A NUMERICAL MODEL FOR UNILATERAL MATERIALS IN FINITE DEFORMATIONS, PART I: LARGE STRAIN MODELLING AND NUMERICAL INTEGRATION SCHEME

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SUMMARY

A non linear finite deformation model for unilateral materials in the field of finite deformations and its implementation in a standard isoparametric finite elements code has been developed. These materials are not able to sustain any tension (resp. compression) stress and represent a special case of generalized elastic materials, such that the anelastic evolution is ruled by a potential. In part one the constitutive model and its numerical integration is described. The model has been developed on the basis of a multiplicative split of the gradient of deformation tensor and assuming that no dissipations occurs in the anelastic process. The prediction of the model for some homogeneous deformation states is reported in the paper.

INTRODUCTION

In this work it is presented an extension to finite deformations of the No Tension Material (NTM) model, characteristic of materials like masonry, unreinforced concrete, mortar, sand, soft rock, that have very low tensile resistance. Commonly their tension resistance is completely neglected and it is assumed that anelastic deformations are possible in the direction of the maximum stress. The model is characterized by absence of dissipation; as a matter of fact in the small deformation range the NTM model falls within the class of conewise elastic materials [1] and the constitutive equations can be derived from a potential of the total deformations. Since in the model a continuous anelastic deformation field accounts for the localized anelastic phenomena like fractures or rupture lines (like in sand) it is likely that large strains develop in concentrated areas. Moreover, many types of NTM structures, like long span shallow arches, large vaults, slender panels, exhibit large rigid motions caused by fractures. In either case, linear kinematics appears insufficient, as was also observed long time ago by Heyman.
The model presented in this paper extends to the finite deformation field the equivalent one formulated in small deformations [2,3,4,5], that is widely used in the analysis of

masonry structures. It has the great advantage of being based on few constitutive parameters, i.e. only the elastic coefficients, and of approximating in a satisfactory way the structural response, provided no dissipation due to reversed cycles of loading occurs. The model is, therefore, reversible, so that it is not able to model damage processes. An useful discussion on the limitations and on the advantages of the model can be found in [2,6,7].

The constitutive relations are developed starting from energy principles so that they turn out to be thermodynamically well defined. The deformation process is decomposed in its elastic and its anelastic part introducing an intermediate (eventually fictitious) stress free configuration, so that the deformation gradient is multiplicatively split in an elastic and an anelastic part, while the velocity of deformation tensor is additively decomposed [8]. The velocity of deformation measures used in the model satisfy the requirements of objectivity and are defined to be dual in power to the stress tensors introduced in each configuration.

The anelastic constitutive relation is directly obtained by the assumption that the process occurs without dissipation while an hyperelastic potential is used for the elastic deformation. From this assumption it follows that the rate of anelastic deformations is orthogonal to the stress tensor. The model so formulated presents the same characteristic of reversibility of the equivalent model formulated in the field of small deformations because of the hypothesis of absence of dissipations. Differently from the small deformation case, it as not been possible to obtain an explicit expression of the deformation potential, and only its rate form has been specified. The question whether a generalised hyperelastic potential exists for this class of materials is still an open matter.

The model has been locally integrated (in the Gauss points of a finite element model) performing a step linearisation of the rate of deformation (material derivative of the right Cauchy-Green deformation tensor) and of the Lie derivative of the Kirchhoff stress tensor. The resultant system of non-linear equations is numerically solved using a classical Newton-Raphson scheme.

Although the paper deals only with no-tension materials, the same procedure would apply to no-compression unilateral materials, that have recently became an important issue in studying the finite deformation of soft membranes, as in the case of living tissue. Most of the results obtained in this paper would hold also for this case just changing the sign of the admissible stresses. However this case will not be explicitly considered in this work.

THE MATERIAL MODEL

Kinematics

No Tension Materials are not able to sustain any tension stress, so, in a general deformation process, fractures occur. Here a continuum approach is used, that is, no discontinuity is allowed, and the effects of fractures are modelled through an additional anelastic deformation field.

Anelastic effects are accounted for decomposing the deformation process in its elastic and anelastic parts introducing an intermediate (eventually fictitious) stress free configuration. Calling B_0 the reference, B_a the intermediate and B_t the final configurations, each one is associated with a reference framework $\{X\},\{\Xi\}$ and $\{x\}$, with metric G, $\hat{g}$, g respectively (see Figure 1).

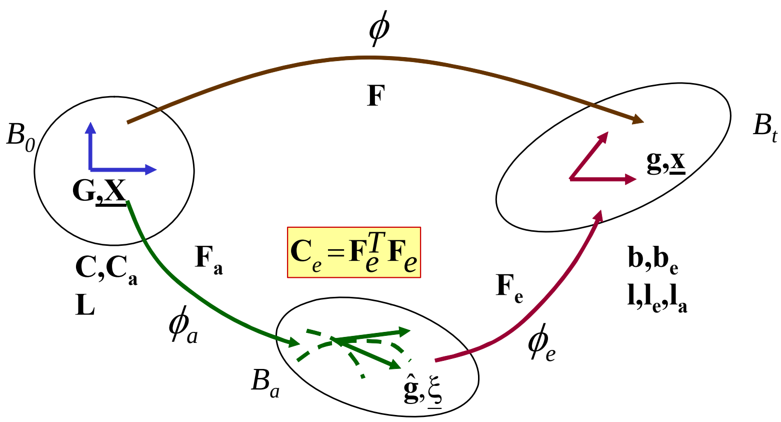


Figure 1. Process Schematization

Of course in the case of Cartesian framework the metric is the identity matrix. By virtue of this schematisation, the gradient of deformation is multiplicatively split in

$$\mathbf{F}=\mathbf{F}_e\mathbf{F}_a \tag{1}$$

Consequently also the volume deformation, given by the determinant of the Jacobian of the deformation process, is given by the product

$$J = J_e J_a \tag{2}$$

The gradients of deformation $\mathbf{F}_e$, $\mathbf{F}_a$ and their inverses operate the "Pull-Back" and "Push-Forward" transformations of the kinematic and equilibrium tensors between the configurations defined in figure 1, according to standard rules [9]. We will denote by ϕ_* and ϕ^* the Pull-Back and Push-Forward operations.

On the intermediate configuration we define the elastic right Cauchy-Green deformation tensor as follows

$$\mathbf{C}_e = \mathbf{F}_e^T\mathbf{F}_e \quad \rightarrow \quad C_{e\alpha\beta} = g_{ij} F_e{}^i{}_\alpha F_e{}^j{}_\beta \tag{3}$$

The component notation shows that $\mathbf{C}_e$ can be interpreted as the pull back on the intermediate configuration of the spatial metric g, i.e. it has the meaning of convective

metric in the intermediate configuration. In a similar manner we can show that the total Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ is the pull back on the reference configuration of the current metric g [10,11].
The tensor $\mathbf{C}_e$, defined in (3), is a measure of elastic deformations. As another measure of elastic deformations we define, in the current configuration, the left elastic Cauchy-Green deformation tensor as follows

$$\mathbf{b}_e = \mathbf{F}_e \mathbf{F}_e^T \tag{4}$$

while the total inelastic deformations are ruled by the anelastic right Cauchy-Green deformation tensor defined as

$$\mathbf{C}_a = \mathbf{F}_a^T \mathbf{F}_a \tag{5}$$

It is easy to prove that the following relation occurs for the elastic and anelastic deformation measures defined in (4) and (5)

$$\mathbf{b}_e = \mathbf{F}\mathbf{C}_a^{-1}\mathbf{F}^T \tag{6}$$

Using Equation (1), the velocity gradient tensor in the final configuration can be additively decomposed in its elastic and anelastic parts as

$$\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{F}}_e \mathbf{F}_e^{-1} + \mathbf{F}_e \dot{\mathbf{F}}_a \mathbf{F}_a^{-1} \mathbf{F}_e^{-1} =: \mathbf{l}_e + \mathbf{l}_a \tag{7}$$

The symmetric part of $\mathbf{l}$ is the velocity of deformation tensor, that can be decomposed as follows

$$\mathbf{d} = sym(\mathbf{l}) = sym(\mathbf{l}_e) + sym(\mathbf{l}_a) =: \mathbf{d}_e + \mathbf{d}_a \tag{8}$$

It is useful to define gradient of velocity in the intermediate configuration. To this end we introduce the mixed variant elastic pull back of the velocity tensors defined in (7) and (8)

$$\hat{\mathbf{l}} = \mathbf{F}_e^{-1}\mathbf{l}\mathbf{F}_e = \mathbf{F}_e^{-1}\dot{\mathbf{F}}_e + \dot{\mathbf{F}}_a\mathbf{F}_a^{-1} =: \hat{\mathbf{l}}_e + \hat{\mathbf{l}}_a \tag{9}$$

The tensor $\hat{\mathbf{l}}_a$ has an important meaning, in that it gives the evolution of the intermediate configuration. In fact it represents the material derivative of the instantaneous increment of the anelastic gradient of deformation $\hat{\mathbf{l}}_a = \left.\dfrac{\partial \mathbf{F}_a(\tau)}{\partial \tau}\right|_{\tau=t}$.
However, the tensor $\hat{\mathbf{l}}$ cannot be used as a measure of velocity of deformation as does the spatial velocity gradient tensor (8) since the symmetric part of $\hat{\mathbf{l}}$ is not the pull-back of the velocity of deformation tensor, but bears also information of the spatial spin. ($sym(\hat{\mathbf{l}}) \neq \mathbf{F}_e^{-1}\mathbf{d}\mathbf{F}_e$)Consequently, the tensor (9) is not dual in power to the second Piola-Kirchhoff stress tensor that will be introduced later (see eq. (20)).

A sound measure for the velocity of deformation of the intermediate configuration is obtained performing a total covariant elastic pull back of the spatial gradient of velocity tensor

$$\hat{\hat{\mathbf{l}}} = \phi_e^*(\mathbf{l}) = \mathbf{F}_e^T \mathbf{l} \mathbf{F}_e = \mathbf{C}_e \hat{\mathbf{l}} = \mathbf{F}_e^T \dot{\mathbf{F}}_e + \mathbf{F}_e^T \mathbf{F}_e \dot{\mathbf{F}}_a \mathbf{F}_a^{-1} =: \hat{\hat{\mathbf{l}}}_e + \hat{\hat{\mathbf{l}}}_a \tag{10}$$

$$\hat{\hat{\mathbf{d}}} = \phi_e^*(\mathbf{d}) = sym\left(\hat{\hat{\mathbf{l}}}\right) = \mathbf{F}_e^T \mathbf{d}_e \mathbf{F}_e + \mathbf{F}_e^T \mathbf{d}_a \mathbf{F}_e =: \hat{\hat{\mathbf{d}}}_e + \hat{\hat{\mathbf{d}}}_a \tag{11}$$

The tensors defined in (10) are the covariant form of the corresponding tensors defined in (9) in the convected metric $\mathbf{C}_e$ (es. $\hat{\hat{l}}_{e_{\alpha\beta}} = C_{e_{\alpha\gamma}} \hat{l}^{\gamma}_{e\,\beta}$). Throughout the paper double hat indicates covariant kinematic objects defined in the intermediate configuration. As indicated in (11) and as can be checked by a direct calculation, the symmetric part of $\hat{\hat{\mathbf{l}}}$ is indeed the covariant elastic pull-back of the velocity of deformation tensor, and can be similarly decomposed in the elastic and anelastic parts. It is worth noting that the covariant velocity of deformation directly translates into the intermediate configuration the properties of the spatial velocity of deformation, particularly it bears the following property:

$$\text{if } \mathbf{n} = \mathbf{F}_e \nu \text{ it is } \mathbf{d}\,\mathbf{n} \cdot \mathbf{n} = \hat{\hat{\mathbf{d}}}\,\nu \cdot \nu \tag{12}$$

that is the velocity of stretch can be directly computed in the intermediate configuration on the pull-back of the direction of the stretch. The tensor $\hat{\mathbf{l}}$ preserves, on the contrary, the equality [11]

$$\mathbf{l}\,\mathbf{n} \cdot \mathbf{m} = \hat{\mathbf{l}}\,\nu \cdot \mu \tag{13}$$

with $\qquad \mathbf{n} = \mathbf{F}_e \nu, \qquad \mathbf{m} = \mathbf{F}_e^{-T} \mu$

so that $\qquad \nu \perp \mu \;\leftrightarrow\; \mathbf{n} \perp \mathbf{m}$

Moreover the convected velocity of deformation tensor $\hat{\hat{\mathbf{d}}}$ coincides with half of the Lie derivative of the elastic right Cauchy-Green deformation tensor. In fact, in virtue of the following relations

$$\hat{\hat{\mathbf{d}}}_e = sym\left(\mathbf{C}_e \hat{\mathbf{l}}_e\right) = \frac{1}{2}\left(\dot{\mathbf{F}}_e^T \mathbf{F}_e + \mathbf{F}_e^T \dot{\mathbf{F}}_e\right) = \frac{1}{2} \overset{\bullet}{\overline{\mathbf{F}_e^T \mathbf{F}_e}} = \frac{\dot{\mathbf{C}}_e}{2} \tag{14}$$

$$\hat{\hat{\mathbf{d}}}_a = sym\left(\mathbf{C}_e \hat{\mathbf{l}}_a\right) \tag{15}$$

one has

$$\frac{1}{2} L_v^p \mathbf{C}_e = \mathbf{F}_a^{-T} \frac{\partial}{\partial t}\left(\mathbf{F}_a^T \mathbf{C}_e \mathbf{F}_a\right) \mathbf{F}_a^{-1} = \frac{\dot{\mathbf{C}}_e}{2} + sym\left(\mathbf{C}_e \hat{\mathbf{l}}_a\right) = \hat{\hat{\mathbf{d}}} \tag{16}$$

In (16) the symbol L_v^p denotes the plastic Lie derivative of an object defined in the intermediate configuration, i.e., for a kinematic tensor

$$\frac{1}{2} L_v^p(\bullet) = \mathbf{F}_a^{-T} \frac{\partial}{\partial t}\left(\mathbf{F}_a^T \bullet \mathbf{F}_a\right)\mathbf{F}_a^{-1} \tag{17}$$

By virtue of the Virtual Power identity, we can introduce in each configuration a stress measure dual in power to the velocity of deformation tensors defined in (8), (11). It is easy in fact to see that the following relations hold

$$P_{vi} = \int_{B_t} \sigma \cdot \mathbf{d}\, dv = \int_{B_0} \tau \cdot \mathbf{d}\, dV = \int_{B_0} \mathbf{S}_e \cdot \hat{\hat{\mathbf{d}}}\, dV = \int_{B_0} \mathbf{S} \cdot \frac{\dot{\mathbf{C}}}{2} dV = \int_{B_0} \mathbf{P} \cdot \dot{\mathbf{F}}^T\, dV \tag{18}$$

where σ is the Cauchy stress tensor and the others stress tensors are defined as follows

$\tau = J\sigma$	Kirchhoff stress tensor	(19)
$\mathbf{S}_e = \phi_a^*(\tau) = \mathbf{F}_e^{-1}\tau\,\mathbf{F}_e^{-T}$	Elastic second Piola-Kirchhoff stress tensor	(20)
$\mathbf{S} = \phi^*(\tau) = \phi_e^*(\mathbf{S}_e) = \mathbf{F}_a^{-1}\mathbf{S}_e\mathbf{F}_a^{-T}$	Second Piola-Kirchhoff stress tensor	(21)
$\mathbf{P} = \mathbf{FS}$	First Piola-Kirchhoff stress tensor	(22)

Note that $\mathbf{S}$ and τ are respectively the pull-back and the push-forward of the elastic stress tensor defined in the intermediate configuration.

Constitutive Model

In the paper it will be analysed a material whose constitutive behaviour is defined by the following assumptions:

1. A general hyperelastic potential exists that satisfies the requirements of locality and objectivity. Locality means that the potential depends only on the local value of its arguments, and objectivity means that its value must be constant under rigid motion over imposed on the deformation process. This implies that the potential must depend on the elastic process only throughout the elastic Cauchy-Green deformation tensor defined in eq. (3), so that

$$W(\mathbf{C}_e, \underline{\xi}) \tag{23}$$

   The elastic potential is defined on the intermediate configuration that is assumed to be the natural state of the material.
2. Motivated by the idea that in a No Tension Material inelastic deformation occurs at zero stress, it is assumed that the anelastic deformation occurs with zero dissipation, that is

$$P_e = P_{tot} \Rightarrow D = 0 \tag{24}$$

   Since the dissipation of mechanical power is given by the difference between the total mechanical power minus the time derivative of the elastic potential, in virtue of (18) its explicit expression in the intermediate configuration is

$$D = \mathbf{S}_e \cdot \hat{\hat{\mathbf{d}}} - \dot{W} \tag{25}$$

For an *elastic increment* of a general deformation process $\dot{\mathbf{F}}_a = 0$, so that, using (10), (14) and (23) the elastic constitutive relations are recovered as

$$\begin{aligned} \hat{\hat{\mathbf{d}}} = \hat{\hat{\mathbf{d}}}_e \rightarrow D = \mathbf{S}_e \cdot \hat{\hat{\mathbf{d}}}_e - 2\frac{\partial W}{\partial \mathbf{C}_e} \cdot \hat{\hat{\mathbf{d}}}_e = 0 \\ \rightarrow \mathbf{S}_e = 2\frac{\partial W}{\partial \mathbf{C}_e} \end{aligned} \tag{26}$$

For a general process, using the previous results, (11) and the symmetry of τ, the mechanical dissipation in the intermediate and in the current configuration has the form

$$D = \mathbf{S}_e \cdot \hat{\hat{\mathbf{d}}}_a = \boldsymbol{\tau} \cdot \mathbf{d}_a = \boldsymbol{\tau} \cdot \mathbf{l}_a = 0 \tag{27}$$

so that the anelastic gradient of velocity and the Kirchhoff stress tensor are orthogonal.

The evolution law of the *anelastic deformations* is defined by the condition that the stress state is admissible if the maximum (true) Cauchy stress is not positive

$$\sigma_{n\ \max} \leq 0 \tag{28}$$

Since $\tau = J\sigma$ and $J \geq 0$ always (28) is equivalent to

$$\tau_{n\ \max} \leq 0 \tag{29}$$

which means that the Kirchhoff stress tensor must belong to the subspace K_σ of the negative semidefinite symmetric tensors.
As in the small deformation case [2], anelastic strains develop only if the stress tensor belongs to the boundary of K_σ.

In order to determine the anelastic evolution equations, let's consider a purely anelastic process for which the stresses evolve on the limit surface. The maximum dissipation principle and the admissibility condition (29) are equivalent to a constrained minimization problem which can be solved using the Lagrange multipliers method imposing the stationarity of the functional

$$L = -D + \dot{\beta}\,\tau_{\max} \tag{30}$$

Stationarity of functional (30) is enforced by the following relations

$$\begin{aligned} \nabla_\tau (L) = 0 \rightarrow -\mathbf{l}_a + \dot{\beta}\nabla_\tau(\tau_{\max}) = 0 \\ \nabla_{\dot{\beta}} (L) = 0 \rightarrow \tau_{\max} = 0 \end{aligned} \tag{31}$$

leading to the evolution relations

$$\begin{aligned} \mathbf{l}_a = \dot{\beta}\,\partial_\tau f(\boldsymbol{\tau}) \qquad & f(\boldsymbol{\tau}) = \tau_{n\max} \\ f(\boldsymbol{\tau}) \leq 0 \qquad & \dot{\beta}\, f(\boldsymbol{\tau}) = 0 \end{aligned} \tag{32}$$

The first of (32) implies that anelastic deformations grow in the direction of the maximum stress. Note that in (32) the Lagrange multiplier is not sign constrained.
Being $\tau_{n\max}$ an isotropic function and in virtue of the symmetry of τ, the anelastic gradient of velocity $\mathbf{l}_a$ defined in the previous equation is symmetric, i.e. the anelastic spin tensor is null:

$$\mathbf{l}_a = \mathbf{d}_a = \dot{\beta}\nabla_\tau \tau_{n\max} \qquad \mathbf{w}_a = skew(\mathbf{l}_a) = 0 \tag{33}$$

Furthermore, using the relations (32), one has $\mathbf{l}_a \boldsymbol{\tau} = \mathbf{0}$

proof.

- if $\mathbf{l}_a = \mathbf{0} \rightarrow$ obvious;
- if $\mathbf{l}_a \neq \mathbf{0} \rightarrow \tau_{\max} = 0$ so the stress state is plane. Calling $\underline{\mathbf{b}}$ the normal unit vector to the plane of stress and using (32), for general vectors one has
  $\mathbf{l}_a \underline{\mathbf{m}} = \underline{\mathbf{v}} = |\underline{\mathbf{v}}|\underline{\mathbf{b}}$ $\quad \forall \underline{\mathbf{m}}, \underline{\mathbf{n}} \rightarrow (\boldsymbol{\tau}\underline{\mathbf{n}}) \cdot (\mathbf{l}_a \underline{\mathbf{m}}) = \underline{\mathbf{n}} \cdot (\boldsymbol{\tau}\mathbf{l}_a \underline{\mathbf{m}}) = 0 \forall \underline{\mathbf{m}}, \underline{\mathbf{n}} \rightarrow \boldsymbol{\tau}\mathbf{l}_a = \mathbf{0}$
  $(\boldsymbol{\tau}\underline{\mathbf{n}}) \cdot \underline{\mathbf{b}} = 0$
  having indicated with $|\underline{\mathbf{v}}|$ the modulus of the vector $\underline{\mathbf{v}}$ ■

The entities in Equation (27) are all defined in the current configuration, but, using the identity $D = \tau \cdot \mathbf{d}_a = \mathbf{S}_e \cdot \hat{\hat{\mathbf{d}}}_a$ and Equation (20), the constitutive relations (32) can be formulated in the intermediate configuration as:

$$\hat{\hat{\mathbf{l}}}_a = \mathbf{C}_e \hat{\mathbf{l}}_a = \dot{\beta}\frac{\partial \hat{f}(\mathbf{S}_e)}{\partial \mathbf{S}_e} = \hat{\hat{\mathbf{d}}}_a \tag{34}$$

since $$sym(\mathbf{d}_a) = \dot{\beta}\frac{\partial \hat{f}(\tau(\mathbf{S}_e))}{\partial \tau} = \dot{\beta}\, \mathbf{F}_e^{-T} \frac{\partial \hat{f}(\mathbf{S}_e)}{\partial \mathbf{S}_e} \mathbf{F}_e^{-1} = \mathbf{F}_e^{-T} \hat{\hat{\mathbf{d}}}_a \mathbf{F}_e^{-1} \tag{35}$$

In (35) $\hat{f}$ is the functional expression of $\tau_{\max}$ in terms of the elastic second Piola Kirchhoff stress tensor $\mathbf{S}_e$.
The anelastic evolution relation (34) for the anelastic rate of deformation will not be used in the numerical implementation.

TIME STEP INTEGRATION OF THE CONSTITUTIVE MODEL

The model will be integrated by time step linearisation, so that rate form of the constitutive equations are required. From (26), the time derivative of $\mathbf{S}_e$ is given by

$$\dot{\mathbf{S}}_e = 4\frac{\partial^2 W}{\partial \mathbf{C}_e \partial \mathbf{C}_e} : \frac{\dot{\mathbf{C}}_e}{2} = \mathsf{C} : \frac{\dot{\mathbf{C}}_e}{2} \tag{36}$$

where C represents the elastic fourth order tensor. Since $\dot{\mathbf{S}}_e$ is not an objective stress measure, the objective rate form of the constitutive equations is obtained introducing the Lie derivative of the stress tensor. It is defined as the push-forward into the configuration where the tensor is defined of the time derivative of the pull-back of the stress tensor into the undeformed configuration. For the stress tensor $\mathbf{S}_e$ defined on the intermediate configuration one has the anelastic Lie rate

$$\begin{aligned} L_v^p \mathbf{S}_e &= \phi_{a*}\left(\frac{\partial}{\partial t}\phi_a^* \mathbf{S}_e\right) = \left\{\mathbf{F}_a \frac{\partial}{\partial t}\left[\mathbf{F}_a^{-1}(\mathbf{S}_e)\mathbf{F}_a^{-T}\right]\mathbf{F}_a^T\right\} = \\ &= \dot{\mathbf{S}}_e - \hat{\mathbf{l}}_a \mathbf{S}_e - \mathbf{S}_e \hat{\mathbf{l}}_a^T \end{aligned} \tag{37}$$

The anelastic pull-back of $L_v^p \mathbf{S}_e$ gives the material derivative of the II Piola - Kirchhoff stress tensor defined in the undeformed configuration,

$$\dot{\mathbf{S}} = \mathbf{F}_a^{-1}\left[\mathsf{C} \cdot \frac{1}{2}\dot{\mathbf{C}}_e - \hat{\mathbf{l}}_a \mathbf{S}_e - \mathbf{S}_e \hat{\mathbf{l}}_a^T\right]\mathbf{F}_a^{-T} \tag{38}$$

and its elastic push forward gives the Lie derivative of the Kirchhoff stress tensor defined in the current configuration

$$\begin{aligned} L_v \boldsymbol{\tau} &= \mathbf{F}_e L_v^p \mathbf{S}_e \mathbf{F}_e^T = \mathbf{F}_e \dot{\mathbf{S}}_e \mathbf{F}_e^T - \mathbf{F}_e \hat{\mathbf{l}}_a \mathbf{F}_e^{-1} \boldsymbol{\tau} - \boldsymbol{\tau} \mathbf{F}_e^{-T} \hat{\mathbf{l}}_a^T \mathbf{F}_e^T = \dot{\boldsymbol{\tau}} - \mathbf{l}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{l}^T = \\ &= \mathsf{c}_{\mathbf{e}} : \mathbf{d}_e - \mathbf{l}_a \boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{l}_a^T \end{aligned} \tag{39}$$

where the result $\dot{\mathbf{S}}_e = \mathbf{F}_e^{-1}\left(\dot{\boldsymbol{\tau}} - \mathbf{l}_e \boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{l}_e^T\right)\mathbf{F}_e^{-T}$ (see [12]) and Equation (7) have been used. The tensors C, c_e are the 4th order material and spatial elastic tensors, related through the transformation

$$\mathsf{c}_{\mathbf{e}}^{\;abcd} = F_e{}^d{}_\delta F_e{}^c{}_\gamma F_e{}^b{}_\beta F_e{}^a{}_\alpha \mathsf{C}^{\alpha\beta\gamma\delta} \tag{40}$$

In (39) the derivative of the Kirchhoff stress tensor is the material one

$$\dot{\tau}(\underline{\mathbf{x}},t) = \left(\frac{\partial}{\partial t}\left(\tau(\underline{\mathbf{x}},t)\circ\varphi(\underline{\mathbf{X}},t)\right)\right)\circ\varphi^{-1}(\underline{\mathbf{x}},t) = \frac{\partial}{\partial t}\tau + \left(\nabla_{\underline{\mathbf{x}}}\tau\right)\underline{\mathbf{v}} \tag{41}$$

Since $\mathbf{l}_a \boldsymbol{\tau} = \mathbf{0}$, using Equation (39) the Lie derivative of the Kirchhoff stress tensor is given by

$$L_v \boldsymbol{\tau} = \mathsf{c}_{\mathbf{e}} \cdot \mathbf{d}_e \tag{42}$$

By virtue of the additive decomposition of the gradient of velocity (therefore also of its symmetric part) expressed in (7), the total velocity of deformation tensor is the sum of its elastic and its anelastic components. The first term can be calculated inverting the constitutive relation (42), while the second is given from Equation (33) as follows

$$\mathbf{d} = \mathbf{d}_e + \mathbf{d}_a = \mathsf{c}_e^{-1} \cdot L_v \tau + \dot{\beta}\nabla_\tau \tau_{\max} \tag{43}$$

Equation (43) is perfectly analogous to the model defined in the field of small deformations where it is assumed the existence of a complementary energy potential defined as

$$\phi^c(\sigma) = \phi_e^c(\sigma) + \phi_a^c(\sigma) = \phi_e^c(\sigma) + ind(K_\sigma) \tag{44}$$

being $K_\sigma = \{\sigma : \sigma_{n\max} \le 0\}$ the domain of the admissible stress tensors and $ind(\cdot)$ the indicator function. From Equation (44) the deformation field is then given by

$$\varepsilon = \frac{\partial \phi^c}{\partial \sigma} = \frac{\partial \phi_e^c}{\partial \sigma} + \frac{\partial (ind\, K_\sigma)}{\partial \sigma} = \varepsilon_e + \beta \nabla_\sigma \sigma_{\max} \tag{45}$$

In the paper the following expression of the elastic potential has been adopted

$$W = \frac{\lambda}{4}\left(J_e^2 - 1\right) - \left(\frac{\lambda}{2} + \mu\right) \lg(J_e) + \frac{1}{2}\mu(I_1 - 3) \tag{46}$$

where μ, λ are material constants, I_1 is the first invariant of $\mathbf{C}_e$ and J_e is the elastic jacobian, that coincides with the volumetric invariant of the elastic deformation. The expression of the material elastic tensor corresponding to the potential (46) is

$$\mathbf{c}_{\mathbf{e}}^{\ abcd} = \lambda J_e^2 g^{ab} g^{cd} + \left[\lambda \frac{1 - J_e^2}{2} + \mu\right]\left(g^{ac} g^{bd} + g^{ad} g^{bc}\right) \tag{47}$$

The model adopted presents, therefore, some characteristics that are not present in the usual elasto-plastic models:

a. the elastic potential depends on the elastic Jacobian J_e;
b. anelastic deformations are not isochoric, in fact, according to the evolution law (34), for a general process we have

$$\dot{J}_a = \frac{\partial J_a}{\partial (F_a)^\alpha{}_A} \left(\dot{F}_a\right)^\alpha{}_A = J_a \left(F_a^{-1}\right)^A{}_\alpha \left(\dot{F}_a\right)^\alpha{}_A = J_a\, tr\left(\hat{\mathbf{l}}_a\right) \ne 0 \tag{48}$$

so that J_e has to be determined iteratively at each step increment;
c. in (32) it must be noted the multiplier β is not sign constrained.

Properties a.-c. do not make possible to extend to the present model one of the standard algorithms developed for finite elasto-plasticity like the exponential integration algorithm [13, 14].

In the following paragraph an original integration procedure is introduced. The general form and a detailed proof of the algorithm are presented in [15] and will be briefly reviewed in part II in connection with the case of no tension material.

COMBINED COMPRESSION-SHEAR DEFORMATION PROCESS

In order to investigate the main properties of the model so far developed, in this section it is applied to a simple homogeneous deformation process. The hyper-elastic potential (46) is used, with the values $\mu = 0.6$, $\lambda = 0.5$ MPa.
A prismatic bar is first compressed in the X_2 direction up to a stretch of 0.50, constraining the lateral deformation to zero. Then a pure shear is applied in the X_1 direction, assigning a linear displacement along the X_2 axis (Figure 2)

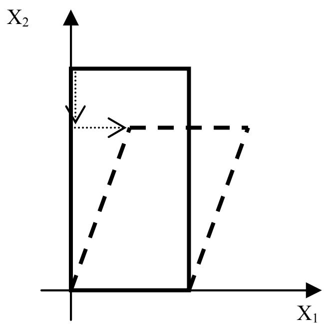


Figure 2. Combined compression-shear deformation process

The compression process is purely elastic. Applying the shear deformation, the process grows elastically until the condition $\tau_{max}=0$ is reached (see Figures 3, 4 and 5). In this part of the deformation process the τ_{12} component grows linearly, τ_{22} remains constant, while $|\tau_{11}|$ decreases according to the hyperelastic constitutive law. When the admissibility stress condition is reached the process became inelastic. The results are shown in Figures 3, 4 and 5, where two values of the pre-compression where used, 10% and 50%. The plots present the evolution of the stress state after the initiation of the fracturing strains. The process is then reversed and during unloading the same stress state of the loading process is recovered. Comparison with the infinitesimal theory is also included in the graphs.
The evolution of the angle of fracture (that is, the direction of the principal value of $\nabla_\tau g$) is presented in Figure 6.a. In this figure, θ_{Bt} is the angle between the normal to the fracture and the x_2 direction in the current configuration, and θ_{B0} is the angle between the pull back on the reference configuration of the normal to the fracture, and the X_2 direction.
The differences with the predictions of the infinitesimal theory, apparent from the plots of Figures 3-6.a, are underlined by the results of Figure 6.b, where the limit elastic shear F_{12} is plotted against the axial pre-compression. In any case, the infinitesimal theory predicts a larger value of the limit shear. The limit shear F_{12} for the model examined in the finite deformations theory is given by

$$F_{12} = \sqrt{\frac{(1-F_{22})^2 \lambda \left[-4 F_{22} (\lambda + 2\mu) - (1-F_{22})^2 (\lambda + 2\mu)\right]}{\mu\left[-2\left(1-F_{22}^2\right)\lambda - 4\mu\right]}} \tag{93}$$

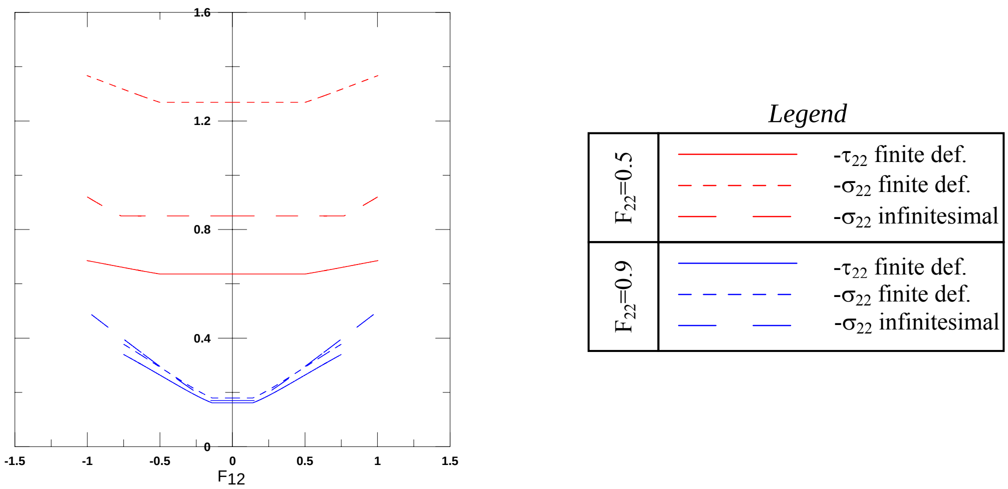


Figure 3. Vertical stress vs. shear deformation

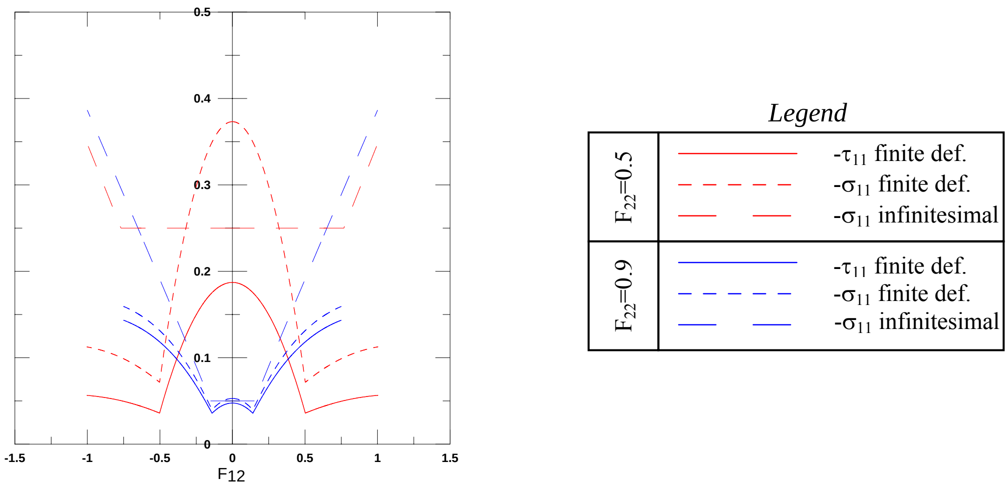
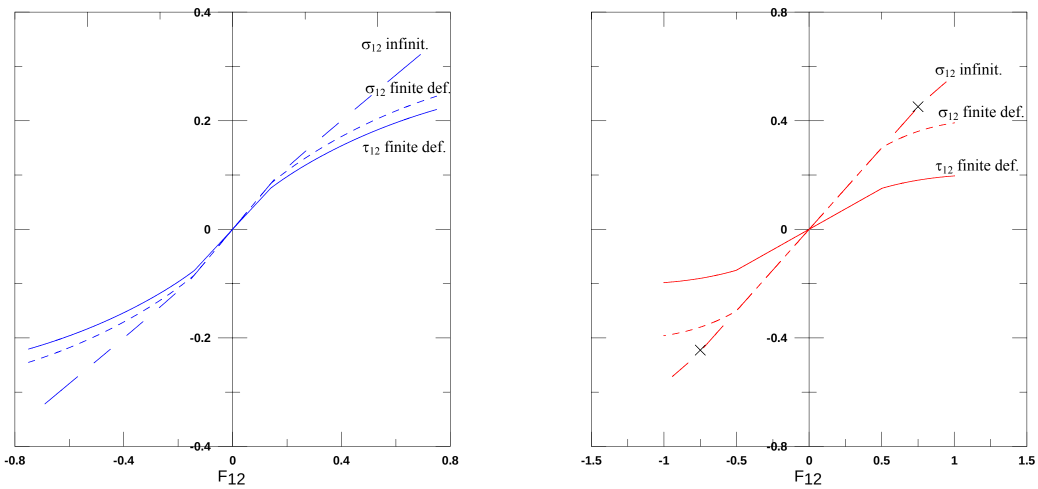
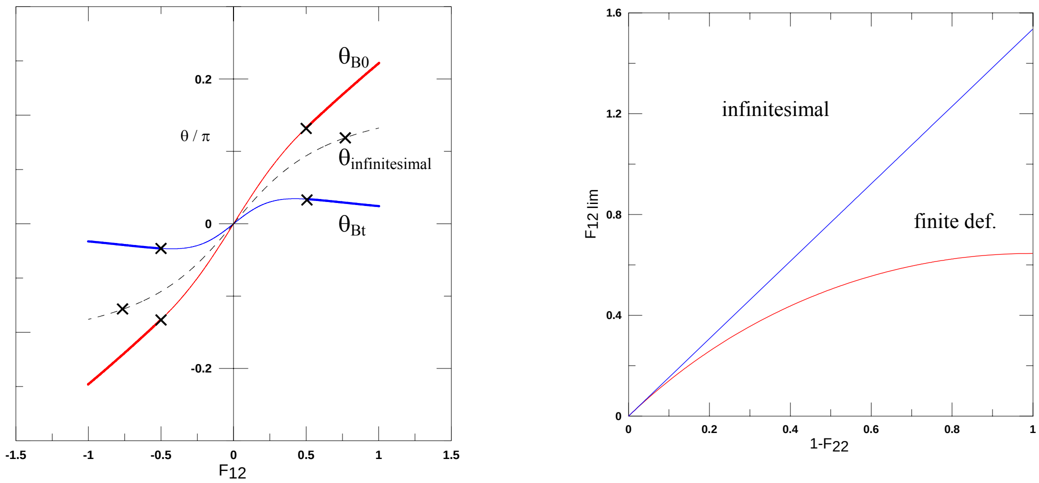


Figure 4. Lateral stress vs. shear deformation



(a) (b)

Figure 5. Shear stress vs. shear deformation. (a) F_{22} = 0.90; (b) F_{22} = 0.50



(a) (b)

Figure 6. (a) Evolution of fracturing angle vs. shear deformation
(b) Limit shear deformation vs. factor of precompression

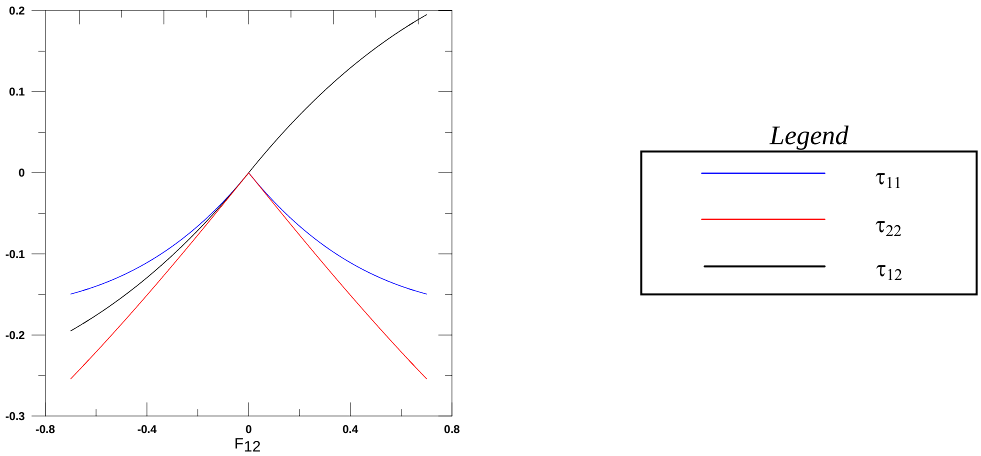


Figure 7. Pure shear deformation process evolution of stress state

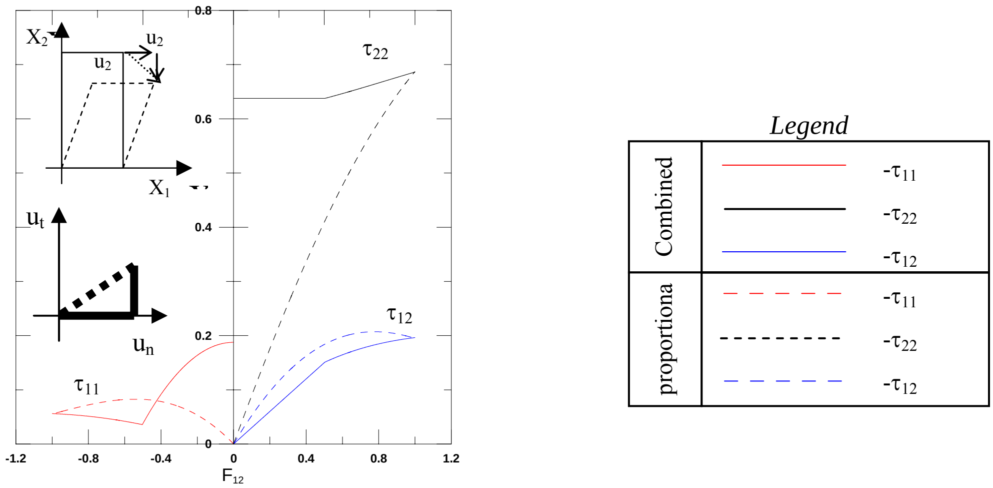


Figure 8. Combined and proportional process

The limit case of a pure shear process without axial compression is presented in Figure 7. Of course, in this case, fracture strains evolve since the beginning of the process. The deformation is isochoric, but $J_e = \sqrt{I_3^e}$ (for Cartesian framework) is not constant.

A final numerical simulation concerns the numerical verification of the path-independency of the stress state (see Figure 8). The same final configuration, characterized by F_{11}=1, F_{12}=1, F_{21}=0, F_{22}=0.5 is reached following two alternative strain-histories. In one case first compression up to its final value is applied as in the simulation reported in Figures 3-6. In the second case compression and shear are made to grow proportionally. The results of Figure 8 show that the stress state in the final stage are the same.

CONCLUSIONS

In this paper a constitutive model for unilateral (no tension) materials has been proposed. The model has been developed assuming that the deformation process can be decomposed in its elastic and anelastic part using the multiplicative decomposition of the gradient of deformation tensor. This decomposition implies an additive split of the velocity of deformation. An hyperelastic potential and an evolution law for the anelastic velocity of deformation have been presented. The hyperelastic potential rules the elastic part of the deformation process and depends only on the elastic deformations and on the Jacobian of the elastic gradient of deformation, in order to take into account non isochoric deformations. The evolution of the anelastic velocity of deformation derives from the assumption of absence of dissipation in the deformation process, so that the anelastic velocity of deformation grows in the direction of the maximum stress. A full set of evolution relations has been proposed with reference either to the final configuration and to the intermediate configuration. To this end covariant velocity of deformation tensors, dual in power to the stress tensors, have been introduced in the final and in the intermediate configuration. It is possible to show that the model so defined is reversible and path independent. [16]

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