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On the derivation of boundary-layer equations

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Summary. Some results which may be useful in teaching boundary-layer momentum equations are derived by employing kinematical relations of flow near a rigid impermeable wall. The derivations are based on Taylor expansions of the velocity components and the boundary form in the neighbourhood of the wall. The Newtonian fluid with the no-slip boundary condition and the inviscid fluid model with the free-slip boundary condition are dealt with. Additionally, kinematical relations resulting from the continuity equation are made use of. The treatment is restricted to steady, incompressible, two-dimensional, laminar flow.

Key words: fluid flow, rigid wall, real fluid, ideal fluid, Taylor expansions, boundary-layer momentum equations

Introduction

The rigid impermeable wall is the most usual boundary type encountered in fluid mechanics. The present article deals with some consequences due to the constraints imposed on the flow by kinematical relations near a rigid wall. The relations employed are the no-slip boundary condition and the continuity equation in the case of the real Newtonian fluid model and the free-slip boundary condition and the irrotational flow condition in the case of the ideal inviscid fluid model.

The treatment is restricted to the steady, incompressible, two-dimensional, laminar flow. The well-known equations governing the flow are the continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{1}$$

the momentum equations (without body forces)

$$\begin{aligned}\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)&=-\frac{\partial p}{\partial x}+\frac{\partial \sigma_x'}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}\,,\\ \rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)&=-\frac{\partial p}{\partial y}+\frac{\partial \sigma_y'}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}\,,\end{aligned} \tag{2}$$

and the constitutive relations

$$\sigma'_x = 2\mu \frac{\partial u}{\partial x},$$
$$\sigma'_y = 2\mu \frac{\partial v}{\partial y}, \tag{3}$$
$$\tau_{xy} \equiv \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

Above, x and y are rectangular Cartesian coordinates, u and v the corresponding velocity components, ρ the (constant) density, p the pressure, σ'_x and σ'_y the deviatoric normal stresses, τ_{xy} the shearing stress and μ the (constant) viscosity. In more detail, the deviatoric normal stresses appear as

$$\sigma_x = -p + \sigma'_x,$$
$$\sigma_y = -p + \sigma'_y, \tag{4}$$

where σ_x and σ_y are the normal stress components.

The constitutive relations are valid for the so-called isotropic Newtonian fluid obeying Stokes' law of friction [6, p. 48]. Here we shall call such a fluid shortly real fluid in contrast to the ideal fluid model considered later.

Basic expansions

Figure 1 depicts a section of a fixed wall in a two-dimensional case. To study the flow near a certain point O at the wall, it is convenient to define a local rectangular Cartesian x, y-coordinate system with its origin at point O. The x-axis is taken along the tangent to the boundary at point O, and the y-axis is directed towards the fluid side of the boundary. We emphasize that during the derivations the selected coordinate system is kept fixed.

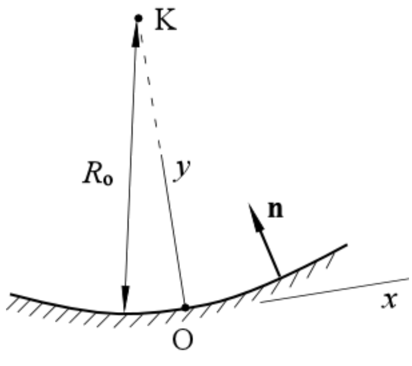

Figure 1. Two-dimensional flow and wall.

It is assumed that the boundary form and the velocity distribution are smooth enough so that they can be expressed as Taylor expansions about point O. We thus obtain

$$y_{\mathrm{b}} = (y_{\mathrm{b}})_{\mathrm{o}} + \left(\frac{\mathrm{d}y_{\mathrm{b}}}{\mathrm{d}x}\right)_{\mathrm{o}} x + \frac{1}{2}\left(\frac{\mathrm{d}^2 y_{\mathrm{b}}}{\mathrm{d}x^2}\right)_{\mathrm{o}} x^2 + \cdots = \frac{1}{2}\left(\frac{\mathrm{d}^2 y_{\mathrm{b}}}{\mathrm{d}x^2}\right)_{\mathrm{o}} x^2 + \cdots$$

$$= \frac{1}{2R_{\mathrm{o}}} x^2 + \cdots \tag{5}$$

$$\begin{aligned} u &= u_{\mathrm{o}} + \left(\frac{\partial u}{\partial x}\right)_{\mathrm{o}} x + \left(\frac{\partial u}{\partial y}\right)_{\mathrm{o}} y + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_{\mathrm{o}} x^2 + \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{\mathrm{o}} xy + \frac{1}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)_{\mathrm{o}} y^2 + \cdots \\ v &= v_{\mathrm{o}} + \left(\frac{\partial v}{\partial x}\right)_{\mathrm{o}} x + \left(\frac{\partial v}{\partial y}\right)_{\mathrm{o}} y + \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2}\right)_{\mathrm{o}} x^2 + \left(\frac{\partial^2 v}{\partial x \partial y}\right)_{\mathrm{o}} xy + \frac{1}{2}\left(\frac{\partial^2 v}{\partial y^2}\right)_{\mathrm{o}} y^2 + \cdots \end{aligned} \tag{6}$$

Here, subscript o refers to the value of a quantity evaluated at point O. Similarly, subscript b refers to the value of a quantity evaluated at the wall boundary. The term $1/R_{\mathrm{o}}$ is the curvature of the boundary at point O:

$$\frac{1}{R_{\mathrm{o}}} = \left(\frac{\mathrm{d}^2 y_{\mathrm{b}}}{\mathrm{d}x^2}\right)_{\mathrm{o}} . \tag{7}$$

It is positive if the centre of curvature K (Figure 1) is on the fluid side of the boundary. Terms up to the second order have been shown in expressions (5) and (6). Terms $(y_{\mathrm{b}})_{\mathrm{o}}$ and $(\mathrm{d}y_{\mathrm{b}}/\mathrm{d}x)_{\mathrm{o}}$ vanish due to the choice of the coordinate system.

When expression (5) is substituted into expressions (6), we obtain

$$\begin{aligned} u_{\mathrm{b}} &= u_{\mathrm{o}} + \left(\frac{\partial u}{\partial x}\right)_{\mathrm{o}} x + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{R}\frac{\partial u}{\partial y}\right)_{\mathrm{o}} x^2 + \cdots \\ v_{\mathrm{b}} &= v_{\mathrm{o}} + \left(\frac{\partial v}{\partial x}\right)_{\mathrm{o}} x + \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{R}\frac{\partial v}{\partial y}\right)_{\mathrm{o}} x^2 + \cdots \end{aligned} \tag{8}$$

In the present article, the nature of the flow near a curved (or in a special case near a flat) wall is explained using rectangular Cartesian coordinates (cf. Figure 1). Later, the results obtained are applied to derive the standard boundary-layer momentum equations. However, in this approach there is initially a discrepancy as with curved boundaries the normal way to present the governing equations is to use an orthogonal curvilinear coordinate system — sometimes called a body intrinsic coordinate system, e.g. [1. p. 197]. This system is commented on further at the end of the article in connection with Figure 3. The symbols for the coordinates and the velocity components are the same as above, but their content is different. The abscissa, *x*, is measured along the curved wall, and the ordinate, *y*, at right angles to it. Symbols *u* and *v* refer now to the velocity components in the directions of the local x- and y-axes.

Using the intrinsic system we still have relations of the type $u = u(x, y)$ and $v = v(x, y)$ and we can write expansions (6) as before. However, we do not need equation (5) as the boundary is given now by $y = 0$ and instead of (8) we have

$$
\begin{aligned}
u_{\mathrm{b}} &= u_{\mathrm{o}} + \left(\frac{\partial u}{\partial x}\right)_{\mathrm{o}} x + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_{\mathrm{o}} x^2 + \cdots \\
v_{\mathrm{b}} &= v_{\mathrm{o}} + \left(\frac{\partial v}{\partial x}\right)_{\mathrm{o}} x + \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2}\right)_{\mathrm{o}} x^2 + \cdots
\end{aligned}
\tag{9}
$$

The analogues of equations (1) to (3) become rather complicated in intrinsic coordinates. For instance, the continuity equation is [6, p. 68]

$$
\frac{R}{R-y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{v}{R-y} = 0, \tag{10}
$$

where the radius of curvature of the wall R depends on x. With the intrinsic coordinates we could try to proceed in the same way as with the Cartesian coordinates used in the present article. However, the derivations would become rather involved for teaching purposes. Thus, at this phase it might be enough to refer to the literature to inform the student that the use of the more involved equations finally give locally a formulation identical to that obtained by using Cartesian coordinates.

Real fluid

For a real fluid, the no-slip condition on a fixed wall requires that the fluid velocity disappears:

$$
\begin{aligned}
u_{\mathrm{b}} &= 0, \\
v_{\mathrm{b}} &= 0.
\end{aligned}
\tag{11}
$$

Comparison of expressions (8) and equations (11) thus gives the following ”generalized no-slip conditions”:

$$
\begin{aligned}
&u_{\mathrm{o}} = 0, \quad \left(\frac{\partial u}{\partial x}\right)_{\mathrm{o}} = 0, \quad \left(\frac{\partial^2 u}{\partial x^2}\right)_{\mathrm{o}} = -\frac{1}{R_{\mathrm{o}}}\left(\frac{\partial u}{\partial y}\right)_{\mathrm{o}}, \quad \cdots \\
&v_{\mathrm{o}} = 0, \quad \left(\frac{\partial v}{\partial x}\right)_{\mathrm{o}} = 0, \quad \left(\frac{\partial^2 v}{\partial x^2}\right)_{\mathrm{o}} = -\frac{1}{R_{\mathrm{o}}}\left(\frac{\partial v}{\partial y}\right)_{\mathrm{o}}, \quad \cdots
\end{aligned}
\tag{12}
$$

These are obtained by equating the coefficient of each power of x to zero in (8).

It is to be noted that no approximations are involved in equations (12) even if they are obtained from the first few terms of the series expansions. To convince ourselves about this we can proceed as follows. For instance, with $u_{\mathrm{b}} = 0$ in the former of expressions (8), we first let x tend to zero. This gives $u_{\mathrm{o}} = 0$. (The terms ignored may be represented by a remainder term which tends to zero as x tends to zero; so we do not need to worry about any convergence questions of infinite series.) Putting now $u_{\mathrm{o}} = 0$ in the equation and dividing it by x gives

$$0 = \left(\frac{\partial u}{\partial x}\right)_{\mathrm{o}} + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{R}\frac{\partial u}{\partial y}\right)_{\mathrm{o}} x + \cdots \tag{13}$$

We let $x \to 0$, which gives $\left(\partial u / \partial x\right)_{\mathrm{o}} = 0$, etc. This procedure is naturally based on the premise that the expansions exist to an order high enough. (For (12) to be valid, it is sufficient that u and v are three times continuously differentiable in the neighbourhood of point O.)

Next we make use of the continuity equation (1). The no-slip condition $\left(\partial u / \partial x\right)_{\mathrm{o}} = 0$ applied in (1) gives the additional constraint

$$\left(\frac{\partial v}{\partial y}\right)_{\mathrm{o}} = 0. \tag{14}$$

When this is put into the last of equations (12), we obtain the further result

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{\mathrm{o}} = 0. \tag{15}$$

By employing all the kinematical results derived above, the original expansions (6) can finally be written as

$$\begin{aligned} u &= \left(\frac{\partial u}{\partial y}\right)_{\mathrm{o}} y - \frac{1}{2R_{\mathrm{o}}}\left(\frac{\partial u}{\partial y}\right)_{\mathrm{o}} x^2 + \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{\mathrm{o}} xy + \frac{1}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)_{\mathrm{o}} y^2 + \cdots \\ v &= \left(\frac{\partial^2 v}{\partial x \partial y}\right)_{\mathrm{o}} xy + \frac{1}{2}\left(\frac{\partial^2 v}{\partial y^2}\right)_{\mathrm{o}} y^2 + \cdots \end{aligned} \tag{16}$$

It is also of some interest to record the stress components on the wall:

$$\begin{aligned} \left(\sigma_x\right)_{\mathrm{o}} &= -p_{\mathrm{o}}, \\ \left(\sigma_y\right)_{\mathrm{o}} &= -p_{\mathrm{o}}, \\ \left(\tau_{xy}\right)_{\mathrm{o}} \equiv \left(\tau_{yx}\right)_{\mathrm{o}} &= \mu\left(\frac{\partial u}{\partial y}\right)_{\mathrm{o}}. \end{aligned} \tag{17}$$

Thus the deviatoric stress components vanish on the wall and the shearing stress expression becomes simpler. These results follow from equations (12) and (14). The second formula (17) in fact indicates that the values measured from a flowing fluid by manometers through small openings on rigid surfaces really give values of pressure and not just values of the corresponding normal stress.

Boundary-layer momentum equations

Let us initially consider the situation according to Figure 1. When we consider flow along the line $x = 0$ near the wall it is obvious from (16) that the flow direction is in general roughly parallel to the wall and thus

$$|v| << |u| . \tag{18}$$

An exceptional case is with $\left(\partial u / \partial y\right)_{\text{o}} = 0$, which is associated with the separation of flow. Then the boundary-layer assumptions are no more valid. The main additional simplifying idea is that the rate of change in values for certain quantities is in general much smaller in the tangential than in the wall normal direction:

$$\left|\frac{\partial(\ )}{\partial x}\right| << \left|\frac{\partial(\ )}{\partial y}\right| . \tag{19}$$

However, these intuitively appealing arguments are not enough for making rational simplifications.

In the literature the most usual way to produce the simplifications is to apply the order-of magnitude analysis which is described, for instance, in [3], [5] and [6]. We present here an alternative way, which is based mostly on the series representations containing the kinematical constraints due to the no-slip boundary condition and the continuity equation.

We need the derivatives of the velocity components. From (16),

$$\begin{aligned}
\frac{\partial u}{\partial x} &= -\frac{1}{R_{\text{o}}}\left(\frac{\partial u}{\partial y}\right)_{\text{o}} x + \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{\text{o}} y + \cdots \\
\frac{\partial u}{\partial y} &= \left(\frac{\partial u}{\partial y}\right)_{\text{o}} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{\text{o}} x + \left(\frac{\partial^2 u}{\partial y^2}\right)_{\text{o}} y + \cdots \\
\frac{\partial v}{\partial x} &= \left(\frac{\partial^2 v}{\partial x \partial y}\right)_{\text{o}} y + \cdots \\
\frac{\partial v}{\partial y} &= \left(\frac{\partial^2 v}{\partial x \partial y}\right)_{\text{o}} x + \left(\frac{\partial^2 v}{\partial y^2}\right)_{\text{o}} y + \cdots
\end{aligned} \tag{20}$$

Using (20) and (16), the acceleration terms are thus

$$
\begin{aligned}
u\frac{\partial u}{\partial x} &= -\frac{1}{R_o}\left(\frac{\partial u}{\partial y}\right)_o^2 xy + \left(\frac{\partial u}{\partial y}\right)_o \left(\frac{\partial^2 u}{\partial x \partial y}\right)_o y^2 + \cdots \\
v\frac{\partial u}{\partial y} &= \left(\frac{\partial u}{\partial y}\right)_o \left(\frac{\partial^2 v}{\partial x \partial y}\right)_o xy + \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)_o \left(\frac{\partial^2 v}{\partial y^2}\right)_o y^2 + \cdots \\
u\frac{\partial v}{\partial x} &= \left(\frac{\partial u}{\partial y}\right)_o \left(\frac{\partial^2 v}{\partial x \partial y}\right)_o y^2 + \cdots \\
v\frac{\partial v}{\partial y} &= 0 + \cdots
\end{aligned} \tag{21}
$$

when terms up to the second order are retained. We compare the terms on the line $x = 0$ to obtain

$$u\frac{\partial u}{\partial x}\Big|_{x=0} = \left(\frac{\partial u}{\partial y}\right)_o \left(\frac{\partial^2 u}{\partial x \partial y}\right)_o y^2 + \cdots \tag{22}$$

$$v\frac{\partial u}{\partial y}\Big|_{x=0} = \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)_o \left(\frac{\partial^2 v}{\partial y^2}\right)_o y^2 + \cdots = -\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)_o \left(\frac{\partial^2 u}{\partial x \partial y}\right)_o y^2 + \cdots \tag{23}$$

$$
\begin{aligned}
u\frac{\partial v}{\partial x}\Big|_{x=0} &= \left(\frac{\partial u}{\partial y}\right)_o \left(\frac{\partial^2 v}{\partial x \partial y}\right)_o y^2 + \cdots = -\left(\frac{\partial u}{\partial y}\right)_o \left(\frac{\partial^2 u}{\partial x^2}\right)_o y^2 + \cdots \\
&= \frac{1}{R_o}\left(\frac{\partial u}{\partial y}\right)_o^2 y^2 + \cdots
\end{aligned} \tag{24}
$$

$$v\frac{\partial v}{\partial y}\Big|_{x=0} = 0 + \cdots \tag{25}$$

To get the final expressions (23) and (24) we have made use of equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} = 0 \tag{26}$$

and

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{27}$$

obtained by differentiation of the continuity equation. Additionally, to get (24), one of the formulae in (12) has been used.

It can be seen that the terms $u\,\partial u/\partial x$ and $v\,\partial u/\partial y$ are of the same order of magnitude. The term $u\,\partial v/\partial x$ is small (actually zero) compared to them when the boundary is straight ($1/R_o = 0$). This may also be considered to be valid when the curvature $1/R_o$ is reasonably small. Now we can first write the momentum equations in the form

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial \sigma'_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y},$$
$$0=-\frac{\partial p}{\partial y}+\frac{\partial \sigma'_y}{\partial y}+\frac{\partial \tau_{xy}}{\partial x} \tag{28}$$

in the neighbourhood of the wall. We then look at the terms containing the viscous stress components. From equations (3) and (20):

$$\frac{\partial \sigma'_x}{\partial x}=2\mu\frac{\partial^2 u}{\partial x^2}=-2\mu\frac{1}{R_o}\left(\frac{\partial u}{\partial y}\right)_o+\cdots$$
$$\frac{\partial \tau_{yx}}{\partial y}=\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)=\mu\left[\left(\frac{\partial^2 u}{\partial y^2}\right)_o+\left(\frac{\partial^2 v}{\partial x\partial y}\right)_o\right]+\cdots$$
$$\frac{\partial \sigma'_y}{\partial y}=2\mu\frac{\partial^2 v}{\partial y^2}=2\mu\left(\frac{\partial^2 v}{\partial y^2}\right)_o+\cdots=-2\mu\left(\frac{\partial^2 u}{\partial x\partial y}\right)_o+\cdots \tag{29}$$
$$\frac{\partial \tau_{xy}}{\partial x}=\mu\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right)=\mu\left(\frac{\partial^2 u}{\partial x\partial y}\right)_o+\cdots$$

Again, equation (27) has been used. If the boundary is straight, $\partial \sigma'_x/\partial x = 0$. We may assume this result to be valid also when the curvature is small. To get further we now make use of relation (19) and drop all terms containing the derivative with respect to *x*. This gives $\partial \sigma'_y/\partial y = 0$, $\partial \tau_{xy}/\partial x = 0$, and the conventional boundary-layer momentum equations

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{yx}}{\partial y},$$
$$\frac{\partial p}{\partial y}=0 \tag{30}$$

are obtained with

$$\tau_{yx}=\mu\frac{\partial u}{\partial y} \tag{31}$$

in laminar flow. Motivation for dropping the term $\partial v/\partial x$ from the full expression (3) is reinforced by representation (17) for τ_{yx} at the wall.

We have left the first equation (30) in such a form that it is valid in the turbulent case, too (cf. [3, p. 44]), then with the expression

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} - \rho \overline{u'v'} \tag{32}$$

for the shearing stress. In this case u and v refer to the mean and u' and v' to the fluctuating values. However, we do not try to derive expression (32) here as the present article is limited to laminar flow.

Ideal fluid

Next we develop some formulae using the frictionless fluid model with respect to the boundary condition. The resulting formulae will also prove useful in boundary-layer considerations.

Under the free-slip boundary condition for an inviscid fluid, the fluid velocity v is tangential to a fixed wall boundary or, equivalently, perpendicular to the normal of the boundary:

$$\mathbf{n} \cdot \mathbf{v}_\mathrm{b} = n_x u_\mathrm{b} + n_y v_\mathrm{b} = 0\,. \tag{33}$$

Here $\mathbf{n}$ is the unit normal vector (with components n_x and n_y) of the boundary, pointing towards the fluid (Figure 1). By analytic geometry we have

$$\mathbf{n} = \frac{-\left(\mathrm{d}y_\mathrm{b} / \mathrm{d}x\right)\mathbf{i} + \mathbf{j}}{\left[\left(\mathrm{d}y_\mathrm{b} / \mathrm{d}x\right)^2 + 1\right]^{1/2}}, \tag{34}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit base vectors. From expression (5) it follows that

$$\frac{\mathrm{d}y_\mathrm{b}}{\mathrm{d}x} = \frac{1}{R_\mathrm{o}} x + \cdots \tag{35}$$

and by using a binomial series expansion and the above result we have

$$\frac{1}{\left[\left(\mathrm{d}y_\mathrm{b} / \mathrm{d}x\right)^2 + 1\right]^{1/2}} = 1 - \frac{1}{2}\left(\frac{\mathrm{d}y_\mathrm{b}}{\mathrm{d}x}\right)^2 + \cdots = 1 - \frac{1}{2R_\mathrm{o}^2} x^2 + \cdots \tag{36}$$

Thus, when only terms up to the first order are retained, we obtain

$$\begin{aligned} n_x &= -\frac{1}{R_\mathrm{o}} x + \cdots \\ n_y &= 1 + \cdots \end{aligned} \tag{37}$$

Substitution of expressions (37) and (8) into equation (33) yields

$$\left(-\frac{1}{R_o}x+\cdots\right)\left[u_o+\left(\frac{\partial u}{\partial x}\right)_o x+\cdots\right]+\left(1+\cdots\right)\left[v_o+\left(\frac{\partial v}{\partial x}\right)_o x+\cdots\right]=0 \tag{38}$$

or

$$v_o+\left[-\frac{1}{R_o}u_o+\left(\frac{\partial v}{\partial x}\right)_o\right]x+\cdots=0\,. \tag{39}$$

Thus the ”generalized free-slip conditions” are

$$v_o=0\,,\quad \left(\frac{\partial v}{\partial x}\right)_o=\frac{1}{R_o}u_o\,,\cdots \tag{40}$$

and the original expansions (6) reduce to (when only terms up to the first order are retained)

$$\begin{aligned} u&=u_o+\left(\frac{\partial u}{\partial x}\right)_o x+\left(\frac{\partial u}{\partial y}\right)_o y+\cdots \\ v&=\frac{1}{R_o}u_o x+\left(\frac{\partial v}{\partial y}\right)_o y+\cdots \end{aligned} \tag{41}$$

If the flow is further assumed to be irrotational — as is usual in ideal fluid (potential) flow analysis applied to determine the impressed pressure on the boundary layer — we have the additional kinematical constraint

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0 \tag{42}$$

and the second condition (40) can be put into the form

$$\left(\frac{\partial u}{\partial y}\right)_o=\frac{1}{R_o}u_o\,. \tag{43}$$

The content of this formula is best understood by looking at Figure 2. For a given u_o the magnitude and sign of the normal derivative $(\partial u/\partial y)$ at the wall are determined by the curvature $1/R_o$. When the wall is concave (convex) as looked from the fluid side, $1/R_o$ is positive (negative), and the magnitude of the tangential velocity component near the wall is decreasing (increasing) when the wall is approached.

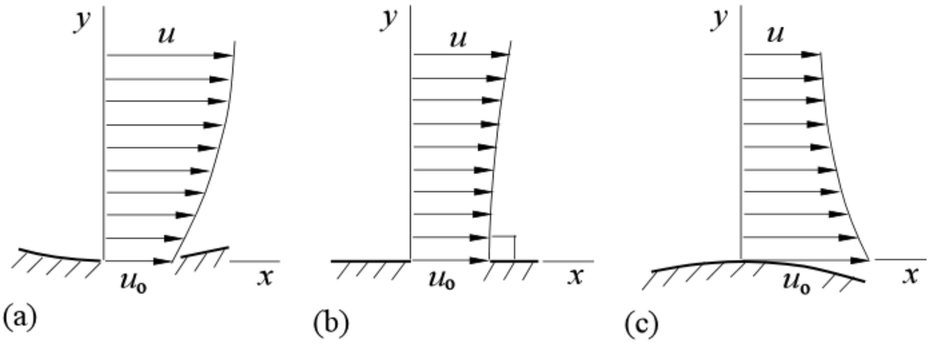

Figure 2. Tangential velocity distribution on the wall normal with (a) concave, (b) straight, (c) convex wall.

Formula (43) explains for instance why high fluid velocities and danger for cavitation can be expected in convex areas with high value of curvature; this point is discussed in reference [2, p. 386], where the equivalent of the formula is derived by a different procedure.

With proper interpretation the free-slip conditions can even be employed in connection with real fluid flow. In reality we should not apply our formulae at the rigid surface, where the velocity is actually zero, but at some distance from it. Especially, formulae (39) to (41) are exactly valid anywhere in the flow if we just consider the boundary replaced by a streamline with a point O on it now used as the expansion centre. (It should be realized that above we have only used kinematical equations and no constitutive relations have been invoked.) The subscript o in formulae (39) to (41) can now be removed to avoid confusion with the case where point O is on the wall.

As an application we consider the assessment of the term $u\,\partial v/\partial x$ which is usually neglected in the boundary layer equations. Application of the second formula in (40) results in the relation

$$u\frac{\partial v}{\partial x}=\frac{1}{R}u^2\,. \tag{44}$$

The expression $1/R$ is the unknown curvature of the streamline. We put $1/R \approx 1/R_\mathrm{b}$ where b refers now to the boundary value and use $v=0$ from the first equation (40). The second momentum equation (2) reduces then in the frictionless case to

$$\frac{\rho}{R_\mathrm{b}}u^2=-\frac{\partial p}{\partial y}\,. \tag{45}$$

This may be considered as a more accurate boundary-layer momentum equation compared to the second of (30) and is called the ”centrifugal force” equation in [3, p. 41].

Final remarks

The derivation of the boundary-layer momentum equations presented in this article was based on the fixed x, y-coordinate system according to Figure 1. If the wall is actually straight, the equations are also valid elsewhere than in the neighborhood of the selected origin of the coordinate system.

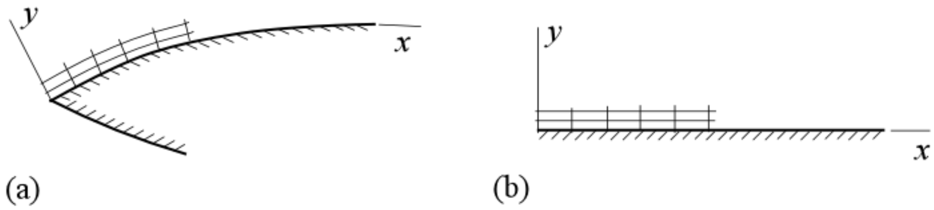

Figure 3. (a) An orthogonal curvilinear and (b) a rectangular coordinate system.

As discussed at the end of Section “Basic expansions”, with a curved wall an intrinsic coordinate system of the type shown in Figure 3 (a) is usually applied in connection with the boundary-layer theory. Some coordinate lines are sketched in the figure. It can be shown, e.g. [5, p. 314], that if the thickness of the boundary layer is essentially smaller than the radius of the curvature of the wall, the equations valid for a straight boundary can also be applied with sufficient accuracy in the curvilinear system. From the mathematical point of view, the treatment can then finally be considered to take place in the rectangular coordinate system of Figure 3 (b). However, when calculating the potential flow needed for impressing the pressure on the boundary layer flow, the original geometry of Figure 3 (a) must be employed.

Reference [4] is concerned with the use of Taylor expansions in the neighborhood of a rigid wall to study the conditions of separation in boundary layers. The present article is similar in approach. In reference [4] the derivations are performed for a plane surface and the free-slip condition is not included.

Finally, it is not claimed that the steps employed in the present article to obtain the boundary-layer momentum equations are rigorous. The exact order of magnitude analysis is, however, rather involved and demanding to the student, and if one cannot spend much time on teaching the subject, the above way of presentation is suggested as a possible alternative.

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