THE CONCEPT OF A SUBSTITUTE ELEMENT

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ABSTRACT

The concept of a substitute element is introduced. The substitute element replaces temporarily an original element in two dimensions with an ellipse and in three dimensions with an ellipsoid. The purpose is to obtain an oriented length measure for an element varying smoothly with direction. This facilitates the extension of sensitizing parameters found in one dimension to two and three dimensions. Two methods to obtain a substitute element are described. They are called the inertia tensor method and the second moment tensor method. In two dimensions the methods are equivalent but in three dimensions they differ. Some example results are given.

INTRODUCTION

In our efforts to extend the sensitizing (stabilizing) parameters employed in one-dimensional cases to two and three dimensions we have found it useful to define oriented linear length measures for finite elements in two and three dimensions. For this purpose a finite element is replaced temporarily by a shape with a smoothly varying boundary; we call this a substitute element. This substitute element represents the shape, size and orientation of the original element in some average manner. Now the oriented length (or diameter) of the substitute element depends smoothly on the direction in which the length is measured.

We are going to describe two methods — called the inertia tensor method and the second moment tensor method, respectively — to generate a substitute element. In both methods, the final shape of the substitute element is given by an ellipsoid (in two dimensions by an ellipse) corresponding to a second order symmetric tensor. In the first method the tensor is a kind of inertia tensor and in the second method the tensor might be called a second moment tensor.

The ellipsoids are generated in different ways in the two methods. We therefore first recall from the literature how the properties of a symmetric second order tensor are generally illustrated geometrically by using an ellipse or ellipsoid. We refer here to [1] and to the familiar application to the stress tensor $\boldsymbol{\tau}$ (τ_{ij}), which is a symmetric second order tensor. The two alternatives are the Cauchy's stress tensor quadric representation and the Lamé stress ellipsoid representation. In the Cauchy's stress quadric representation, a quadratic surface (when the eigenvalues, which are the principal stresses σ_1, σ_2, σ_3, are all positive, an ellipsoid) is defined by an equation

$$\mathbf{x}\cdot\boldsymbol{\tau}\cdot\mathbf{x} \equiv \tau_{ij}x_i x_j = k^2, \tag{1}$$

where **x** is the position vector to the surface and *k* is a constant which has the correct physical dimension and which controls the size of the surface. It can be shown that the semiaxes of the ellipsoid are

$$\mathrm{OP}_1 = \frac{k}{\sqrt{\sigma_1}}, \quad \mathrm{OP}_2 = \frac{k}{\sqrt{\sigma_2}}, \quad \mathrm{OP}_3 = \frac{k}{\sqrt{\sigma_3}}. \tag{2}$$

These lengths are measured in the corresponding directions (principal directions) of the eigenvectors of component representation $[\tau]$ of $\boldsymbol{\tau} = \{\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}\}[\tau]\{\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}\}^{\mathbf{T}}$ in the usual $\mathbf{i}, \mathbf{j}, \mathbf{k}$ basis of a Cartesian coordinate system. Thus perhaps not so obviously, the higher the eigenvalue (principal stress), the shorter the corresponding semiaxis.

In the Lamé stress ellipsoid representation, the semiaxes are taken directly to be the eigenvalues (principal stresses). If these are measured along the principal directions we obtain an ellipsoid with the same axis directions as in the Cauchy representation but the resulting shapes of the two ellipsoids are in general completely different.

INERTIA PROPERTIES OF A BODY

Reference [2] contains a lucid description of inertia properties of bodies and we refer here to it. The inertia tensor is defined as

$$\mathbf{I} = \iiint \rho\left(\mathbf{r}\cdot\mathbf{r}\,\mathbf{1} - \mathbf{r}\mathbf{r}\right)\mathrm{d}V, \tag{3}$$

where the integral is over the body volume in question and where ρ is the density, **r** the radius vector from the point (reference point) with respect to which the tensor is evaluated, $\mathbf{1}$ is the unit tensor, $\mathbf{r}\cdot\mathbf{r}$ is a scalar product and $\mathbf{rr}$ a tensor product. Our original starting point was the intuitive observation that the corresponding inertia ellipse representation of the inertia tensor more or less seemed to resemble the shape and orientation of the plane body in question when the inertia quantities were evaluated about the mass center and when the density was constant.

When dealing with the shape of a body the density distribution is of no relevance and we may put formally $\rho = 1$ in (3). It transforms to

$$\mathbf{I} = \iiint \left(\overline{\mathbf{r}}\cdot\overline{\mathbf{r}}\,\mathbf{1} - \overline{\mathbf{r}}\,\overline{\mathbf{r}}\right)\mathrm{d}V. \tag{4}$$

In Cartesian coordinates with the origin at the reference point, the diagonal components of (4) are the moments of inertia about the three axes:

$$I_{xx} = \iiint \left(\overline{y}^2 + \overline{z}^2\right)\mathrm{d}V,$$

$$I_{yy} = \iiint \left(\bar{z}^2 + \bar{x}^2\right) \mathrm{d}V , \tag{5}$$

$$I_{zz} = \iiint \left(\bar{x}^2 + \bar{y}^2\right) \mathrm{d}V .$$

The off-diagonal components are called often the products of inertia:

$$I_{xy} = I_{yx} = -\iiint \bar{x}\,\bar{y} \mathrm{d}V ,$$

$$I_{yz} = I_{zy} = -\iiint \bar{y}\bar{z} \mathrm{d}V , \tag{6}$$

$$I_{zx} = I_{xz} = -\iiint \bar{z}\,\bar{x} \mathrm{d}V .$$

The overbar notation is used here to refer to the fact that the coordinates are measured from the volume centroid. In the plane case, the definitions corresponding to (4) to (6) are obvious, the integrals are over the plane domain in question and the volume centroid is replaced by the area centroid. From now on, we will consider finite elements and instead of a body we speak about an element.

Before evaluating (5) and (6), the position of the volume centroid $\mathbf{r}_\mathrm{C}$ of the element must be determined. It is obtained using the first moment of the element from (V is the volume of the element)

$$\mathbf{r}_\mathrm{C} = \frac{1}{V} \iiint \mathbf{r} \,\mathrm{d}V \tag{7}$$

having the Cartesian components

$$x_\mathrm{C} = \frac{1}{V} \iiint x \mathrm{d}V , \qquad y_\mathrm{C} = \frac{1}{V} \iiint y \mathrm{d}V , \qquad z_\mathrm{C} = \frac{1}{V} \iiint z \mathrm{d}V . \tag{8}$$

Thus,

$$\bar{\mathbf{r}} = \mathbf{r} - \mathbf{r}_\mathrm{C} \tag{9}$$

or in the Cartesian component form

$$\bar{x} = x - x_\mathrm{C} , \qquad \bar{y} = y - y_\mathrm{C} , \qquad \bar{z} = z - z_\mathrm{C} . \tag{10}$$

In the plane case the counterpart of (7) is (A is the area of the element)

$$\mathbf{r}_\mathrm{C} = \frac{1}{A} \iint \mathbf{r} \,\mathrm{d}A$$

and if we operate in the xy-plane, the two first of formulas (10) apply.

INERTIA TENSOR METHOD

Following now the Cauchy type representation we define the surface of the substitute element as the ellipsoid

$$\bar{\mathbf{x}} \cdot \mathbf{I} \cdot \bar{\mathbf{x}} = (k_{\mathrm{I}})^2, \tag{11}$$

where k_{I} is a constant which has the correct physical dimension and which controls the size of the ellipsoid. We set the size here so that the volume of the ellipsoid (in two dimensions the area of the ellipse) becomes equal to the volume V (in two dimensions the area A) of the element. The volume of an ellipsoid is

$$\frac{4}{3}\pi abc, \tag{12}$$

where a, b, and c are the semiaxes. In two dimensions, the area of an ellipse is correspondingly

$$\pi ab. \tag{13}$$

Now referring to formulas (1) and (2), we find that the volume is given here by

$$\frac{4}{3}\pi \frac{k_{\mathrm{I}}}{\sqrt{I_3}} \frac{k_{\mathrm{I}}}{\sqrt{I_2}} \frac{k_{\mathrm{I}}}{\sqrt{I_3}} = \frac{4}{3}\pi \frac{(k_{\mathrm{I}})^3}{\sqrt{I_1 I_2 I_3}}, \tag{14}$$

where I_1, I_2, I_3 are the eigenvalues of $[\mathrm{I}]$ called also the principal moments of inertia and $[\mathrm{I}]$ is the component form of $\mathbf{I} = \{\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}\}[\mathrm{I}]\{\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}\}^{\mathbf{T}}$ represented in the usual $\mathbf{i}, \mathbf{j}, \mathbf{k}$ basis of a Cartesian coordinate system. In fact, according to literature the product under the square root is an invariant and in detail

$$I_1 I_2 I_3 = \det[\mathrm{I}] \tag{15}$$

so that for the volume evaluation we do not actually need to determine the eigenvalues. Equating expression (14) with the volume V of the element gives

$$k_{\mathrm{I}} = \frac{V^{1/3}\left(\det[\mathrm{I}]\right)^{1/6}}{\left(\frac{4}{3}\pi\right)^{1/3}}. \tag{16}$$

In two dimensions we obtain correspondingly

$$k_{\mathrm{I}} = \frac{A^{1/2}\left(\det[\mathrm{I}]\right)^{1/4}}{\pi^{1/2}}. \tag{17}$$

Finally, the end purpose is to find the oriented length h_{I} of the ellipsoid in the direction given by a unit vector $\mathbf{n}$. Let the corresponding position vector on the surface be $\overline{\mathbf{x}} = c_{\mathrm{I}}\mathbf{n}$, where c_{I} is the length of the position vector. Substitution of this in (11) gives

$$\left(c_{\mathrm{I}}\right)^2 \mathbf{n}\cdot\mathbf{I}\cdot\mathbf{n} = \left(k_{\mathrm{I}}\right)^2 , \tag{18}$$

from which

$$h_{\mathrm{I}}\left(\mathbf{n}\right) = 2\,c_{\mathrm{I}} = 2\frac{k_{\mathrm{I}}}{\sqrt{\mathbf{n}\cdot\mathbf{I}\cdot\mathbf{n}}} . \tag{19}$$

SECOND MOMENT TENSOR METHOD

The inertia tensor method described above was arrived at by making use of familiar concepts from mechanics texts. Some further considerations lead to an alternative and geometrically more direct approach. Let us define a symmetric second order tensor

$$\mathbf{S} = \iiint \overline{\mathbf{r}\mathbf{r}}\,\mathrm{d}V \tag{20}$$

having the Cartesian components

$$\begin{aligned} S_{xx} &= \iiint \overline{x}^2\mathrm{d}V , \\ S_{yy} &= \iiint \overline{y}^2\mathrm{d}V , \\ S_{zz} &= \iiint \overline{z}^2\mathrm{d}V , \end{aligned} \tag{21}$$

and

$$\begin{aligned} S_{xy} &= S_{yx} = \iiint \overline{x}\,\overline{y}\,\mathrm{d}V , \\ S_{yz} &= S_{zy} = \iiint \overline{y}\,\overline{z}\,\mathrm{d}V , \\ S_{zx} &= S_{xz} = \iiint \overline{z}\,\overline{x}\,\mathrm{d}V . \end{aligned} \tag{22}$$

S might be called second moment tensor. In the inertia tensor method, the ellipsoid was generated by a Cauchy type representation. Here we will employ a Lamé type representation.

It is recalled that the equation of an ellipsoid with the coordinate axes along its axes is

$$\frac{\overline{x}^2}{a^2} + \frac{\overline{y}^2}{b^2} + \frac{\overline{z}^2}{c^2} = 1 . \tag{23}$$

If the ellipsoid is in an oblique orientation with respect to the coordinate axes, its equation is seen from (23) to become

$$\frac{(\bar{\mathbf{x}}\cdot\mathbf{e}_1)^2}{a^2}+\frac{(\bar{\mathbf{x}}\cdot\mathbf{e}_2)^2}{b^2}+\frac{(\bar{\mathbf{x}}\cdot\mathbf{e}_3)^2}{c^2}=1, \tag{24}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the unit vectors in the direction of the axes of the ellipsoid.

Let the eigenvectors and eigenvalues of $[\mathrm{S}]$ be $\mathbf{v}_1^{\mathrm{S}}$, $\mathbf{v}_2^{\mathrm{S}}$, $\mathbf{v}_3^{\mathrm{S}}$ and S_1, S_2, S_3, respectively (again $[\mathrm{S}]$ is the component representation of $\mathbf{S}$ in Cartesian basis). We form the ellipsoid

$$\frac{(\bar{\mathbf{x}}\cdot\mathbf{e}_1)^2}{S_1}+\frac{(\bar{\mathbf{x}}\cdot\mathbf{e}_2)^2}{S_2}+\frac{(\bar{\mathbf{x}}\cdot\mathbf{e}_3)^2}{S_3}=(k_{\mathrm{S}})^2. \tag{25}$$

The unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are now in the directions of $\mathbf{v}_1^{\mathrm{S}}$, $\mathbf{v}_2^{\mathrm{S}}$, $\mathbf{v}_3^{\mathrm{S}}$ and k_{S} is a constant which has the correct physical dimension and which controls the size of the ellipsoid. It is realized that we have not followed strictly the Lamé representation recipe as the semiaxes are here taken to be $k_{\mathrm{S}}\sqrt{S_1}$, $k_{\mathrm{S}}\sqrt{S_2}$, $k_{\mathrm{S}}\sqrt{S_3}$, that is, square roots of the eigenvalues are employed. The reason is that $\mathbf{S}$ represents the measure of length squared rather than the length itself which is obvious from definition (20).

Equation (25) can be transformed into a more illuminating form. For instance, manipulation of the first term on the left-hand side gives

$$\frac{(\bar{\mathbf{x}}\cdot\mathbf{e}_1)^2}{S_1}=\frac{(\bar{\mathbf{x}}\cdot\mathbf{e}_1)(\bar{\mathbf{x}}\cdot\mathbf{e}_1)}{S_1}=\frac{(\bar{\mathbf{x}}\cdot\mathbf{e}_1)(\mathbf{e}_1\cdot\bar{\mathbf{x}})}{S_1}=\bar{\mathbf{x}}\cdot\frac{\mathbf{e}_1\mathbf{e}_1}{S_1}\cdot\bar{\mathbf{x}}. \tag{26}$$

Proceeding similarly with the other two terms we find the left-hand side of (25) to become

$$\bar{\mathbf{x}}\cdot\left(\frac{\mathbf{e}_1\mathbf{e}_1}{S_1}+\frac{\mathbf{e}_2\mathbf{e}_2}{S_2}+\frac{\mathbf{e}_3\mathbf{e}_3}{S_3}\right)\cdot\bar{\mathbf{x}}. \tag{27}$$

The expression inside the parentheses is seen to be the inverse of $\mathbf{S}$ represented in the basis of the unit eigenvectors. Thus, equation (25) becomes simply

$$\bar{\mathbf{x}}\cdot\mathbf{S}^{-1}\cdot\bar{\mathbf{x}}=(k_{\mathrm{S}})^2. \tag{28}$$

The volume of the ellipsoid is obtained again according to (12) as

$$\frac{4}{3}\pi\cdot k_{\mathrm{S}}\sqrt{S_1}\cdot k_{\mathrm{S}}\sqrt{S_2}\cdot\sqrt{S_3}=\frac{4}{3}\pi(k_{\mathrm{S}})^3\sqrt{S_1S_2S_3}=\frac{4}{3}\pi(k_{\mathrm{S}})^3\sqrt{\det[\mathrm{S}]}. \tag{29}$$

Putting this equal to the volume V of the element, we find that

$$k_{\mathrm{S}} = \frac{V^{1/3}}{\left(\frac{4}{3}\pi\right)^{1/3}\left(\det[\mathrm{S}]\right)^{1/6}}. \tag{30}$$

In two dimensions, the corresponding result is

$$k_{\mathrm{S}} = \frac{A^{1/2}}{\pi^{1/2}\left(\det[\mathrm{S}]\right)^{1/4}}. \tag{31}$$

Finally, comparing (11) and (28) with the result (19) we obtain the oriented length

$$h_{\mathrm{S}}(\mathbf{n}) = 2c_{\mathrm{S}} = 2\frac{k_{\mathrm{S}}}{\sqrt{\mathbf{n}\cdot\mathbf{S}^{-1}\cdot\mathbf{n}}}. \tag{32}$$

RELATIONS BETWEEN THE METHODS

It is known that the quadratic form $\mathbf{n}\cdot\mathbf{I}\cdot\mathbf{n}$ gives the moment of inertia of a system with respect to an axis along $\mathbf{n}$ ([2]). Thus component representation $[\mathrm{I}]$ of $\mathbf{I}$ in a Cartesian system is a positive definite matrix as the moment of inertia about an axis is positive. The only situation when a positive semidefinite case can occur is in the case where the element volume is distributed on a line through the centroid. With a reasonable element this is not the case.

Let us consider similarly the quadratic form

$$\begin{aligned}\mathbf{n}\cdot\mathbf{S}\cdot\mathbf{n} &= \mathbf{n}\cdot\iiint \mathbf{r}\mathbf{r}\,\mathrm{d}V\cdot\mathbf{n} = \iiint(\mathbf{n}\cdot\mathbf{r})(\mathbf{r}\cdot\mathbf{n})\,\mathrm{d}V = \iiint(\mathbf{n}\cdot\mathbf{r})(\mathbf{n}\cdot\mathbf{r})\,\mathrm{d}V\\ &= \iiint(\mathbf{n}\cdot\mathbf{r})^2\,\mathrm{d}V.\end{aligned} \tag{33}$$

The manipulations show that the result is the second moment of the system with respect to a plane through the volume centroid and perpendicular to **n**. This is always non-negative, so the second moment tensor is also positive definite. The only situation when a positive semi-definite case can occur is in the case where the element volume is distributed on a plane through the centroid. Again, for a reasonable element this cannot occur.

Because the geometric measures **I** and **S** considered are symmetric, the eigenvectors of $[\mathrm{I}]$ and $[\mathrm{S}]$ are orthogonal (or can be chosen so) and because the tensors are positive semi-definite, the eigenvalues are non-negative as has been tacitly assumed above.

Geometric measures **I** and **S** have some simple relationships. Realizing that the term

$$\bar{\mathbf{r}}\cdot\bar{\mathbf{r}} = \bar{x}^2 + \bar{y}^2 + \bar{z}^2, \tag{34}$$

we obtain from (4) the connection

$$\mathbf{I} = \iiint \overline{\mathbf{r}} \cdot \overline{\mathbf{r}} \, \mathrm{d}V \, \mathbf{1} - \iiint \overline{\mathbf{r}} \, \overline{\mathbf{r}} \, \mathrm{d}V = \iiint \left(\overline{x}^2 + \overline{y}^2 + \overline{z}^2 \right) \mathrm{d}V \, \mathbf{1} - \mathbf{S}$$

$$= \left(S_{xx} + S_{yy} + S_{zz} \right) \mathbf{1} - \mathbf{S} = \left(S_1 + S_2 + S_3 \right) \mathbf{1} - \mathbf{S} = \mathrm{tr}\, \mathbf{S} - \mathbf{S} \,. \tag{35}$$

Let the eigenvectors of $[\mathrm{I}]$ be $\mathbf{v}_1^{\mathrm{I}}$, $\mathbf{v}_2^{\mathrm{I}}$, $\mathbf{v}_3^{\mathrm{I}}$. Multiply both sides of (35) from the right by $\mathbf{v}_j^{\mathrm{I}}$:

$$\mathbf{I} \cdot \mathbf{v}_j^{\mathrm{I}} = \left(S_1 + S_2 + S_3 \right) \mathbf{1} \cdot \mathbf{v}_j^{\mathrm{I}} - \mathbf{S} \cdot \mathbf{v}_j^{\mathrm{I}} \tag{36}$$

or

$$I_j \mathbf{v}_j^{\mathrm{I}} = \left(S_1 + S_2 + S_3 \right) \mathbf{v}_j^{\mathrm{I}} - \mathbf{S} \cdot \mathbf{v}_j^{\mathrm{I}} \tag{37}$$

or further, by arranging terms,

$$\mathbf{S} \cdot \mathbf{v}_j^{\mathrm{I}} = \left(S_1 + S_2 + S_3 - I_j \right) \mathbf{v}_j^{\mathrm{I}} \,. \tag{38}$$

This shows that component representations $[\mathrm{I}]$ and $[\mathrm{S}]$ have identical eigenvectors and the corresponding eigenvalues are related by

$$\begin{aligned} S_1 &= S_1 + S_2 + S_3 - I_1 \,, \\ S_2 &= S_1 + S_2 + S_3 - I_2 \,, \\ S_3 &= S_1 + S_2 + S_3 - I_3 \,, \end{aligned} \tag{39}$$

or

$$\begin{aligned} I_1 &= S_2 + S_3 \,, \\ I_2 &= S_3 + S_1 \,, \\ I_3 &= S_1 + S_2 \,. \end{aligned} \tag{40}$$

TWO DIMENSIONS

In two dimensions (put $S_3 = 0$ in the two first equations (40))

$$\begin{aligned} I_1 &= S_2 \,, \\ I_2 &= S_1 \,. \end{aligned} \tag{41}$$

Thus the eigenvalues so to say switch places and for example

$$\det[\mathrm{I}] = \det[\mathrm{S}] \equiv D. \tag{42}$$

Let us consider **I** and **S** here in some more detail. Using the *xy*-plane, the component representation $[\mathrm{I}]$ of $\mathbf{I} = \{\mathbf{i} \quad \mathbf{j}\}[\mathrm{I}]\{\mathbf{i} \quad \mathbf{j}\}^{\mathbf{T}}$ have the elements

$$I_{xx} = \iint \bar{y}^2 \mathrm{d}A, \qquad I_{yy} = \iint \bar{x}^2 \mathrm{d}A, \qquad I_{xy} = I_{yx} = -\iint \bar{x}\,\bar{y}\,\mathrm{d}A. \tag{43}$$

The component representation $[\mathrm{S}]$ of $\mathbf{S} = \{\mathbf{i} \quad \mathbf{j}\}[\mathrm{S}]\{\mathbf{i} \quad \mathbf{j}\}^{\mathbf{T}}$ becomes thus

$$[\mathrm{S}] = \begin{bmatrix} I_{yy} & -I_{xy} \\ -I_{xy} & I_{xx} \end{bmatrix}. \tag{44}$$

Its inverse is found to be

$$[\mathrm{S}]^{-1} = \frac{1}{D}\begin{bmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{bmatrix} = \frac{1}{D}[\mathrm{I}] \tag{45}$$

or in the coordinate system invariant notation, we have

$$\mathbf{S}^{-1} = \frac{1}{D}\mathbf{I}. \tag{46}$$

Comparing formulas (17) and (31) it is found that

$$k_{\mathrm{I}} = \sqrt{D}\, k_{\mathrm{S}}. \tag{47}$$

Using relations (46) and (47) in expressions (19) and (32) it is seen that they produce identical oriented lengths. Thus, perhaps surprisingly, the two methods are found to be equivalent in two dimensions but not in general in three dimensions.

SOME EXAMPLES

In the examples, the integrals needed to find the geometric measures — centroids, the components of **I** or **S** and the volumes or areas — of the elements were evaluated by quadratures for triangles and tetrahedrons giving exact integrals up to and including second order polynomials. For this purpose, four-node quadrilateral and eight-node hexahedron elements were divided into two triangles and six tetrahedrons, respectively. Use of a quadrature does not introduce any modification over the scheme discussed, as we will apply the sensitized (stabilized) finite element method only in connection with the simplest elements, that is, with three-node triangles and four-node quadrilaterals and in three dimensions the four-node tetrahedrons and eight-node hexahedrons. A quadrature is needed anyway in any practical implementation and, therefore a quadrature was used also in the examples.

To give an idea of the geometrical relationship between the actual and the substitute element, we show the results of some calculations. The first example in Figure 1 shows the original and substitute element in the two-dimensional case, for two distorted element shapes. In the two dimensions, the two methods discussed produce identical substitute elements. The three dimensional examples of Figure 2 and Figure 3 show that the substitute element generated by the second moment method is closer to the original shape in orientation and size than the one by the inertia method.

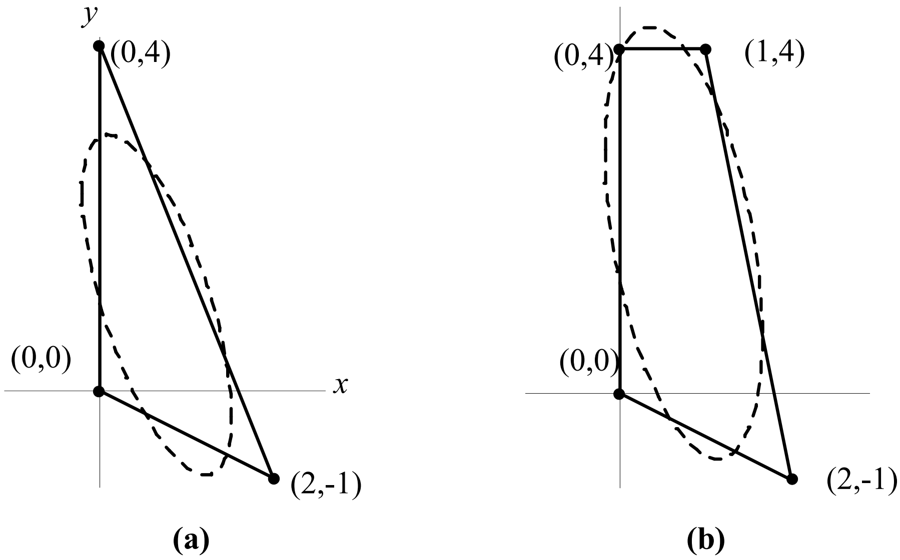

Figure 1 Substitute element by the second moment tensor method **(a)** for a triangular element and **(b)** for a quadrilateral element.

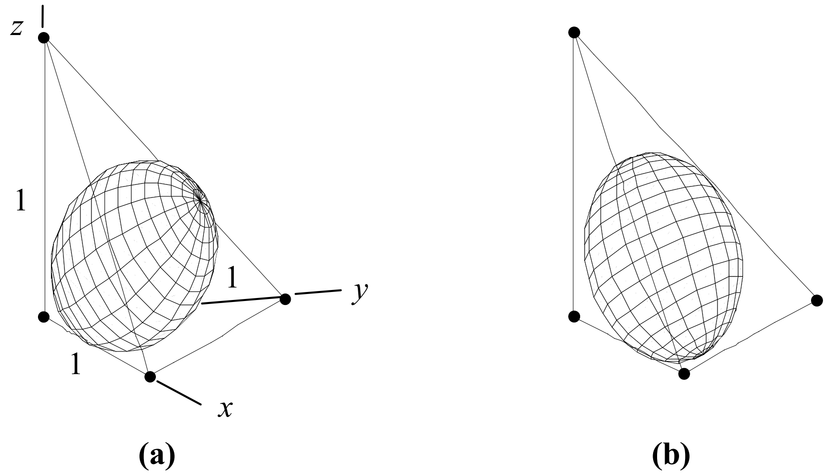

Figure 2 Substitute element for a tetrahedron element **(a)** by the inertia tensor method and **(b)** by the second moment tensor method.

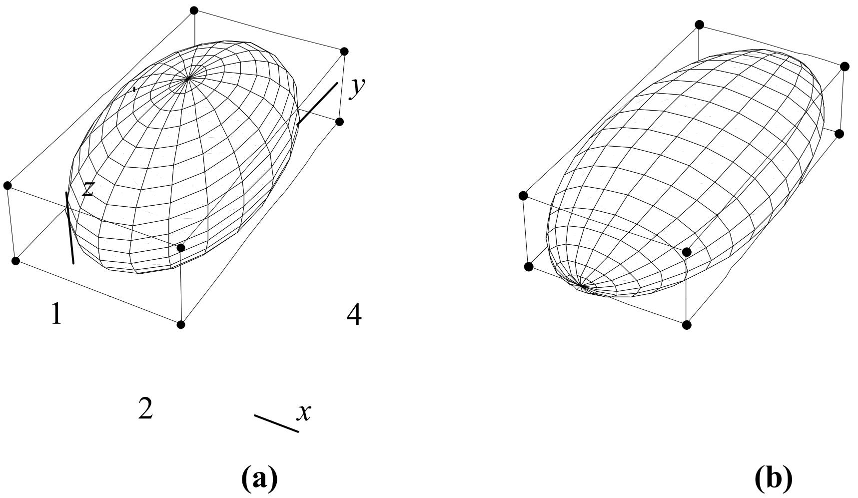

Figure 3 Substitute element for a eight-node hexahedron element **(a)** by the inertia tensor method and **(b)** by the second moment tensor method.

CONCLUDING REMARKS

It is rather obvious that the inertia moment method starts to produce somewhat odd results in three dimensions if the element considered is "thin" in some direction. One criterion for comparing the two methods could perhaps be the following: for an initially ellipsoid shaped element the methods should obviously reproduce the original element as the substitute element. Of course, in reality, no practical element is an ellipsoid but we can still use this thought experiment here. Employing the inertia and the second moment tensor methods in this case it is found that the second moment tensor method indeed reproduces the ellipsoid but the inertia tensor method does not. As an example, when applying the inertia tensor method for an ellipsoid with the shape $b/a = 0.5$, $c/a = 0.25$, the semiaxes *a*, *b*, *c* are reproduced multiplied roughly by the factors 0.77, 0.84, 1.54, respectively. For the shape $b/a = 0.8$, $c/a = 0.6$, somewhat milder factors 0.89, 0.96, 1.16, respectively, are obtained. In any case, it seems that in three dimensions, at least when rather elongated elements are used, the second moment tensor method is to be preferred.

Considering the second moment method from a general point of view we note that we have in fact operated with the zeroth moment (scalar)

$$\iiint \mathrm{d}V = V \tag{48}$$

the first moment (vector)

$$\iiint \mathbf{r}\,\mathrm{d}V = \mathbf{r}_{\mathrm{C}} V \tag{49}$$

and the second moment (second order tensor)

$$\iiint \mathbf{r}\mathbf{r}\,\mathrm{d}V = \mathbf{r}_{\mathrm{C}}\mathbf{r}_{\mathrm{C}} V + \iiint \overline{\mathbf{r}}\,\overline{\mathbf{r}}\,\mathrm{d}V \tag{50}$$

and demanded these quantities, respectively, to be the same for the original element and for the substitute element. The right-hand sides of (49) and (50) follow from the property that the first moment with respect to the centroid disappears. One could naturally speculate on more complicated substitute elements so that even higher moments could be imitated when more adjustable parameters would be available to alter the shape. However, using an ellipse or ellipsoid means already an improvement over the one-parameter circle or sphere approximation of the mathematical theory of the finite element method.

A certain further equivalence between an original element and the corresponding substitute element appears emphasized when the shape functions of the element are linear in the coordinates, that is, for the three-node triangle and for the four-node tetrahedron. Let us consider for shortness of presentation the three-node triangle. When the Galerkin finite element method is applied in a linear problem (with assumed constant data in an element), the weak form integrand is seen to be of the form

$$\alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2 , \tag{51}$$

where the coefficients α are constants. For instance, the constant term may emerge from diffusion type contributions, the linear terms from convection type contributions and the quadratic terms from reaction type contributions. Integrating (51) over the element domain produces the result

$$\begin{aligned} &\alpha_0 \iint \mathrm{d}A + \alpha_1 \iint x\mathrm{d}A + \alpha_2 \iint y\mathrm{d}A + \alpha_3 \iint x^2\mathrm{d}A + \alpha_4 \iint xy\mathrm{d}A + \alpha_5 \iint y^2\mathrm{d}A \\ &= \alpha_0 A + \alpha_1 x_\mathrm{C} A + \alpha_2 y_\mathrm{C} A + \alpha_3 x_\mathrm{C}^2 A + \alpha_3 S_{xx} + \alpha_4 x_\mathrm{C} y_\mathrm{C} + \alpha_4 S_{xy} + \alpha_5 y_\mathrm{C}^2 + \alpha_5 S_{yy} . \end{aligned} \tag{52}$$

The steps used should be rather obvious. Now, the substitute element has by design the same area, the same centroid coordinates (midpoint of the ellipse) and the same second moment components about the centroid as the original element. Thus integration of (51) over the substitute element domain produces again the result (52). Of course, we do not perform this kind of integrations over the substitute element as the substitute element is employed for other purposes but in any case, the result increases the credibility of the substitute element concept.

As a maybe the simplest example of the application of the substitute element concept we could mention its use in the solution of the diffusion-convection equation by the sensitized (stabilized) finite element method in two or three dimensions. By taking the oriented length in the flow direction as the characteristic length to be used in the corresponding one-dimensional sensitizing parameter expression, rather satisfactory solutions are obtained. We intend to report results from more complicated applications in the future.

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