NONLINEAR THIN BEAM BENDING MODEL

Jorma Kinnunen

Journal of Structural Mechanics,
Vol. 37, No. 1, 2004, pp. 3-11

ABSTRACT

The nonlinear thin beam bending model presented in the article uses a relatively simple numerical method for solving highly nonlinear planar beam problems. In this model, the curvature of the beam is not approximated by the second derivative of the deflection as in conventional small deflection models. The amount of the rotation of the beam is not limited. The deformation caused by normal and shear forces is neglected.

INTRODUCTION

When the curvature of an initially straight beam gets high, the conventional straight beam formulas can not be used anymore. When the problem gets very nonlinear, the final geometry may depend of the loading history. If the loading history is unknown, we obtain multiple solutions. Nonlinear beam problems do not usually have analytical solutions. The elastic curve of the beam has to be solved numerically, and the solution has to be iterated in most cases, as the forces acting on the beam may depend on the shape of the beam. An example of a nonlinear problem is illustrated in figure 1. While rotating the beam around the point A at angle α,the beam slides over the support B. At the supporting point we have a reaction force with normal component F and the friction force μF. The beam is also loaded by the weight of the beam mg and a pure bending moment M.

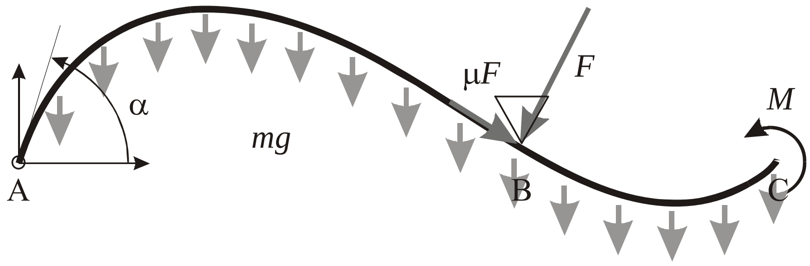

Figure 1. Bending of a beam with nonlinear geometry

FORMULATION OF THE MODEL

A nonlinear beam model can be formulated by dividing the beam in short sections and assuming that the bending moment *M* is constant inside a section. If the beam is thin, we can ignore the deformation caused by normal and shear forces.

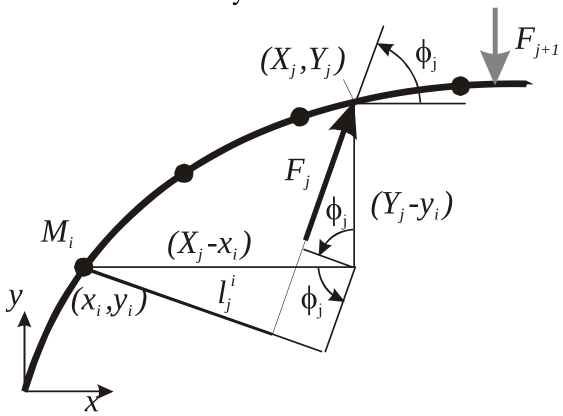

Figure 2. Bending moment at a node caused by a force.

To determine the shape of the beam we have to calculate the bending moments at the nodes (endpoints of the sections). The total moment at a node is the sum of the moments of the forces acting on the right-hand side of the node:

$$M_i = \sum_j l_j^i F_j \tag{1}$$

Inspecting the geometry in figure 2 we can find out the expression of the distance l_j^i :

$$l_j^i = (X_j - x_i)\sin(\phi_j) - (Y_j - y_i)\cos(\phi_j) \tag{2}$$

If an initially straight beam (or a short section of the beam) with length s is loaded by a constant bending moment M and the beam has bending stiffness B, the elastic curve of the beam or the beam section is a circular arc with radius of curvature $R = B/M$. We take the average of the values at the end nodes as the bending moment of the beam section *e* between the nodes *i* and *i*+1 :

$$M_e = (M_i + M_{i+1})/2 \tag{3}$$

If the coordinates and the inclination angle at the left end are x_i, y_i and θ_i we obtain from figure 3 the equations (4a)…(4c). These equations can be simplified to shorter form, but the form presented here behaves well in numerical calculations when the curvature of the beam section is very low (radius R_e is very high).

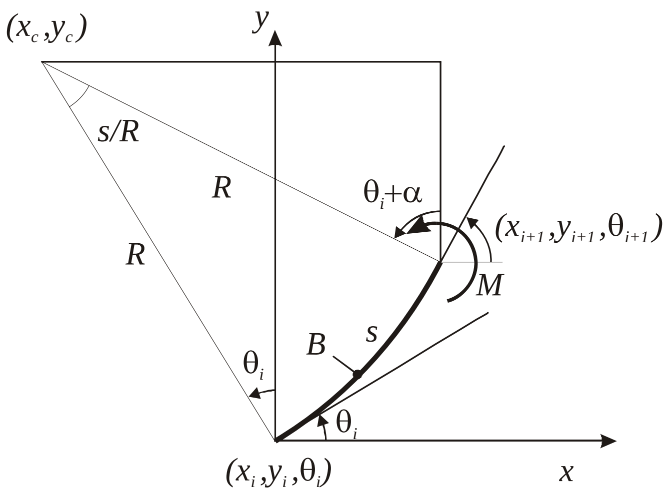

Figure 3. Geometry of a circular arc with radius R

$$x_{i+1} = x_i + R_e\left(\sin(s_e / R_e)\cos(\theta_i) + \cos(s_e / R_e)\sin(\theta_i) - \sin(\theta_i)\right) \quad (4\text{ a})$$
$$y_{i+1} = y_i + R_e\left(\sin(s_e / R_e)\sin(\theta_i) - \cos(s_e / R_e)\cos(\theta_i) + \cos(\theta_i)\right) \quad (4\text{ b})$$
$$\theta_{i+1} = \theta_i + s_e / R_e \quad (4\text{ c})$$

where

$$R_e = B_e / M_e \quad (4\text{ d})$$

The subscript e refers to the element number between nodes. Equations (4) are valid if the beam section has curvature. If there is no bending moment and the beam section is straight

$$x_{i+1} = x_i + s_e \cos(\theta_i) \quad (5\text{ a})$$
$$y_{i+1} = y_i + s_e \sin(\theta_i) \quad (5\text{ b})$$
$$\theta_{i+1} = \theta_i \quad (5\text{ c})$$

After calculating the bending moments we can join the arcs of the short sections to form the elastic curve of the beam.

ITERATION PROCEDURE

If only pure bending moments are acting on the beam, the shape of the beam has no effect to the bending moments of the beam sections. In all other cases, we have to iterate the calculation until the solution is acceptable. We can take the average displacement of nodes between iteration steps as the measure

$$e = \frac{1}{n}\sum_{i=1}^{n}\sqrt{\left(x_i^t - x_i^{t-1}\right)^2 + \left(y_i^t - y_i^{t-1}\right)^2} \quad (6)$$

of the acceptability of the solution. Here e is the error norm, n is the number of nodes and t denotes the current iteration step. When the shape of the beam does not change anymore the system is in equilibrium. In numerical calculations, we have to allow some small number, which is greater than zero, as an error tolerance e_{tol}.

The main problem in the solution is the convergence of the iteration process. Usually, we have to increase the loads gradually so that the system does not go too far from the equilibrium state. When our problem is highly nonlinear, we have to use some damping between iteration steps to achieve stability. In the beginning of the iteration step t, we have the nodal coordinates of the beam and the bending moments at the nodes M_i^{t-1} from the previous iteration step. In the current iteration step, we calculate an estimate of the new bending moments $\hat{M}_i^t$ at the nodes, using the geometry of the previous step by equations (1) and (2). In very nonlinear problems we have to interpolate between the old and new bending moments using the damping factor α:

$$M_i^t = (1-\alpha)\hat{M}_i^t + \alpha M_i^{t-1} \tag{7}$$

If the damping factor is zero, we do not use bending moment information from the previous step. If the damping factor is 1, the solution does not change between iterations. In very nonlinear problems the damping factor has to be quite high to achieve stability. In example 2 of this paper, damping factor α=0.95 has been used.

An example of an iteration procedure for problems where forces are known in advance, and the shape of the beam is calculated, can be described roughly as follows:

```
Set initial values of forces applied to the beam (usually zero)
Calculate initial geometry (usually straight beam)
LOOP 1 : ( Iteration of the external forces)
        Increase forces applied to the beam
        LOOP 2: (Iteration of the beam geometry)
                Calculate bending moments of the nodes. Equations (1),(2),(7)
                Calculate new geometry of the beam. Equations (3), (4) and (5)
                Calculate difference between new and old geometry. Eq. (6)
        END:    If the difference is greater than tolerance, go to LOOP 2
END: If the forces do not have the final values, go to LOOP 1
Print and plot the results
```

EXAMPLE 1: BEAM LOADED BY OWN WEIGHT

We calculate the end deflection of a straight cantilever beam loaded by its own weight starting from the classic small deflection Euler-Bernoulli beam solution. Solving it for the bending stiffness[1] B, and multiplying by a nonlinearity correction factor k, we get equation

[1] For a homogenous beam, made of isotropic material, the bending stiffness is $B=EI$, where E is the elastic modulus of the material and I is the moment of inertia of the cross section.

$$B = \frac{mgl^4}{8d} k \qquad (8)$$

where m is the mass off the beam per unit length, g is the acceleration of gravity and other dimensions are according to figure 4 a. If the elastic curve of the deflected beam is normalized by beam length, the geometric nonlinearity depends only of the normalized deflection d/l. Then $k=k(d/l)$.

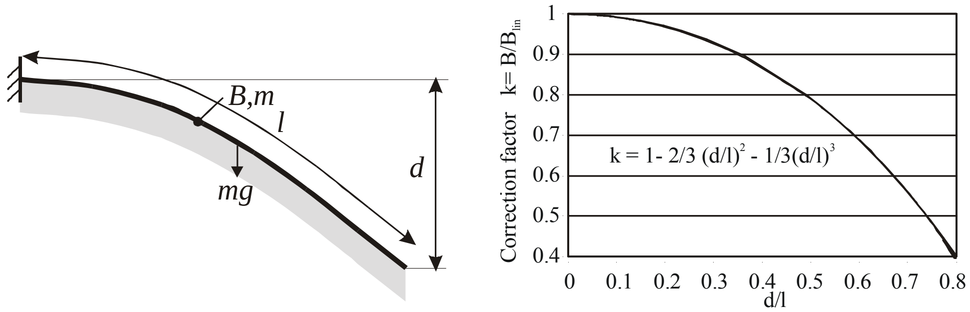

Figure 4. Beam loaded by own weight (a). Correction factor for equation (8)

Using the nonlinear thin beam model we calculate the numerical values of k for different values of normalized deflection. Fifty elements have been used in this calculation. The numerical values can be fitted quite accurately to the polynomial

$$k = 1 - \frac{2}{3}\left(\frac{d}{l}\right)^2 - \frac{1}{3}\left(\frac{d}{l}\right)^3 \qquad (9)$$

The numerical solution and the graph of the polynomial are shown in figure 4 b. The maximum difference in k between curves is 0.005, when the deflection is less than 80% of the beam length. However, this polynomial is *not* the analytical solution of the problem. Equation (8) with this correction factor (9) can be used as a simple method for measuring the bending stiffness of a thin flexible beam. Those equations have also one real solution for deflection d. This solution is quite complicated and it is not presented here.

EXAMPLE 2: PROBLEM WITH MULTIPLE SOLUTIONS

An initially straight beam according to figure 5 a is loaded by a bending moment M and a force **F** at the right end. B is the bending stiffness of the beam. What is the elastic curve of the beam when the x- and y-direction components of the force are 100 and -100, and the bending moment $M = 4\pi$?

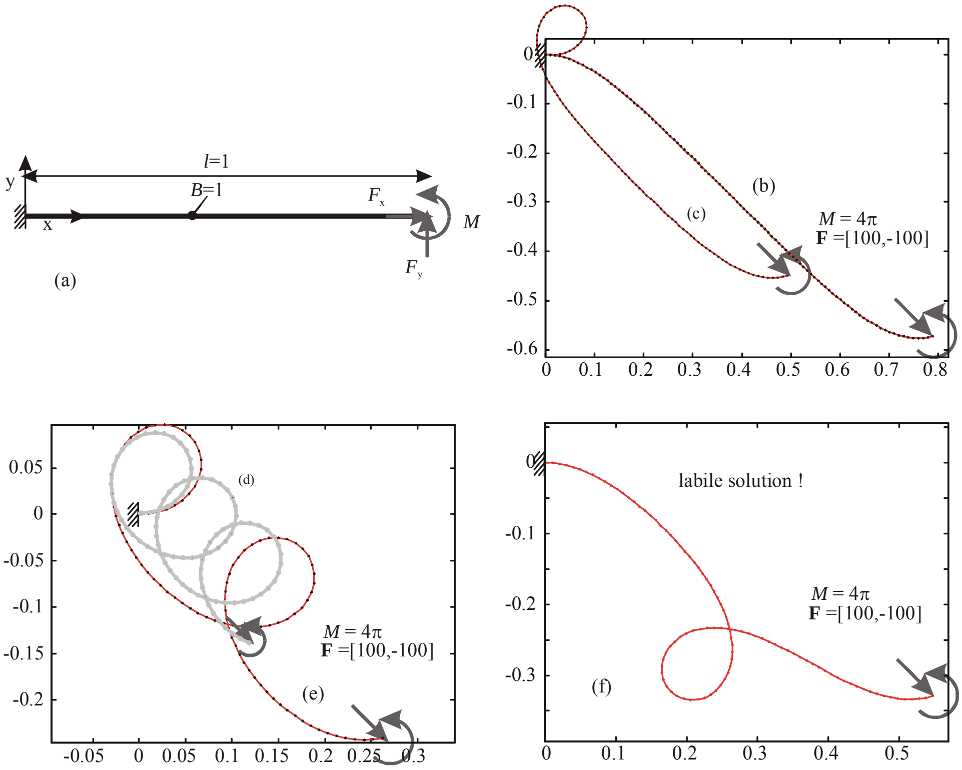

Figure 5. The initial configuration (a) , and the solutions (b …f) of the spring problem.

To solve this problem, 100 elements have been used. The solution depends on the path of loading. If we apply the moment and the force at the same time, we end up to the simple solution (b). If we first load the spring by the moment $M=4\pi$, the spring makes two full circles. If we then add the force, we obtain the solution (e). The solutions (c) and (d) have been found by applying first half of the moment, then double the final moment. The problem might also have labile solutions. If we first make a loop by using a bending moment at the opposite direction ($M=-2\pi$) , and then apply the final force and moment, we find a solution (f), which is very close to be in equilibrium. It is hard to find the stationary point using numerical iteration. However, this might be a true solution in real life, if the material is not perfectly elastic.

COMPARISON WITH ANALYTICAL RESULTS

Some elementary cases of nonlinear beam problems have been solved by analytical methods using elliptic functions and integrals. Frisch-Fay (1962) gives the solution for "Vertical strut under vertical load" according to figure 6. This example has been used as a benchmark for the numerical method in this article.

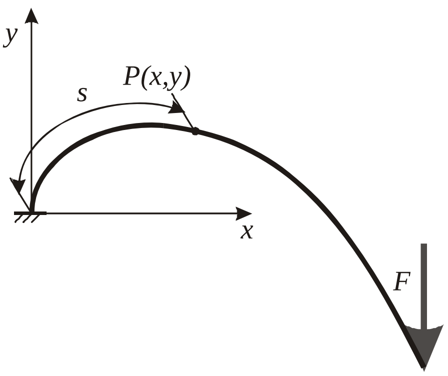

Figure 6. Vertical strut under vertical load.

An initially vertical bar having length L and bending stiffness B , fixed at the bottom, is subject to a vertical load F at the top. If the bar is sufficiently flexible it will buckle and take the shape in figure 6. First we have to solve the modulus p governing the shape of the bar from equation

$$L = \sqrt{\tfrac{B}{F}}\, K(p) \tag{10}$$

where $K(p)$ is the complete elliptic integral of the first kind. This equation can not be solved for p but the value can be calculated by numerical iteration. After solving the modulus p, the coordinates $x(s)$ and $y(s)$ of the elastic curve can be calculated from

$$x = 2p\sqrt{\tfrac{B}{F}}\left(1 - cn\left(p, s\sqrt{\tfrac{F}{B}}\right)\right) \tag{11}$$

$$y = 2\sqrt{\tfrac{B}{F}}\, E\left(p, am\left(p, s\sqrt{\tfrac{F}{B}}\right)\right) - s \tag{12}$$

where s is the distance along the bar, $E(p,\phi)$ is the elliptic integral of the second kind [2], $am(p,u)$ is the amplitude of u and cn denotes the Jacobi's elliptic function $cn(p,u) = \cos(am(p,u))$.

Figure 7 shows the elastic curves calculated using analytical formulas (10…12), and exploiting the numerical method presented in this article. The parameter values for this benchmark have been: L = 1, B = 1 and F=10. The critical buckling load of this beam is F_{cr} = 2.4674. In numerical calculations we have used a lateral excitation force in the first loading step to make the beam buckle. Different number of elements between 1 and 1000 have been tested. At least in this case, the error compared to analytical results decreases, when the number of elements increases. The average position error of nodes 2…NE+1 is shown in figure 8.

[2] This article uses the notation of Frisch-Fay for the elliptic functions. *Mathematica* ® program, for instance, uses different convention: $K(p)$=EllipticK[p^2] , $E(p,\phi)$=EllipticE[ϕ, p^2], $cn(p,u)$= JacobiCN[u,p^2] and $am(p,u)$= JacobiAmplitude[u,p^2]

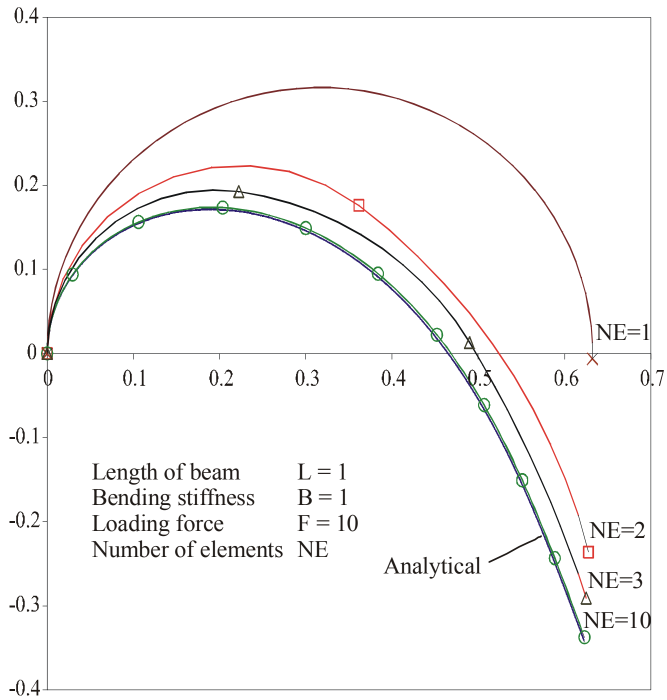

Figure 7. Analytical and numerical solution for vertical strut under vertical load.

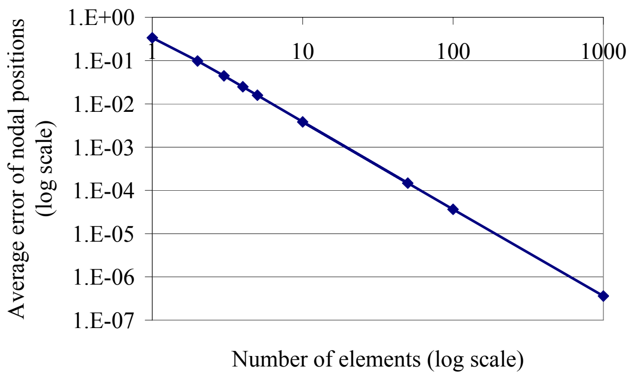

Figure 8. Average position error of nodes compared to analytical results

CONCLUSIONS

This paper, for simplicity, has concentrated on relatively simple statically determinate cases of beam problems where the initial shape of the beam has been straight. Arbitrary initial curvature of the beam can be easily added to the method. The procedure can also be expanded to statically indeterminate cases, where the forces acting to the beam depend on the shape of the beam. In those cases we need an extra loop for iterating the forces wrapped around the procedure explained in the chapter: "iteration procedure".

The accuracy of this method can be increased by adding the number of elements in the model. Adding the number of elements increases the computing time linearly, whereas in some other methods the computing time and memory requirements grow exponentially.

Author has used this procedure also as an embedded subroutine in a simulation program simulating complicated mechanisms including flexible beams.

REFERENCES

Frisch-Fay, R., 1962, *Flexible Bars*, Butterworth & Co. Limited, London

Jorma Kinnunen, development manager

Metso Paper Inc.
Wärtsilänkatu 100, 04400 Järvenpää
jorma.kinnunen@metso.com