AN AUTOMATIC STRAIN-BASED INCREMENTAL-ITERATIVE TECHNIQUE FOR ELASTO-PLASTIC BEAM-COLUMNS

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SUMMARY

Existing incremental-iterative techniques usually make an effort to be applicable for all kind of FE (finite element) model, thus do not take into account the advantageous specialties of the actually used FE models. The proposed method is developed especially for elasto-plastic thin-walled beam-columns (a general model for steel structures). The procedure integrates the three – generally separately handled – main parts: incrementation, iteration and convergence checking using some measure of the strain field to control the whole solution. The result is a totally automatic and efficient solution technique, which does not require any additional change in different kind of problems.

INTRODUCTION

In nonlinear structural analysis it is always an important question how to advance the solution to reach an accurate, reliable and effective method. The widely used Newton-type solution methods for solving problems described by a set of nonlinear equations generally have three main parts: incrementation (prediction), iteration (correction) and convergence check (termination test). The practical application in a structural FE code has proper requirements, which determine the desirable features of the three main parts. The incrementation strategies usually have to be fully automatic, including some knowledge about the possible future behavior (geometrical and material nonlinearity). The predicted step size has to be not too small to achieve a reasonable computing time if possible, but not too large in order to converge reliably in a few iteration steps. The iteration process should converge to the correct solution even in case of larger load increment, and avoid jumping to an unwanted equilibrium path. Finally unified convergence criteria should be applied, which produces approximately equal efficiency for different type of problems without unnecessary iterations. The importance of these requirements always depends on the actual goal of the computation.
The commonly used techniques in structural analysis usually try to be universal, in other words make an effort to be applicable for all kind of FE models. In order to reach this goal they use only the primary type of variables, like displacements and forces, when

controlling the incremental-iterative solution. Such advanced solution strategies are said to be applicable for various types of purposes (following difficult equilibrium paths, determining critical states etc.) using various types of finite element models (elastic, plastic, 1D, 2D, 3D etc.). However, it is obvious, that in case of a determined problem or FE code the chosen type of general technique needs to be calibrated and improved to make the run automatic and efficient. Moreover, even in case of a calibrated general solution technique, the advantageous specialties of the FE model are not taken into account. On the other hand the main common disadvantage of solution controlling methods, defined in the displacement-load space, is the unit discrepancy. During the solution process it is necessary to compute scalar estimators of the displacement-load vector field using some kind of measure (arc-length, Euclidean norm etc.). Because of the different dimensions the effects and significance in the estimators would be inadequate [1]. It may lead to inaccuracy, to very slow convergence or often to divergence even in case of a stable range of solutions. To reduce this problem the solution is generally controlled in the scaled displacement-load space making the dimensions more or less homogeneous. Also an important but rarely mentioned topic is the unified handling mode of the three main part of the solution process. There exist a great number of different techniques for incrementation, iteration and convergence tests [2], but generally developed, proved and used separately from each other. It results in a more reliable and efficient method if they are handled together unifying the measure modes.

The purpose of this paper is to propose a fully automatic, reliable and efficient solution technique developed especially for elasto-plastic problems in case of thin-walled beam-columns (a general model for steel structures). The main idea of the method is that it uses the deformation or strain space for all control measures to govern the solution. Manipulating the strain field has the following advantages [3]:

- in case of a beam-column model only the longitudinal strain is the relevant one
- it describes supremely the structural behavior (including sensors for geometrical and material non-linearities)
- it has no scaling problem, thus containing all information about the structure adequately
- it unifies the three main parts of the solution in a natural way

Thus the proposal takes the advantages of the beam-column FE model to make a consistent control strategy. The automatic application can be used for various sorts of research aims; the authors applied the technique for examination of the stochastic resistance of steel structures [4]. In stochastic processes the requirements mentioned at the beginning have much greater significance because of the great number of repetition within different initial conditions. It is highly essential that the solution process should be automatic and reliable, or else the user should encroach at the intermissions what makes the calculation inefficient. The solutions also should be very accurate otherwise the sense of stochastic results fades away.

The paper is organized in the following way. In the next section the Newton-type solution method is briefly outlined, then we describe some widely used techniques for incrementation, iteration and convergence checks. Afterwards the proposed strain based solution method is introduced, and two examples are tested from the topic of behavior of elasto-plastic thin-walled beam-columns.

NEWTON-TYPE SOLUTION METHOD

General

The performance of a conservative, one-parameter, nonlinear structural system can be always described by the total potential energy $\Pi(\mathbf{u}, \lambda)$ depending on the state vector $\mathbf{u}$ and the control parameter λ. The state vector generally represents the displacements, which characterize the actual configuration of the structure, and the control parameter determines the magnitude of the load acting on the structure [5]. The structure is in equilibrium if the potential energy is stationary with respect to the state field, consequently

$$\frac{\partial \Pi(\mathbf{u}, \lambda)}{\partial \mathbf{u}} = \mathbf{r}(\mathbf{u}, \lambda) = \mathbf{0} \tag{1}$$

where $\mathbf{r}$ is the gradient vector of the potential energy also called residual or out-of-balance force vector which is always zero in equilibrium state. If Eq. (1) is separable it can be decomposed:

$$\mathbf{r}(\mathbf{u}, \lambda) = \mathbf{p}(\mathbf{u}) - \mathbf{f}(\lambda) \tag{2}$$

where $\mathbf{p}$ is the internal force vector dependent on the displacements, and $\mathbf{f}$ is the applied external force vector which is only depends on the load parameter. In the Newton-method the equilibrium path is linearized by the first-order path derivative, used for the approximation process of the non-linear solution. Furthermore, if the loading is proportional, i.e. the external load vector is linear in λ, then the rate form of Eq. (2) takes the form:

$$\mathbf{K}_T \Delta\mathbf{u} = \mathbf{q}\Delta\lambda \tag{3}$$

$$\mathbf{K}_T = \frac{\partial \mathbf{p}}{\partial \mathbf{u}} \text{ and } \mathbf{q} = \frac{\partial \mathbf{f}}{\partial \lambda} \tag{4-5}$$

where $\mathbf{K}_T$ is the tangent stiffness matrix of the point, and $\mathbf{q}$ is the reference external load. Equation (1) represents a set of non-linear equations in the N–dimensional state field, which is not fully defined because the number of unknowns: N+1 including the control parameter. To make the problem determinate an additional constraint scalar equation is adopted usually also defined in the state-control space:

$$c(\mathbf{u}, \lambda) = 0 \tag{6}$$

The Newton-method solves this problem by continuation, in which the equilibrium states are followed as the control and state parameters vary by a small amount. This step by step method is not the fastest solution but – besides that generally there is no direct solution – has several advantages. The set of non-linear equations has a great number of mathematically correct solutions but only one solution has physical meaning. The continuation method can follow this correct equilibrium path, and can avoid extraneous roots. Although the user generally needs only distinct solution points, solving the whole equilibrium path can provide a good insight look into the structural behavior. In case of presence of plasticity the path dependence also requires the continuation. Newton-method based on continuation has two phases: prediction and correction. Prediction provides a properly chosen increment from the last solution point. This also denotes the starting point for the corrective iteration, which has to reduce the drifting error of the – generally linearized – prediction arisen from structural non-linearity. In such a way we look for a series of state vector and corresponding control parameter solutions that characterizes numerically the structural response satisfying the residual equation (1).

Incrementation

The challenge of the incrementation is to determine a point in the displacement-control space, which is in the local convergence area of the correct equilibrium path [6]. To predict such a point we can use information from previously computed and converged equilibrium points. In order to increase efficiency, the prediction should be located not too close to the previous point to reduce the number of necessary increments. However, it should also not be too far to be able to restore equilibrium within a few iteration steps. It is evident that the incrementation strategy is highly responsible for the stability and efficiency of the solution. The information of possible forthcoming structural behavior is generally considered through the linearized path equation (3) at the previous point. Although there are some attempts to improve the linearized prediction [7], or approach the problem in a totally different manner [8], the stability and the efficiency of these methods are questionable. The linearized prediction is used to determine the direction of the increment in the displacement space by the tangent displacement vector **v**, which is calculated, at the non-singular points of the path, from the tangent stiffness and reference load vector:

$$\mathbf{v} = \left(\mathbf{K}_T\right)^{-1}\mathbf{q} \tag{7}$$

Afterwards the magnitude of the prediction is adjusted by the proper constraint equation (6). Nevertheless, the user should always define the very first size of the increment, because there is no previous solution from which the constraint would take a value. In case of automatic incrementation this magnitude should be chosen carefully, because it will effect all subsequent predictions. This is an important drawback of all incrementation methods, but an improved version of it will be proposed later. Some of the most popular constraints for incrementation are reviewed below (determination of the i-th increment size $\Delta\lambda_i$) [2]:

- constant load increment

$$c = \Delta\lambda_1 - \Delta\lambda_i \tag{8}$$

- constant increment in displacement component

$$c = \Delta\lambda_1 \mathbf{b}^T \mathbf{u}_1 - \Delta\lambda_i \mathbf{b}^T \mathbf{u}_i \tag{9}$$

  where **b** is the vector which assigns the required component

- constant increment in the arc-length

$$c = \left(\Delta\lambda_1\right)^2 \mathbf{v}_1^T \mathbf{v}_1 - \left(\Delta\lambda_i\right)^2 \mathbf{v}_i^T \mathbf{v}_i \tag{10}$$

- constant 'current stiffness parameter'

$$c = \Delta\lambda_1 \mathbf{v}_1^T \mathbf{q} - \Delta\lambda_i \mathbf{v}_i^T \mathbf{q} \tag{11}$$

The calculated increment size can be improved by the iteration ratio, which is the quotient of a user defined desired number of iterations, and the actual number of iterations required for convergence in the previous step. The common effort of the above techniques – except the first one – is that they try to predict the prospective structural behavior by a measure of the last solution point, and moderate the increment size accordingly. The first problem is that these measures mix the dimensionless load factor with the displacements and loads having physical dimensions. The second problem is the different dimensions in the displacement and load vector of which values are numerically handled together. Accordingly the translational values have less effect on the resulting increment size then the rotational or warping values. Some scaling technique can reduce but not eliminate this problem.

Iteration

In non-linear analysis after the prediction Eq. (1) does not hold, that means the structure is not in equilibrium with the external loads:

$$\mathbf{r}\left(\mathbf{u}_i + \Delta\lambda_i \mathbf{v}_i, \lambda_i + \Delta\lambda_i\right) = \mathbf{r}_i^0 \neq \mathbf{0} \tag{12}$$

Henceforth the subscript i denotes the number of increment, and the superscript j denotes the number of iteration step. The Newton-method for correction is based on the first-order Taylor expansions of the system about the previous solution point (since the following equations are applied only to the corrections after the i-th increment, the subscripts are omitted):

$$\mathbf{r}^{j+1} = \mathbf{r}^j + \frac{\partial \mathbf{r}}{\partial \mathbf{u}}\left(\mathbf{u}^{j+1} - \mathbf{u}^j\right) + \frac{\partial \mathbf{r}}{\partial \lambda}\left(\lambda^{j+1} - \lambda^j\right) = \mathbf{0} \tag{13}$$

which can rewritten in the forms

$$\mathbf{K}_T^j \delta\mathbf{u}^j - \mathbf{q}\delta\lambda^j = -\mathbf{r}^j \quad \text{or} \quad \delta\mathbf{u}^j = -\left(\mathbf{K}_T^j\right)^{-1}\mathbf{r}^j + \mathbf{v}^j \delta\lambda^j \tag{14}$$

where $\mathbf{K}_T^j$, the Jacobian matrix of the residual, is the tangent stiffness after the j-th iteration step, $\mathbf{v}$ is the actual tangent displacement vector calculated from Eq. (7), $\delta\mathbf{u}^j$ and $\delta\lambda^j$ are the iterative changes in the displacement vector and load factor. Thus the variation of the displacement is always a linear combination of the tangent displacement and the residual displacement vectors, latter is calculated from the actual, non-zero residual force vector. In the modified Newton-Raphson method the tangent stiffness matrix is held fixed during the iterative process. Here the constraint equation is also used for the calculation of the iterative change in the load factor $\delta\lambda^j$. Hereinafter the most widely used constraints for iteration are summarized:

- iteration at constant load level

$$c = \delta\lambda^j \tag{15}$$

- iteration at constant displacement component

$$c = \mathbf{b}^T \mathbf{u}_r^j + \delta\lambda^j \mathbf{b}^T \mathbf{v}^j \tag{16}$$

where $\mathbf{u}_r$ is the residual displacement vector
- iteration at constant arc-length [9]

$$c = \Delta\mathbf{u}_a^{jT} \Delta\mathbf{u}_a^j - l^2 = \left(\sum_0^j \left(\mathbf{u}_r^j + \delta\lambda^j \mathbf{v}^j \right) \right)^T \left(\sum_0^j \left(\mathbf{u}_r^j + \delta\lambda^j \mathbf{v}^j \right) \right) - \Delta\lambda^2 \mathbf{v}^{jT} \mathbf{v}^j \tag{17}$$

where $\Delta\mathbf{u}_a$ is the accumulated displacement vector until the j-th step, l is the arc-length calculated at the incrementation
- iteration at constant external work [10]

$$c = \mathbf{q}^T \mathbf{u}_r^j + \delta\lambda^j \mathbf{q}^T \mathbf{v}^j \tag{18}$$

- iteration at minimum residual displacement norm [11]

$$c = \frac{\partial}{\partial \delta\lambda} \left(\delta\mathbf{u}^{jT} \delta\mathbf{u}^j \right) = \mathbf{v}^{jT} \mathbf{u}_r^j + \delta\lambda^j \mathbf{v}^{jT} \mathbf{v}^j \tag{19}$$

These iteration techniques also have the above mentioned scaling problems, which in this case can lead to divergence, so the proper scaling of the variables is again required.

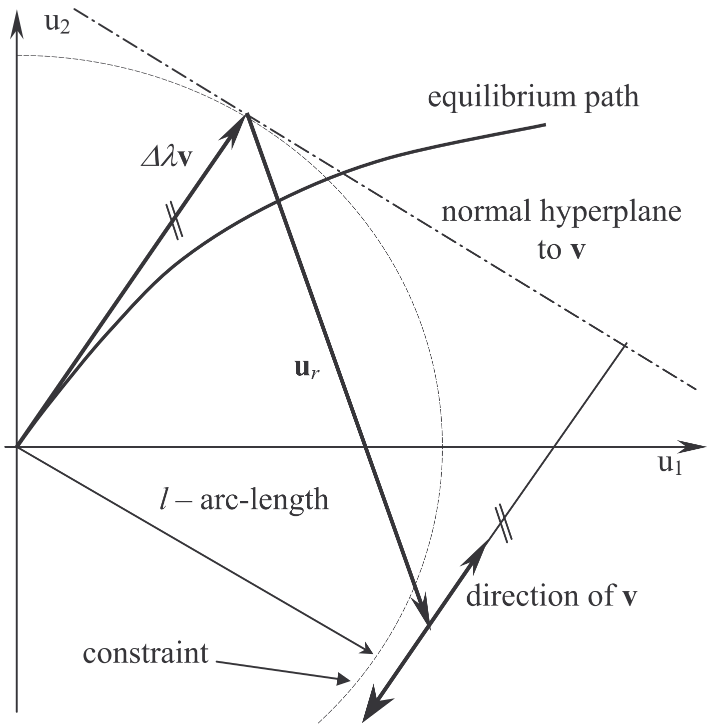

Figure 1. Failing of the arc-length method

Additionally the most popular arc-length method has another problem: the iterative change is one of the two solutions of a second-order equation, which may have no real solution. It is the case when the length of the residual displacement vector projected to the normal hyperplane of the tangent displacement vector is greater then the arc-length in the constraint equation, so the constraint hypersphere has no intersection with the direction parallel to the tangent displacement (Fig. 1.). In algebraic form that means that the discriminant of the second-order equation is less then zero. In case of inelastic stability calculations this problem can occur often, so the arc-length method may fail in the most important region.

Convergence criteria

The iteration process will never yield an absolutely correct solution, so it is continued until the solution error decreases under an acceptable level. To check it a suitable convergence criteria is constructed which is tested after all iteration cycles. The tests should correctly denote the magnitude and the progress of the errors to avoid divergence. The convergence criteria should be particularly homogeneous or else different problems must have different error limits. The general convergence tests are built upon some measure of the residual field. Categorically three different tests can be distinguished [5]:

- displacement test

$$\frac{\left\|\mathbf{u}_r^j\right\|}{\left\|\mathbf{u}\right\|} = \frac{\sqrt{\mathbf{u}_r^{jT}\mathbf{u}_r^j}}{\sqrt{\mathbf{u}^T\mathbf{u}}} \leq \gamma_d \tag{20}$$

- force test

$$\frac{\left\|\mathbf{r}^j\right\|}{\lambda\left\|\mathbf{q}\right\|} \leq \gamma_r \tag{21}$$

- work test

$$\frac{\left|\mathbf{r}^{jT}\mathbf{u}_r^j\right|}{\lambda\left|\mathbf{uq}\right|} \leq \gamma_w \tag{22}$$

where the γ_d, γ_r, γ_w are carefully chosen error tolerances. Obviously the scaling problem is again an important fact which reduces the consistency and homogeneity of the tests. The normalizing denominators make the criteria dimensionless and can be diverse then the introduced ones. However there are essential differences even in the above expressions. In the force test the denominator does not contain the effect of non-linear behavior. The norm of total displacement contains it and when the non-linear effects start to dominate the residual tolerance would increase. In addition the total external force norm decreases in the post-critical range aggravating the criteria, the total displacements increase significantly after loss of stability making easier to satisfy the displacement based convergence test.

STRAIN BASED SOLUTION METHOD

For the proposed improvement of the incremental-iterative solution method it is important to note that in the general strategy – discussed above – the constraint equation is used only to determine the size of the increment or the iterative change. The actual tangent displacement vector defines the direction of the increment and the iterative change. To construct a constraint condition it seems obvious to define it in the same control-state (load-displacement) field although it is not the only possibility. Furthermore the constraint equation is a scalar equation, that is why the appropriate scaling of the different type of variables is always an important task. To eliminate this problem totally, the proposed method transfers the domain field of the constraint equation into the strain or deformation space of the structure. In case of beam-column structures where the only relevant strain is the longitudinal strain, it does not require any complicated or large additional computational effort. After all the strain space contains homogeneously all important properties of the behavior of the structure, and thus it is well suited to control the solution. Additionally it is also evident and convenient to handle the mentioned three phases of the Newton-type process uniformly.

The total normal strain field of spatial thin-walled beam-columns can be expressed by the displacement components [12]:

$$\varepsilon(x, y, z) = u' - zv'' - yw'' + \omega\theta_x'' \tag{23}$$

where ε is the continuous normal strain field along the three dimensional structural body, x,y,z are the coordinate axes (x is the longitudinal, y,z form the plane of cross section), u,v,w are the continuous displacements of the beam-column axis, θ_x is the rotation of the axis, and ω is the sectorial coordinate. In such a way from the continuous displacement field the longitudinal strain at any point of the structure can be unambiguously determined. As the displacement field is discretized into nodal displacement vectors by the finite element method, the strain field is also discretized into average strains at a subregion of a cross-section [13]. After approximating the displacements by suitable shape functions the discretized form of Eq. (23) is:

$$\underset{(2n \cdot nr)}{\boldsymbol{\varepsilon}} = \underset{(2n \cdot nr \times (n+1) \cdot nc)}{\mathbf{A}} \; \underset{((n+1) \cdot nc)}{\mathbf{u}} \tag{24}$$

where n, nr, nc are the sizes of the vectors and matrix (number of elements, number of subregions of a cross-section, number of degrees of freedom), $\boldsymbol{\varepsilon}$ is the strain vector of the structure, and $\mathbf{A}$ is a coefficient matrix containing only terms belonging to the geometry of the structure [14]. Since Eq. (24) is a linearized set of equations all kind of displacement vector have the strain vector pair (total, tangent, residual displacements). In that way the above outlined computations of the constraint equations can be done in the strain space. In the opinion of the authors in case of other types of finite elements (where not only the longitudinal strains are relevant) the idea can also be applied if the compatibility equations are linearized in the program code.

Automatic incrementation

The incrementation strategy is developed for the determination of resistance of steel beam-column structures within stochastic processes. However, it can be also useful for defining the optimal step size in deterministic problems. The base of the proposed method is an average behavior of a real, imperfect steel beam-column member, from the beginning of loading through some loss of stability until the post-critical range. The final state corresponds generally to the total yielding in a cross-section, which is practically the end of the computations. In the first loading range the geometric second-order effects start to develop on the imperfections, but an average member will not reach the critical state in the elastic range. After the first yielding the structure starts to soften progressively, until the loss of stability. In the post-critic range the load generally decreases slowly with greater and greater displacement. In that case the incrementation should predict the progress in geometrical softening, the first yielding point and the loss of stability. Using the strain field these effects can be controlled effectively. However, the next question is how to handle the strain field, how to represent it during the computation. As in case of the general methods the plausible choice is some kind of norm of the discretized field. To govern the incrementation, the infinity norm (or maximum norm) of the strain field was chosen, since it denotes exactly the first yielding point. After that the whole incrementation process is working as follows:

- the equilibrium path in the load control-maximum strain space is approximated by a suitable analytical formula based on the previous equilibrium points and the last derivative (tangent maximum strain),
- the maximum strain is incremented roughly smoothly and the load step size is adjusted suitably using the approximation.

In that way the automatic selection of the very first load size is possible, starting the calculation with an arbitrary small load step. The equilibrium path is divided practically into three ranges – elastic, elasto-plastic, and post-critical –, and based on theoretical considerations, three different approximations are used (Fig. 2., λ_y, λ_u – load parameters according to the first yielding and the loss of stability respectively, ε_r, ε_y – residual (due to fabrication process) and yielding strain respectively).
In the purely elastic range the equilibrium path is approximated with a parabolic function, due to the geometrical second-order effect. Therefore the relation between the load control and maximum strain will be the following:

$$\lambda_i = a_i \|\boldsymbol{\varepsilon}\|_\infty^2 + b_i \|\boldsymbol{\varepsilon}\|_\infty + c_i \tag{25}$$

where $\|\boldsymbol{\varepsilon}\|_\infty$ is the maximum norm of the discretized strain field; a_i, b_i, c_i are the actual coefficients calculated from the previous two solutions, and from the last derivative (tangent), explicitly defined in [3]. The constant term c_i is needed because of the possible existence of residual stress (what is handled as residual strain, ε_r, see Fig. 2.), the linear coefficient b_i represents the initial linear elastic behavior and the coefficient of the quadratic term must be negative due to the geometrical softening effect. At the beginning of the calculation the first approximation point belongs to the zero load level. The second point is computed from an arbitrary small load increment, and the tangent is also calculated at this point which already contains the second order effects. Afterwards the first parabolic approximation can be constructed, which is a good prediction for the

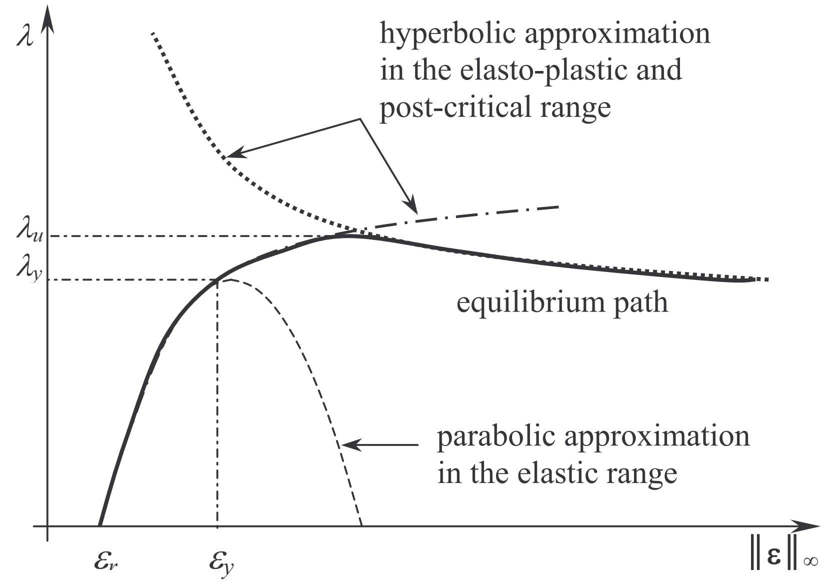

Figure 2. Approximations for the equilibrium path

initial range of the equilibrium path in the load control-maximum strain space. According to a suitable maximum strain step size the load increment is now adjustable, and the coefficients of the approximation are updated in every solution point containing the latest information. Since the maximum strain indicates the actual structural behavior in all kind of problems, the whole load incrementation, including the very first step size, can be always automatically determined by the strain steps. This advantage is very important in the case of stochastic calculations as will be presented in the example. The maximum strain step sizes are practically some divisors of the yielding strain (ε_y):

$$\Delta\|\boldsymbol{\varepsilon}\|_\infty = \|\boldsymbol{\varepsilon}\|_{\infty,i+1} - \|\boldsymbol{\varepsilon}\|_{\infty,i} = \frac{\varepsilon_y}{5 \div 10} = \frac{\sigma_y}{E(5 \div 10)} \tag{26}$$

where σ_y is the yielding stress and E is the elastic modulus.

After the maximum strain reaches the yielding strain a new, hyperbolic approximation is used (Fig. 2.):

$$\lambda_i = a_i + \frac{b_i}{\|\boldsymbol{\varepsilon}\|_\infty^2} \tag{27}$$

where the a_i, b_i coefficients are again determined from the last solution point and last derivative (tangent), and explicitly defined in [3]. The hyperbolic function was chosen to represent the asymptotic convergence of the equilibrium path to the limit point, and the form was adopted from the general relationship of moment-curvature of beam-columns [12]. The incrementation goes on in the same way, the maximum strain is incremented with the determined step size, and the load size is adjusted using the hyperbolic approximation. After the limit point the stiffness matrix becomes negative definite, so the tangent maximum displacement changes sign and accordingly the loading direction turns back as the hyperbolic approximation has descending branch (Fig. 2.), so the method does not need any further estimator to determine the direction of the loading [15].

In the proposed incrementation scheme the load size is totally automatically regulated from the very first step, following the non-linear effects and post critical behavior. The rapidity and correctness of the solution can be fully controlled by the strain step size.

Iterative technique

Three iteration strategies are proposed all based on ideas already discussed earlier, but defined in the strain field , thus totally eliminating scaling problems. The first one uses henceforward the maximum norm of the discretized strain field. The principle of the technique is the same as in the iteration at constant displacement component, defined by Eq. (16). Instead of a defined displacement component, the actual maximum strain is held constant during the iteration. In the method of Eq. (16) a relevant component should be always defined, which is obviously dependent on the actual initial conditions, making thus the automatic selection impossible. This main problem is eliminated with the proposed technique. So the constraint equation of the first method takes the form:

$$c = \left\|\boldsymbol{\varepsilon}_r^j\right\|_\infty + \delta\lambda^j \left\|\boldsymbol{\varepsilon}_t\right\|_\infty = \left\|\mathbf{Au}_r^j\right\|_\infty + \delta\lambda^j \left\|\mathbf{Av}^j\right\|_\infty \tag{28}$$

where $\boldsymbol{\varepsilon}_r$ and $\boldsymbol{\varepsilon}_t$ are the residual and tangent strain field respectively.
The second proposed iteration technique is an arc-length method, Eq. (17), defined in the strain space. The constraint equation is also in the same form:

$$c = \Delta\boldsymbol{\varepsilon}_a^{jT}\Delta\boldsymbol{\varepsilon}_a^j - l_\varepsilon^2 = \left(\sum_0^j \left(\boldsymbol{\varepsilon}_r^j + \delta\lambda^j \boldsymbol{\varepsilon}_t\right)\right)^T \left(\sum_0^j \left(\boldsymbol{\varepsilon}_r^j + \delta\lambda^j \boldsymbol{\varepsilon}_t\right)\right) - \Delta\lambda^2 \boldsymbol{\varepsilon}_t^T \boldsymbol{\varepsilon}_t \tag{29}$$

where the notation is the same as in Eq. (17), but instead of displacement vectors all quantities are calculated from the corresponding discretized strain vectors. This constraint contains all non-linear effects in a natural manner, thus has much less possibility of failing as depicted and explained in Fig. 1.
The third method can have the name of iteration at minimum residual strain norm, for which the model is Eq. (19). The constraint equation has the form:

$$c = \boldsymbol{\varepsilon}_t^T \boldsymbol{\varepsilon}_r^j + \delta\lambda^j \boldsymbol{\varepsilon}_t^T \boldsymbol{\varepsilon}_t \tag{30}$$

The above-introduced constraints have the main advantage of consistency, which is a very important property of iteration governing processes based on distances or measures in a vector field.

Convergence criteria

Two convergence tests are proposed based on two types of norm of the residual strain field. The stricter one uses the maximum norm:

$$\frac{\left\|\boldsymbol{\varepsilon}_r^j\right\|_\infty}{\left\|\boldsymbol{\varepsilon}\right\|_\infty} \le \gamma_{\varepsilon 1} \tag{31}$$

The second one applies the Euclidean norm thus formulating a more general criteria:

$$\frac{\left\|\boldsymbol{\varepsilon}_r^j\right\|}{\left\|\boldsymbol{\varepsilon}\right\|} \le \gamma_{\varepsilon 2} \tag{32}$$

Both types of tests include homogeneously all the effects causing errors in the solution, thus the well-chosen error tolerances can ensure uniform accuracy for different types of problems.

APPLICATION

The main benefits of the proposed strain based solution technique are that it finds automatically the optimal size of increments, and is free of the errors caused by inconsistency in dimensions. The ultimate load level and the shape of the equilibrium path are the properties mostly affect the optimal size of increments, since the increments define a sort of measure of distance along the path. The examples show the influence of different behavior on the effectiveness of the calculation. In stochastic calculations the change in loading, geometrical and material properties, boundary conditions and imperfections can cause significantly different behavior modes. During the stochastic process there is no opportunity to set the solution governing parameters in order to make the run effective. There are two possibilities to compare the different techniques during this automatic run: examination of the cumulative number of iterations for a given initial load step, which produces accurate solution for all calculations, or determination of the optimal initial increment results in a consistent, effective solution method for the different problems. In the examples the first method was used, comparing the proposed method with three another techniques summarized in Table 1. In the following examples a special beam-column finite element model with thin-walled cross-section is used, which was first published by Rajasekaran [12], and implemented with a developed cross-section model by Papp [14].

Table 1. Description of the four compared technique

technique	incremental method	iterative method	convergence test
1	arc-length	constant arc-length	work
2	current stiffness parameter	minimum residual displacement norm	work
3	displacement component	constant displacement component	work
4	maximum strain	constant arc-length in strain space	Euclidean norm of residual strains

Change in loading

In completly probabilistic calculations (e.g. in Monte-Carlo method), where the loading and type of load as well are random variables, a compressed member can easily become a bent or twisted element during the calculation. To illustrate the effect on the efficiency a simply supported beam-column, with a hot-rolled I section (HEA200) is examined for compression and changing bending moment about the major axis (Fig. 3.).

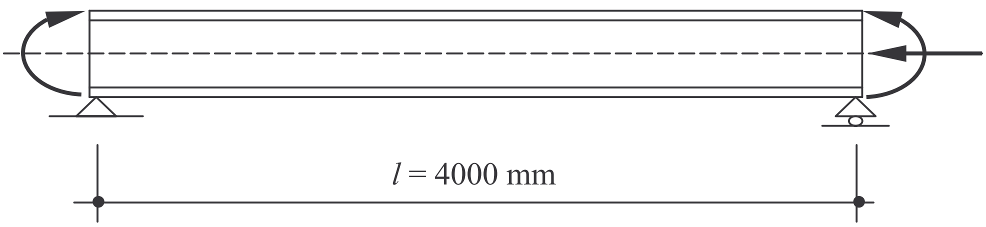

Figure 3. Model of the examples

The material is elastic-perfectly plastic, and the member contains some common imperfections (initial out-of-straightness, residual stresses). Thus the proper failure mode is the interaction of plastic buckling and lateral-torsional-buckling. In Fig.4. the different equilibrium paths are plotted until loss of stability with changing loading conditions (the notations are: N –compressive force, $m = M/M_{cr}$ – bending effect, where M_{cr} is the pure lateral-torsional-buckling resistance of the member, u_y – lateral displacement of the middle cross-section). It can be seen that if the load factor is connected to compressive force, its ultimate value significantly decreases with increasing bending effect, as long as the displacements change only slightly. In Fig.5. the cumulative iteration number of the four techniques is plotted against the bending effect. While the proposed strain based technique is insensitive for the change in loading, the others can do the calculations with increasing iteration number, as the ultimate compressive force is higher. The third technique gives quite good results, but an appropriate displacement component has to be chosen carefully (it was the lateral displacement of the middle of the member).

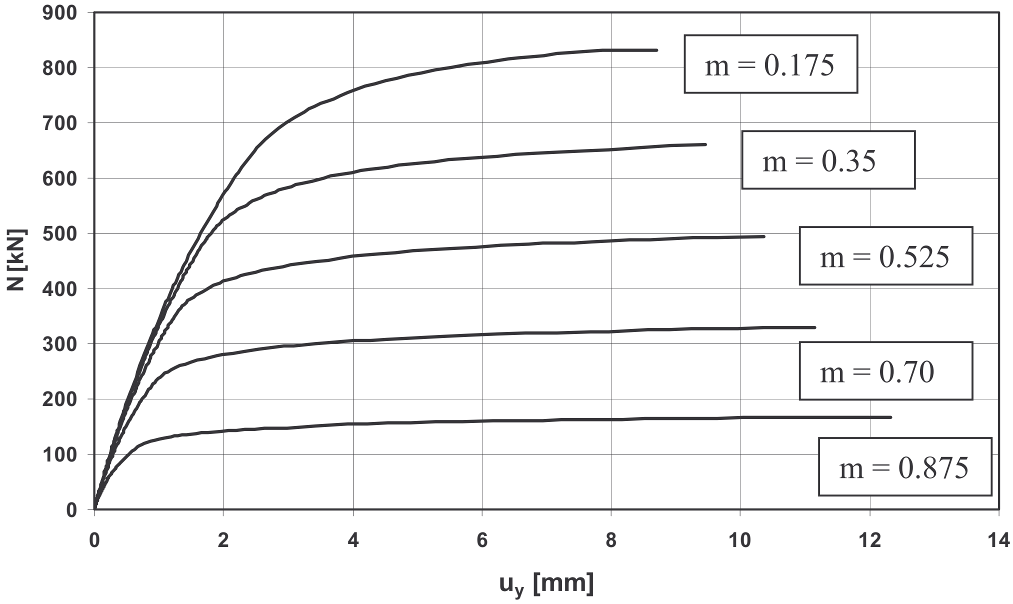

Figure 4. Equilibrium paths in case of interaction

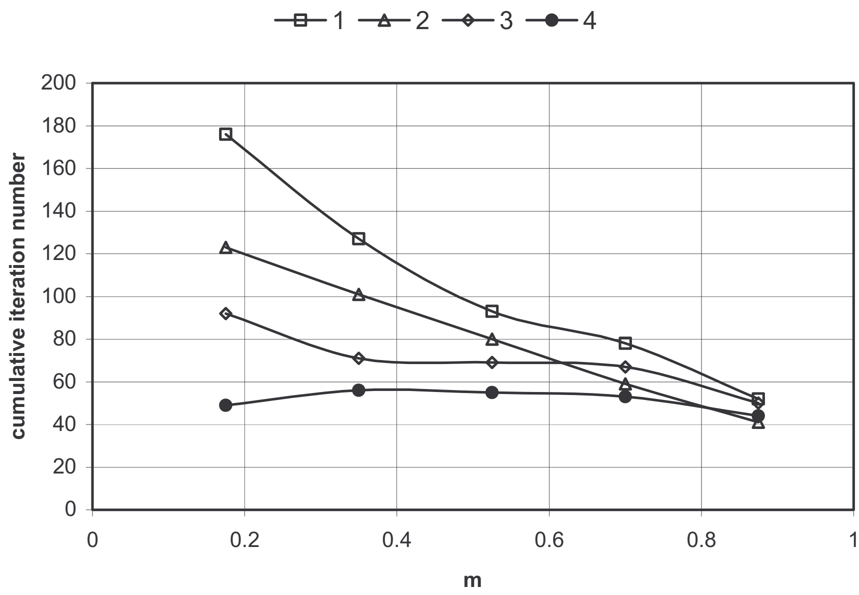

Figure 5. Changes in the number of iterations

Change in imperfections

Initial imperfections of beam-columns have a great effect on the shape of the equilibrium path, which highly influence the performance of the incremental-iterative methods based on some measure along the path. In this example the same model is used as in the previous one, but the load is only a compressive force, so the failure mode is always a simple plastic buckling. We analyze the effect of two imperfections: initial out-of-straightness about the minor axis, and residual stresses along the cross section. The out-of-straightness is sine shaped (the first eigenshape), and the middle amplitude is changing. The residual stress is linearly distributed along the flanges, and constant in the web of the cross-section (Galambos-type), and the variable is the stress value at the tip of the flange. Four coherent cases were examined, with the aid of a practically introduced imperfection parameter:

$$\begin{aligned} e_0 &= l/(3500-500p) \\ \sigma_{r,\max} &= 0.2pf_y \end{aligned} \tag{33-34}$$

where e_0 is the amplitude of the out-of-straightness, l is the length of the member, $\sigma_{r,\max}$ is the mentioned residual stress value, f_y is the yielding stress and p is the imperfection parameter. In Fig. 6. also the equilibrium paths are plotted until the loss of stability, in case of different imperfections. There are also differences between the resistance values, but now the shapes of the paths are very different as well. Consequently the influence on the cumulative number of iterations can be observed in case of the first technique (arc-length), at which the greater imperfections (and the greater displacements) the greater number of iterations, see Fig.7. The second technique seems to behavior conversely, as it is more sensitive with respect to the ultimate load value. The proposed method is again insensitive with respect to the imperfections, and thus for the shape of the equilibrium path.

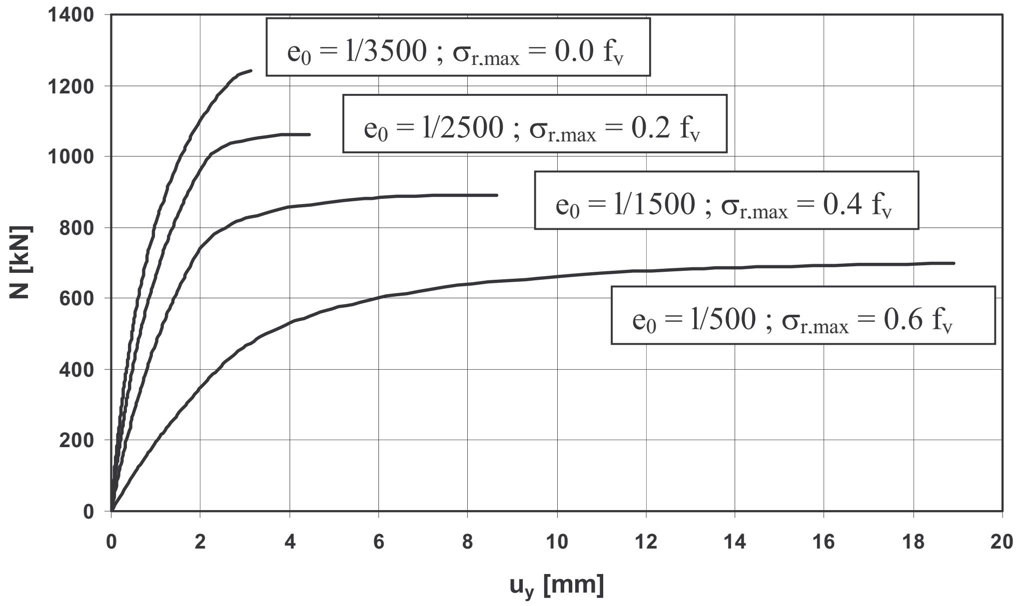

Figure 6. Equilibrium paths with different imperfections

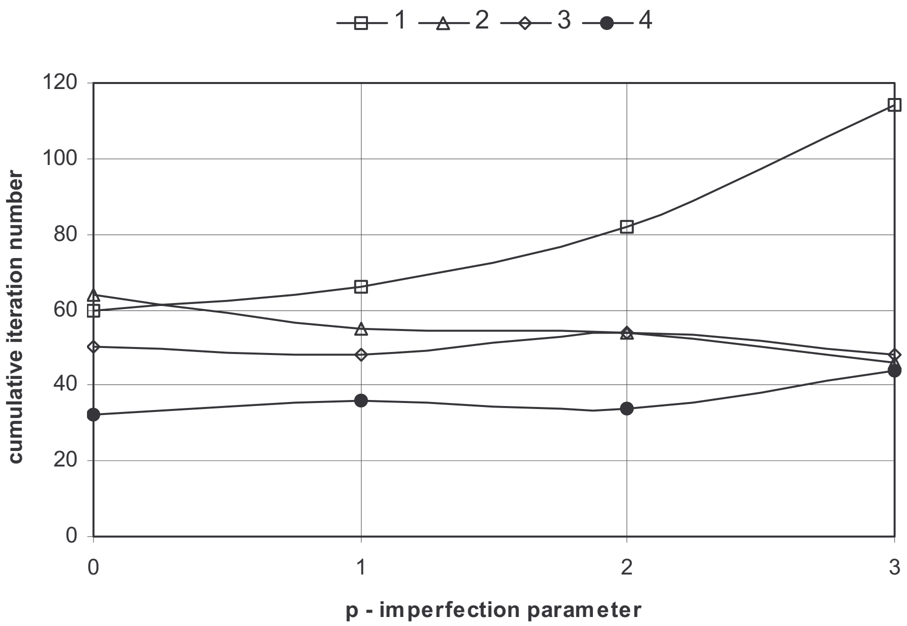

Figure 7. Changes in the number of iterations

CONCLUSIONS

An automatic and uniform incremental-iterative solution method has been presented for elasto-plastic beam-columns suitable for efficient stochastic analysis. The changes in loading conditions, geometrical and material properties, boundary conditions and imperfection can alter significantly the final load level, and the shape of the equilibrium path. In case of existing techniques this generally requires different solution governing parameters – considering particularly the very first size of load parameter – to reach an effective run. Since this is impossible during continuous stochastic calculations, there is a need for making the deterministic solution process insensible for changes in the equilibrium path. Transforming the domain of the constraint equation from the displacement to the strain field can solve this problem. In this way the errors caused by dimension differences (scaling problems) are naturally eliminated. Monitoring the strain field allows of predicting exactly the first yielding and the rate of yielding for any type of equilibrium path, thus the solution governing parameters can be adjusted independently (including the very first size of load parameter). Furthermore the three main parts of the solution process – incrementation, iteration and convergence test – can be integrated in the strain space. In this manner a fully automatic and efficient incremental-iterative technique is developed and successfully applied in the stochastic analysis of the real resistance of thin-walled beam-columns.

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