}} \vec{F} \cdot \delta\vec{u}' \mathrm{d}\Omega, \tag{A.9}$$

$$\delta W^{\text{EXT}} = \int_{\Omega \backslash \Gamma_{\text{I}}} \vec{B}^{\circ} \cdot \delta\vec{u} \mathrm{d}\Omega + \sum_{\Gamma \backslash \Gamma_u} (\vec{T}^{\circ} \cdot \delta\vec{u}^{\,*}). \tag{A.10}$$

Above we have assumed that given tractions are acting on the edge surfaces of the bars. The jump contributions take the forms

$$\delta W^{\text{JI}} = \sum_{\Gamma} \sum_e \delta[\vec{T}^{\,e} \cdot (\vec{u}^{\,e} - \vec{u}^{\,*})], \tag{A.11}$$

$$\delta W^{\text{JB}} = \sum_{\Gamma_{\text{u}}} \sum_e \delta[\vec{T}^{\,e} \cdot (\vec{u}^{\,*} - \vec{u}^{\,\circ})]. \tag{A.12}$$

Integration over the cross-sections above needs to done inside the summing as joints are idealized as points. Finally, the least-squares terms associated with the jump contributions become

$$\delta W^{\text{LSI}} = -\sum_{\Gamma} \sum_e \frac{1}{2} \delta[(\vec{u}^{\,e} - \vec{u}^{\,*}) \cdot \vec{\tau}^{\,e} \cdot (\vec{u}^{\,e} - \vec{u}^{\,*})], \tag{A.13}$$

$$\delta W^{\text{LSB}} = -\sum_{\Gamma_{\text{u}}} \sum_e \frac{1}{2} \delta[(\vec{u}^{\,*} - \vec{u}^{\,\circ}) \cdot \vec{\tau}^{\,e} \cdot (\vec{u}^{\,*} - \vec{u}^{\,\circ})], \tag{A.14}$$

in which the cross-sectional areas are embedded in $\vec{\tau}^{\,e}$. For consistency the stability dyad should be of the form

$$\vec{\tau} = \tau \vec{i}\vec{i} \tag{A.15}$$

in which τ is arbitrary in this context.

Appendix B

We assume a Cartesian material coordinate system so that $\vec{i}$ is constant in a beam. The variations of the element and interface displacement and the gradient of element displacement are

$$\delta\vec{u} = \delta[\vec{u}] = \delta\vec{u}_0 + \delta\vec{\theta} \times \vec{\rho}, \tag{B.1}$$

$$\delta\vec{u}^{\,*} = \delta[\vec{u}^{\,*}] = \delta\vec{u}_0^{\,*} + \delta\vec{\theta}^{\,*} \times \vec{\rho}, \tag{B.2}$$

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$$\nabla \delta \vec{u} = \vec{i}\,\delta \vec{u}_0' + \vec{i}\,\delta \vec{\theta}' \times \vec{\rho} - (\vec{I} - \vec{ii}) \times \delta \vec{\theta}\,, \tag{B.3}$$

in which the gradient operator, the unit dyad, and the relative position vector in the basis of material coordinate system are $\nabla = \vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z$, $\vec{I} = \vec{ii} + \vec{jj} + \vec{kk}$, and $\vec{\rho}(y,z) = \vec{j}y + \vec{k}z$, respectively.

The generic virtual work densities (virtual work per unit volume or area indicated by a subscript) can be manipulated into the forms

$$\delta w_V^{\mathrm{INT}} = -\vec{\sigma} : \delta \vec{\varepsilon}_{\mathrm{c}} = -[(\vec{i} \cdot \vec{\sigma}) \cdot \delta \vec{u}_0' - (\vec{i} \cdot \vec{\sigma} \times \vec{\rho}) \cdot \delta \vec{\theta}' + (\vec{i} \cdot \vec{\sigma} \times \vec{i}) \cdot \delta \vec{\theta}]\,, \tag{B.4}$$

$$\delta w_V^{\mathrm{EXT}} = \vec{f}^{\circ} \cdot \delta \vec{u} = \vec{f}^{\circ} \cdot \delta \vec{u}_0 + (\vec{\rho} \times \vec{f}^{\circ}) \cdot \delta \vec{\theta}\,, \tag{B.5}$$

$$\delta w_A^{\mathrm{EXT}} = \vec{t}^{\circ} \cdot \delta \vec{u}^* = \vec{t}^{\circ} \cdot \delta \vec{u}_0^* + (\vec{\rho} \times \vec{t}^{\circ}) \cdot \delta \vec{\theta}^*\,, \tag{B.6}$$

$$\delta w_A^{\mathrm{JI}} = \sum_e \delta[\vec{t}^{\,e} \cdot (\vec{u}^{\,e} - \vec{u}^*)]\,, \tag{B.7}$$

$$\delta w_A^{\mathrm{JB}} = \delta[\vec{t} \cdot (\vec{u}^* - \vec{u}^{\circ})]\,. \tag{B.8}$$

In (B.4) we have used the vector identities $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$ and $\vec{\sigma} : (\vec{I} \times \delta \vec{\theta}) = 0$ (as $\vec{I} \times \delta \vec{\theta}$ is anti-symmetric) and assumed that the balance law of moment of momentum $\vec{\sigma} = \vec{\sigma}_{\mathrm{c}}$ is satisfied 'a priori'. In (B.5) and (B.6) we have used the vector identity $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$ in the second term on the right hand side.

After integration over cross-section, the virtual work expressions of internal and external forces become

$$\delta W^{\mathrm{INT}} = -\int_{\Omega \backslash \Gamma_{\mathrm{I}}} [\vec{F} \cdot \delta \vec{u}_0' + \vec{M} \cdot \delta \vec{\theta}' + (\vec{F} \times \vec{i}) \cdot \delta \vec{\theta}]\mathrm{d}\Omega\,, \tag{B.9}$$

$$\delta W^{\mathrm{EXT}} = \int_{\Omega \backslash \Gamma_{\mathrm{I}}} (\vec{B}^{\circ} \cdot \delta \vec{u}_0 + \vec{C}^{\circ} \cdot \delta \vec{\theta})\mathrm{d}\Omega$$

$$+ \sum_{\Gamma \backslash \Gamma_{\mathrm{u\theta}}} (\vec{T}^{\circ} \cdot \delta \vec{u}_0^*) + \sum_{\Gamma \backslash \Gamma} (\vec{S}^{\circ} \cdot \delta \vec{\theta}^*)\,. \tag{B.10}$$

Above we have assumed that given tractions are acting on the edge surfaces of the bars. The jump contributions take the forms

$$\delta W^{\mathrm{JI}} = \sum_{\Gamma} \sum_e \delta[\vec{T}^{\,e} \cdot (\vec{u}_0^e - \vec{u}^*)] + \sum_{\Gamma} \sum_e \delta[\vec{S}^{\,e} \cdot (\vec{\theta}^e - \vec{\theta}^*)]\,, \tag{B.11}$$

$$\delta W^{\mathrm{JB}} = \sum_{\Gamma_{\mathrm{u\theta}}} \sum_e \delta[\vec{T}^{\,e} \cdot (\vec{u}_0^* - \vec{u}^{\circ})] + \sum_{\Gamma} \sum_e \delta[\vec{S}^{\,e} \cdot (\vec{\theta}^* - \vec{\theta}^{\circ})]\,, \tag{B.12}$$

where integration over the cross-sections above is performed inside the summing as joints are idealized as points. The unknown functions $\vec{u}_0$ and $\vec{\theta}$ are constants with respect to integration over the cross-section. Finally, the least-squares terms associated with the jump contributions become

$$\delta W^{\text{LSI}} = -\sum\nolimits_{\Gamma} \sum\nolimits_{e} \frac{1}{2} \delta[(\vec{u}_0^e - \vec{u}^*) \cdot \vec{\bar{\tau}}_{\text{u}}^e \cdot (\vec{u}_0^e - \vec{u}^*)]$$

$$-\sum\nolimits_{\Gamma} \sum\nolimits_{e} \frac{1}{2} \delta[(\vec{\theta}^e - \vec{\theta}^*) \cdot \vec{\bar{\tau}}_{\theta}^e \cdot (\vec{\theta}^e - \vec{\theta}^*)] \,, \tag{B.13}$$

$$\delta W^{\text{LSB}} = -\sum\nolimits_{\Gamma_{\text{u}}} \sum\nolimits_{e} \frac{1}{2} \delta[(\vec{u}_0^* - \vec{u}^\circ) \cdot \vec{\bar{\tau}}_{\text{u}}^e \cdot (\vec{u}_0^* - \vec{u}^\circ)]$$

$$-\sum\nolimits_{\Gamma_{\theta}} \sum\nolimits_{e} \frac{1}{2} \delta[(\vec{\theta}^* - \vec{\theta}^\circ) \cdot \vec{\bar{\tau}}_{\theta}^e \cdot (\vec{\theta}^* - \vec{\theta}^\circ)] \,. \tag{B.14}$$

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