ELEMENT BY ELEMENT POSTPROCESSING IN THE FINITE ELEMENT METHOD

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ABSTRACT

This paper deals with postprocessing method by which the result of a conventional finite element analysis can be improved. If the improved solution is good enough, difference between the improved solution and the exact solution can be used as an approximaton for the actual error. Thus the error of the conventional finite element analysis can be computationally estimated /3/.

INTRODUCTION

An efficient, element by element recovery procedure, which uses local polynomial representation containing the information from the field equations using a technique based on truncated polynomial series of the residual of the field equation, has recently been presented by the authors in reference /1/. This paper presents a similar, slightly different procedure, in which the information from the field equations has been put into the representation using the method of weighted residuals.

BOUNDARY VALUE PROBLEM

Consider a boundary value problem in two dimensions governed by a set of n second order linear partial differential equations of form

$$\mathcal{L}u + f = 0, \qquad \text{in } \Omega, \tag{1}$$

where $u(x,y)$ is $n\times1$ vector of unknown functions of the problem, $\mathcal{L}$ is $n\times n$ differential operator matrix and $f(x,y)$ is $n\times1$ vector of known functions.

LOCAL POLYNOMIAL REPRESENTATION

Let us first represent the unknown $u(x,y)$ locally within the recovery element considered using a polynomial of degree p as

$$\begin{aligned}\tilde{u} &= u_{00}\\ &+u_{10}\lambda + u_{11}\mu\\ &+u_{20}\lambda^2 + u_{21}\lambda\mu + u_{22}\mu^2\\ &\cdots\\ &+u_{p0}\lambda^p + u_{p1}\lambda^{p-1}\mu + \cdots + u_{p(p-1)}\lambda\mu^{p-1} + u_{pp}\mu^p\end{aligned} \tag{2}$$

where

$$\lambda = \frac{x - x_0}{h}, \qquad \mu = \frac{y - y_0}{h} \tag{3}$$

are dimensionless co-ordinates, x and y are physical co-ordinates, x_0 and y_0 are co-ordinates of the element center and h is a characteristic length of the element. The parameters u_{rs} are unknown.

The representation (2) can be expressed in matrix form as

$$\tilde{u} = P^p U^p, \tag{4}$$

where

$$P^p = \begin{bmatrix} I & \lambda I & \mu I & \lambda^2 I & \lambda\mu I & \mu^2 I & \cdots & \lambda^p I & \lambda^{p-1}\mu I & \cdots & \lambda\mu^{p-1} I & \mu^p I \end{bmatrix} \tag{5}$$

and

$$U^{p} = \begin{Bmatrix} u_{00} \\ u_{10} \\ u_{11} \\ u_{20} \\ u_{21} \\ u_{22} \\ \vdots \\ u_{p0} \\ u_{p1} \\ \vdots \\ u_{p(p-1)} \\ u_{pp} \end{Bmatrix}, \tag{6}$$

where $\boldsymbol{I}$ is a $n \times n$ unit matrix and the superscripts in matrices $\boldsymbol{P}^{p}$ and $\boldsymbol{U}^{p}$ refer to the degree of the polynomial representation of $\tilde{\boldsymbol{u}}$.

"BUILDING-IN" THE FIELD EQUATIONS

It is demanded that the local representation (4) satisfies the field equation (1) within the recovery element in an average sense. This is done by first forming residuals of the field equations

$$\mathcal{R}(\tilde{\boldsymbol{u}}) = \mathcal{L}\tilde{\boldsymbol{u}} + \boldsymbol{f} \tag{7}$$

and then writing weighted residual equations of form

$$\int_{A^e} \left(\boldsymbol{P}^{p-2}\right)^{T} \mathcal{R}(\tilde{\boldsymbol{u}}) dA = \boldsymbol{0}. \tag{8}$$

With the help of equations (4) and (7) these equations reduce to

$$\boldsymbol{H}\boldsymbol{U}^{p} + \boldsymbol{F} = \boldsymbol{0}, \tag{9}$$

where

$$H = \int_{A^e} \left(P^{p-2}\right)^T \mathcal{L} P^p \, dA \tag{10}$$

and

$$F = \int_{A^e} \left(P^{p-2}\right)^T f \, dA. \tag{11}$$

Equations (9) are $n(p-1)p/2$ constraint equations between the $n(p+1)(p+2)/2$ parameters U^p. Thus it is possible to choose $n(p+1)(p+2)/2-n(p-1)p/2=n(2p+1)$ of these parameters as independent parameters a^p. The remaining $n(p-1)p/2$ of the parameters U^p can be solved in terms of a^p using equations (9). This solution is possible, if the independent parameters a^p have been chosen appropriately /2/ and requires inversion of a square matrix of dimensions $n(p-1)p/2$. Consequently the original parameters U^p can be expressed in terms of the new, independent parameters a^p by

$$U^p = S^p a^p + T^p, \tag{12}$$

where S^p and T^p are matrices of constants /2/. Substituting the result (12) into equation (4) finally gives

$$\tilde{u} = N^p a^p + u_0^p, \tag{13}$$

where

$$N^p = P^p S^p \tag{14}$$

and

$$u_0^p = P^p T^p. \tag{15}$$

Equation (13) is local polynomial representation of the unknown, which contains the information from the field equation 'built-in". The corresponding representation for the derivative quantities of interest $\gamma \equiv \nabla u$ is obtained straightforwardly and is

$$\tilde{\gamma} = B^p a^p + \gamma_0^p, \tag{16}$$

where

$$B^p = \nabla N^p \tag{17}$$

and

$$\gamma_0^p = \nabla u_0^p. \tag{18}$$

EQUATIONS FOR THE UNKNOWNS

Two types of equations for the unknown parameters a^p have been used.

First it is demanded that at each node i of the recovery element the smoothed unknown $\tilde{u}$ is equal to the corresponding nodal value u_i (of the original finite element approximation $\hat{u}$). Thus we get equations

$$\tilde{u}(x_i, y_i) = u_i, \quad i = 1, \cdots, m, \tag{19}$$

where m is the number of element nodes.

Second it is demanded that the local derivative quantities $\tilde{\gamma}$ coincide with the corresponding consistent ones $\hat{\gamma}$ at special sampling points (superconvergent points of the derivatives or other suitable sampling points) within the element. Thus we get equations

$$\tilde{\gamma}(x_i, y_i) = \hat{\gamma}, \quad k = 1, \cdots, n, \tag{20}$$

where n is the number of sampling points of the element.

The least squares technique is used to solve this overdetermined set of equations. The equations are first made dimensionally homogeneous by multiplying the second equation type with the characteristic length h. Thus the corresponding least sqares function is

$$\begin{aligned} \Pi = & \sum_{i=1}^{m} w_1^2 [\tilde{u}(x_i, y_i) - u_i]^T [\tilde{u}(x_i, y_i) - u_i] \\ & + \sum_{k=1}^{n} w_2^2 h^2 [\tilde{\gamma}(x_k, y_k) - \hat{\gamma}(x_k, y_k)]^T [\tilde{\gamma}(x_k, y_k) - \hat{\gamma}(x_k, y_k)] \end{aligned} \tag{21}$$

where the summation indices i and k refer to nodes and derivative sampling points, respectively and w_1 and w_2 are dimensionless weighting factors, whose values can be found by numerical experiments.

GLOBAL SMOOTHED DERIVATIVE QUANTITIES

The global smoothed derivative quantities $\bar{\gamma}(x,y)$ are expressed by using the same C°-continuous finite element approximation as the original finite element solution $\hat{u}(x,y)$ of the problem. The nodal values γ_i of $\bar{\gamma}$ are obtained by evaluating element by element the values of the local representation $\tilde{\gamma}$ at the element nodes and taking nodal averages (weighted by element areas) of them.

NUMERICAL EXAMPLE

Plane elasticity problem

An infinite plate with a circular hole (figure 1) subjected by unidirectional tension σ_∞ was dealt with.

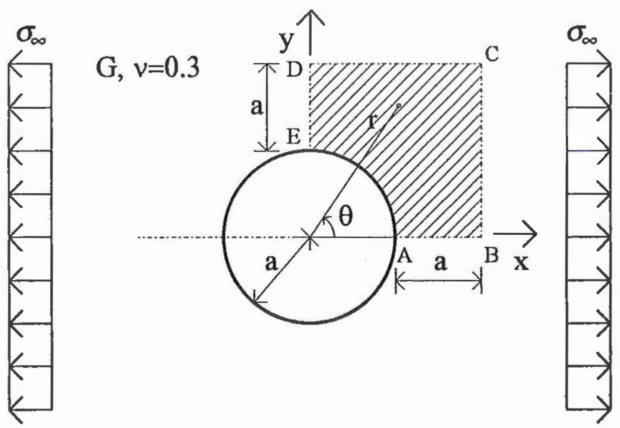

Figure 1: Infinite plate with a circular hole.

The analysis was performed using quadratic Lagrange and Serendip quadrilaterals. Qubic local representation (*p=3*) for the displacements corresponding to quadratic representations for the strains (or stresses) was used. In connection with quadrilaterals the *2×2* Gauss points (which are superconvergent points of the derivatives) were used as derivaties sampling points. Based on preliminary numerical tests the ratio $w_1 : w_2$ =1:1 of the weighting factors was used /2/.

Figure 2 presents result of experimental convergence study using quadratic Lagrange quadrilaterals and Figure 3 presents result of quadratic Serendip quadrilaterals. In figure the left one, (a) shows a comparison of the error of the smoothed solution obtained with the present technique and with the Zienkiewicz-Zhu Superconvergent Patch Recovery (SPR) technique /4/ and the error of the consistent (original finite element) solution. The right one, (b) shows a comparison of the corresponding error estimates and the exact error of the consistent solution.

The error of the smoothed solution is remarkably smaller than the error of the consistent solution with both coarse and dense grid. The corresponding rate of convergence is also excellent and comparable with that of the SPR technique. The error estimate remains good with both element types and with all grids which were used.

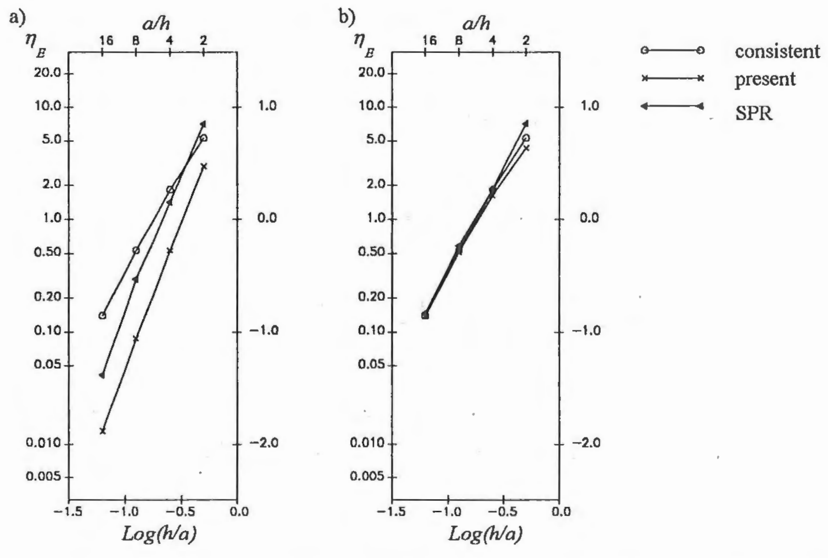

Figure 2: The circular hole problem with quadratic Lagrange quadrilaterals: (a) error of the smoothed and consistent solution (b) error estimate and exact error.

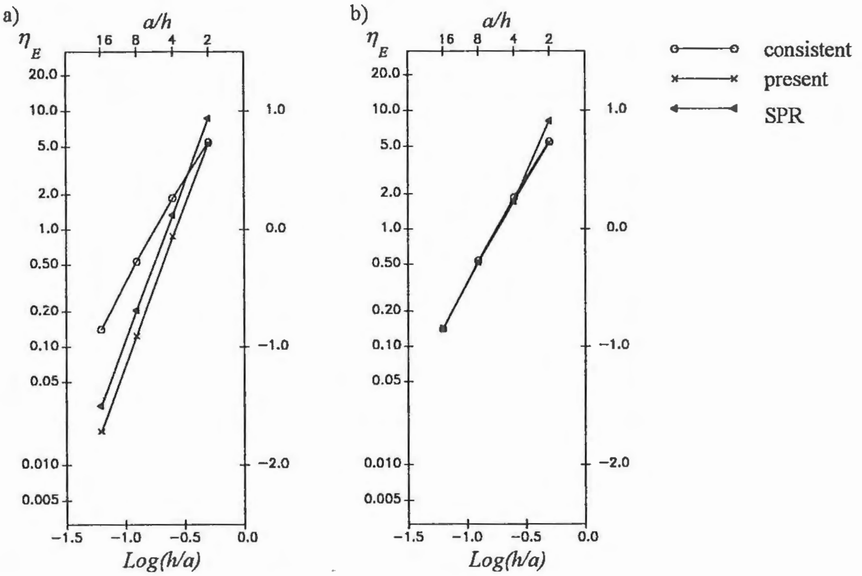

Figure 3: The circular hole problem with quadratic Serendip quadrilaterals: (a) error of the smoothed and consistent solution (b) error estimate and exact error.

Two-dimensional quasi-harmonic equation

Let us consider the two-dimensional quasi-harmonic equation

$$k_{xx}\frac{\partial^2\phi}{\partial x^2}+2k_{xy}\frac{\partial^2\phi}{\partial x\partial y}+k_{yy}\frac{\partial^2\phi}{\partial y^2}+Q=0, \tag{22}$$

where $\phi(x,y)$ is the unknown potential, k_{xx}, k_{xy} and k_{yy} are the conductivities and $Q(x,y)$ is the source term, as our second model problem. The function

$$\phi=\phi_0\left(\frac{x^2}{a^2}+\frac{y^3}{a^3}\right)sin\left(\frac{xy}{a^2}\right) \tag{23}$$

was chosen to be the analytical solution of the problem in square domain (figure 4). The corresponding source term $Q(x,y)$ was obtained using the field equation (1).

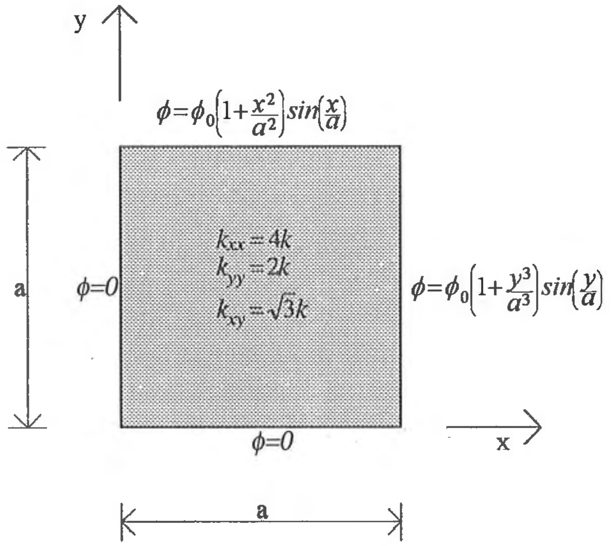

Figure 4: Quasi-harmonic problem in square domain.

The analysis was performed using quadratic Lagrange and Serendip quadrilaterals. Qubic local representation ($p=3$) for the potential corresponding to quadratic representations for the derivatives were used. The same derivative sampling points and the same weighting factors as in the case of plane elasticity problem were used.

Figure 5 presents result of experimental convergence study using quadratic Lagrange quadrilaterals and figure 6 presents result of quadratic Serendip quadrilaterals.

The error of the smoothed solution is remarkably smaller than the error of the consistent solution especially with coarse grids. The corresponding rate of convergence is excellent and comparable with that of the SPR technique in connection with quadratic quadrilaterals. The error estimate, however, seems to be good with both element types and with all grids which were used.

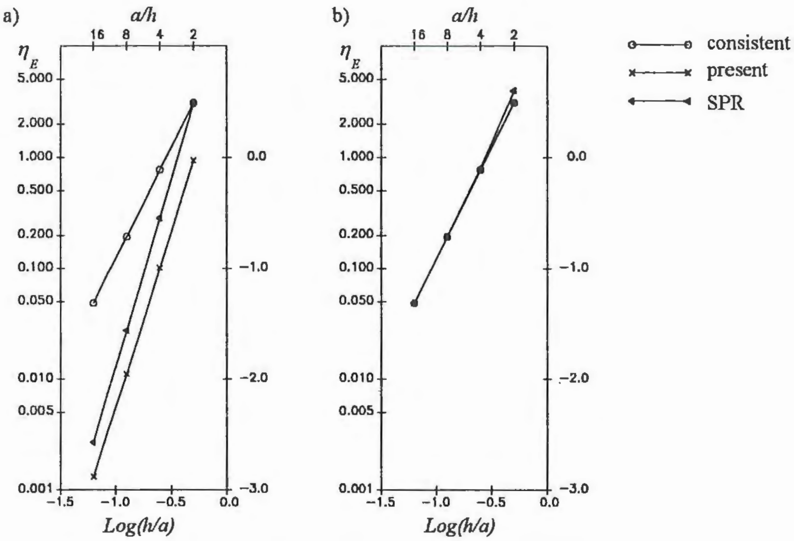

Figure 5: The square domain problem with quadratic Lagrange quadrilaterals: (a) error of the smoothed and consistent solution (b) error estimate and exact error.

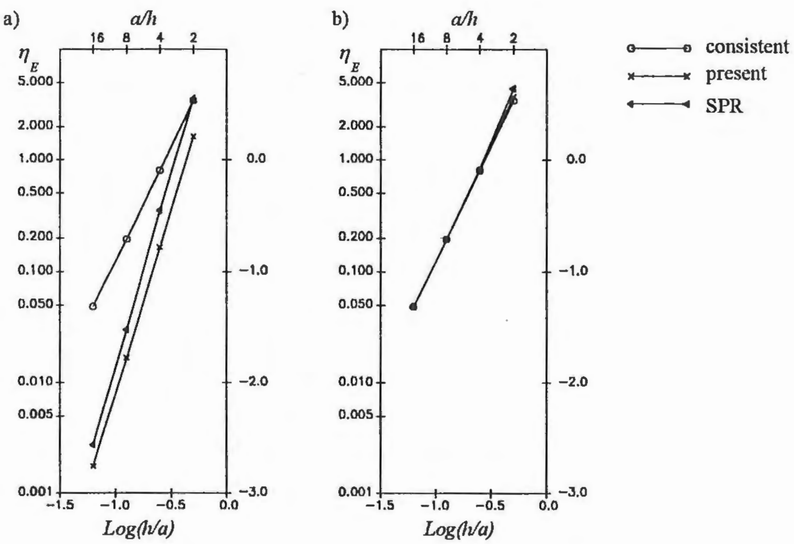

Figure 6: The square domain problem with quadratic Serendip quadrilaterals: (a) error of the smoothed and consistent solution (b) error estimate and exact error.

Finally it is interesting to study the cost involved in using these recovery procedures. The cost of the computation of present and SPR-method is compared to original finite element solution (*FEM*) in figure 7.

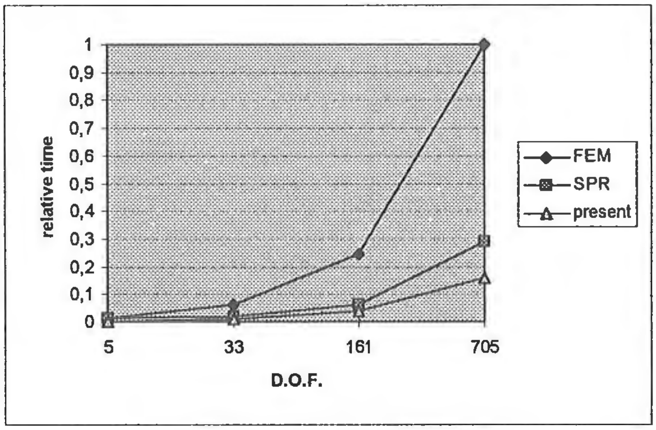

Figure 7: Relative computation time in quasi-harmonic square domain problem present and SPR-method compared to original finite element solution (*FEM*) using quadratic Serendip quadrilaterals.

CONCLUSIONS

An efficient and computationally simple element by element derivative smoothing technique has been presented. It works well with quadratic isoparametric elements. In connection with quadratic quadrilaterals (Lagrange and Serendip) numerical results for smoothed derivatives are excellent and comparable with those of the SPR-technique. Zienkiewicz-Zhu error estimates /3/ based on this procedure seem to be sufficiently accurate for practical purposes even with very coarse grids.

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