FEM ALGORITHMS FOR LINEAR ELASTICITY DYNAMICAL PROBLEMS

Victor N. Terpugov

Rakenteiden mekaniikka,Vol. 27
No 1, 1994, pp. 26-35

SUMMARY: Two algorithms for dynamical problems in linear elasticity based on one of the variational principles derived by Gurtin are investigated. Discretization is based on the Kantorovich approximation and the finite element method (FEM) in space variables. One-step, implicit, unconditionally stable, without approximation viscosity algorithms as a result of the time discretization are considered for solving the problem in time. The numerical dispersion and some other properties of the algorithms are discussed too. Both algorithms are analyzed from the point of view of methods of solution. It is shown that the best way to solve the first algorithm is to use a direct method, and in the case of the second one is to use an iterative method. A joint iterative procedure for optimal computer solution of different linear elasticity dynamical problems with the help of both algorithms is given.

PRELIMINARY REMARKS

I In dynamical problems solved by the FEM the approximation of Kantorovich [1] is mostly used for the representation of unknown functions:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{U}(\mathbf{x})\,\mathbf{q}(t)\ . \tag{1}$$

In equation (1) $\mathbf{u}(\mathbf{x},t)$ in an unknown K component vector-valued function; $\mathbf{q}(t)$ is an unknown N component vector of nodal values of $\mathbf{u}$; $\mathbf{U}(\mathbf{x})$ is a known (given) K×N dimensional matrix of interpolation; t is time; $\mathbf{x}$ is a vector of space variables, i.e., for example, in 3D problems: $\mathbf{x}=(x_1,x_2,x_3)^T$; $(\ )^T$ means a matrix or a vector transpose. Vectors and matrices are denoted by the boldface type.

The hypothesis (1) has the fundamental meaning for both discretization of the problem and the investigation of the properties of the algorithms and the numerical solution. For the first reason it means that the discretization procedure will be divided into two

practically independent parts: the construction of the FEM in space variables and the numerical solution of the finite element (FE) matrix equation of motion in time. On the other hand after using (1) it is easy to investigate the obtained algorithms because such kind of investigation is based on the analysis of spectral properties of the calculation schemes.

Let us think that **u** means the displacement in 3D linear elasticity problems (K=3, $\mathbf{u}=\mathbf{u}(x_1,x_2,x_3,t)=(u_1,u_2,u_3)^T$), when we'll consider the theoretical aspects, and in 2D problems (k=2, $\mathbf{u}=\mathbf{u}(x_1,x_2,t)=(u_1,u_2)^T$), when we'll speak about a program realization and numerical examples. Other vectors and matrices shall be changed respectively.

II The first step of the discretization the construction of the FEM in space variables) is practically the same as in the static problems. Usually it is applied to the virtual displacement principle designed for the dynamical problem [2] which means the realization of the FEM in the form of the Galerkin method and leads to the well known FE equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{F}\ , \tag{2}$$

$$\dot{\mathbf{q}}(0)=\mathbf{q}_0,\quad \mathbf{v}(0)=\mathbf{v}_0\ , \tag{3}$$

where **M** and **K** are mass and stiffness matrices, respectively; $\mathbf{F}=\mathbf{F}(t)$ is the nodal point force vector, consisting volume and surface nodal point force vectors; $\mathbf{v}(t)=\dot{\mathbf{q}}(t)$ is the nodal point velocity vector; upper dots denote the time derivatives; $\mathbf{q}_0,\mathbf{v}_0$ are the nodal point initial displacement and velocity vectors.

Application of the FEM procedure in space variables to the Gurtin's functionals [3,4] is another way. It means the construction of the FEM in the form of the Ritz method and leads to the following matrix FE integral equation of motion (if Gurtin's functional of the Lagrange type is used):

$$\mathbf{K}(g*\mathbf{q})+\mathbf{M}\mathbf{q}=g*\mathbf{F}+\mathbf{M}(\mathbf{q}_0+t\,\mathbf{v}_0)\ , \tag{4}$$

where: $g=g(t)\equiv t$; "*" denotes the convolution operator. For example, for arbitrary functions $\varphi(t)$ and $\phi(t)$ the $\varphi*\phi$ means the following integral:

$$\varphi*\phi=\int_0^t \varphi(t-\tau)\phi(\tau)dt\ .$$

Equations (2) and (4) are considered at the arbitrary moment of time t. It's easy to prove that one can get equation (4) by integrating equation (2) twice in time and taking into account (3) and vice versa: differentiating (4) twice one can get the statement (2)-(3).

III. The second step of the discretization is connected with the construction of the calculation scheme in time. Direct integration schemes are applied more often for the wave problems which leads to step by step solution procedures.

Because of the discretization in time and space variables the calculation schemes (computer algorithms) have their own dynamical properties which differ from the same properties of the corresponding continuum problem. Though we consider dynamical problems without viscosity and dispersion the derived algorithms can possess the approximation viscosity and the numerical dispersion changing both the amplitude and the phase of the numerical solution.

In order to control the viscous properties of algorithms terms **Cv** and **C**(g*v) are added to equations (2) and (4) correspondingly:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{v} + \mathbf{K}\mathbf{q} = \mathbf{F} \tag{5}$$

and

$$\mathbf{K}(g*\mathbf{q}) + \mathbf{C}(g*\mathbf{v}) + \mathbf{M}\mathbf{q} = g*\mathbf{F} + \mathbf{M}(\mathbf{q}_0 + t\mathbf{v}_0)\ . \tag{6}$$

Let us take the artificial damping matrix **C** in the form:

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}\ , \tag{7}$$

where α and β are arbitrary parameters which will be used to control the approximation viscosity. The dispersion properties of the algorithms will be estimated with the help of numerical experiments on modal tasks.

CONSTRUCTION OF ALGORITHMS

I Let us describe the general way of the construction of FE schemes for dynamical problems. If they are linear, implicit and one-step algorithms they can be represented in the following form [1,5]:

$$\mathbf{T}_\eta\mathbf{Q}_\eta = \mathbf{R}_\eta + \mathbf{T}_{\eta-1}\mathbf{Q}_{\eta-1}\ , \tag{8}$$

where $\mathbf{T}_\eta, \mathbf{T}_{\eta-1}$ are matrices, which can depend on the matrices **K**,**M** and time-step Δt_η; $\mathbf{R}_\eta = \mathbf{R}(t_\eta)$ is the nodal point loading vector, constructed with the help of the vector **F**; $\mathbf{Q}_\iota = (\mathbf{q}(t_\iota), \mathbf{v}(t_\iota), \mathbf{v}(t_\iota))^T$ is the nodal point state vector at the time point t_ι; $\iota = \eta, \eta-1$; $n = 1,2,...$ Assuming that the inverse matrix of the $\mathbf{T}_\eta$ exists one can write (8) in the form:

$$\mathbf{Q}_\eta = \mathbf{G}_\eta + \mathbf{T}\mathbf{Q}_{\eta-1}\ , \tag{9}$$

where $\mathbf{G}_\eta = (\mathbf{T}_\eta)^{-1}\,\mathbf{R}_\eta$; $\mathbf{T} = (\mathbf{T}_\eta)^{-1}\mathbf{T}_{\eta-1}$ is the amplification matrix.

With respect to the general theory of numerical methods [1,5] numerical stability and approximation viscosity depend on the properties of some norm of the amplification matrix **T**. If one considers the spectral norm of the **T** then it leads only to necessary conditions and it becomes the sufficient criteria when the spectral norm of **T** equals to any other norm of this matrix [5]. For example, it is true if **T** is a positive definite matrix which is usually satisfied in the FEM and it will be true in our case.

Let us interprete the method of the discretization of the FE equation of motion (5) or (6) as the finite element procedure in time (finite element on a time step). (Fig. 1) Every interval will be considered as independent from others; the solution from the given time step will be the initial condition for the next one, i.e. the arbitrary time interval $[t_{\eta-1},t_\eta]$ can be considered as the first interval [0,t], where t denotes here and further the value of the time step of the integration process in time. From equation (2) or (4) one can conclude that two vectors $\mathbf{q}(\tau)$ and $\mathbf{v}(\tau)$ are enough to describe the motion of the finite element system in time: see Fig.1, where $\tau\in[0,t]$; $\mathbf{q}_0=\mathbf{q}(0)$, $\mathbf{v}_0=\mathbf{v}(0)$, $\mathbf{q}_\gamma=\mathbf{q}(\gamma t)$, $\mathbf{v}_\gamma=\mathbf{v}(\gamma t)$; $\gamma\in[0,1]$.

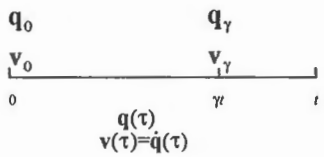

Fig.1

The arbitrary parameter γ will be used to change the stability and viscosity properties of the algorithm. One can write equation (8) in the form

$$\mathbf{T}_1\begin{Bmatrix}\mathbf{q}_\gamma\\ \mathbf{v}_\gamma\end{Bmatrix}=\mathbf{R}+\mathbf{T}_0\begin{Bmatrix}\mathbf{q}_0\\ \mathbf{v}_0\end{Bmatrix}, \tag{10}$$

where $\mathbf{R}=\mathbf{R}_1$.

The matrices $\mathbf{T}_1$ and $\mathbf{T}_0$ have the following form:

$$\mathbf{T}_1=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix};\qquad \mathbf{T}_0=\begin{bmatrix}\mathbf{B}_{11} & \mathbf{B}_{12}\\ \mathbf{B}_{21} & \mathbf{B}_{22}\end{bmatrix}, \tag{11}$$

where each block $\mathbf{A}_{i,j}$, $\mathbf{B}_{i,j}$ is a linear combination of the matrices **K**, **M** and **C** with some coefficients which can depend on the time step t and on the parameter γ. Thus the amplification matrix **T** of the algorithm (10) depends on the arbitrary parameters α, β, γ and by changing them one can maintain the numerical stability and exclude the approximation viscosity from the algorithm.

II Consider two approximation polynomials for our time element shown in the Fig.1 - the third order polynomial

$$\mathbf{q}(\tau) = (1-3\theta^2+2\theta^3)\mathbf{q}_0 + \gamma t(\theta-2\theta^2+\theta^3)\mathbf{v}_0 + \\ +(3\theta^2-2\theta^3)\mathbf{q}_\gamma + \gamma t(-\theta^2+\theta^3)\mathbf{v}_\gamma , \tag{12}$$

where $\theta = \tau/(\gamma t)$, and the second order polynomial:

$$\mathbf{q}(\tau) = (1-\theta^2)\mathbf{q}_0 + \gamma t(\theta-\theta^2)\mathbf{v}_0 + \theta^2\mathbf{q}_\gamma . \tag{13}$$

III Consider the algorithm for the second order polynomial (13) first. If one substitutes (13) into (6) one can be obtain the system of the linear algebraic equations (SLAE) concerning with the nodal point displacement vector $\mathbf{q}_\gamma$ and depending on the parameters α, β, γ. For concrete value of these parameters one can solve the SLAE and find $\mathbf{q}_\gamma$. Vector $\mathbf{v}_\gamma$ can be found with the help of the differentiation of the approximation polynomial (13) in the following form:

$$\mathbf{v}_\gamma = \frac{2\theta}{\gamma t}(\mathbf{q}_\gamma - \mathbf{q}_0) + (1-2\theta)\mathbf{v}_0 . \tag{14}$$

Both equations are the pair composing the algorithm in the form of (10) where the matrix block $\mathbf{A}_{12}$ equals zero.

One can find the values of parameters α, β, γ giving the unconditional stability and excluding the approximation viscosity from the algorithm. They are

$$\alpha = 0, \quad \beta = -\tfrac{t}{6}, \quad \gamma = \tfrac{1}{6}. \tag{15}$$

And thus the algorithm for the second order approximation polynomial (13) takes the following form:

$$\begin{bmatrix} \mathbf{M}+\frac{t^2}{36}\mathbf{K} & 0 \\ \frac{6}{t}\mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_{1/3} \\ \mathbf{v}_{1/3} \end{Bmatrix} = \frac{1}{9} \begin{Bmatrix} g^*\mathbf{F} \\ \mathbf{0} \end{Bmatrix} + \begin{bmatrix} \mathbf{M}-\frac{t^2}{36}\mathbf{K} & \frac{t}{3}\mathbf{M} \\ \frac{6}{t}\mathbf{I} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_0 \\ \mathbf{v}_0 \end{Bmatrix}, \tag{16}$$

where $\mathbf{I}$ is the unit matrix.

Such kind of an algorithm for the quadratic polynomial assumption of the displacement field was discussed first in [6,7]. But in the form (16) it was given in [8].

IV Let us consider the algorithm for the third order polynomial (12). If one substitutes (12) in the FE equation of motion (6) one can observe the SLAE with two nodal point vectors $\mathbf{q}_\gamma$ and $\mathbf{v}_\gamma$: N algebraic equations with 2N unknowns. To construct the

algorithm let us take with FE equations of motion (5) and (6) two equations more obtained after differentiation of equations (5) and (6)

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{C}\dot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \dot{\mathbf{F}} \tag{17}$$

and

$$\mathbf{M}\mathbf{v} + \mathbf{C}\mathbf{q} + \mathbf{K}\dot{g}^{*}\mathbf{q} = \dot{g}^{*}\mathbf{F} + \mathbf{M}\mathbf{v}_0 + \mathbf{C}\mathbf{q}_0 , \tag{18}$$

where $\dot{g} = 1$.

Arbitrary two from these four equations (5), (6),(17) and (18) or arbitrary linear combinations from these equations leading to the two FE equations can be used for the construction of the algorithm. Let us take equation (18) and a linear combination of other equations with arbitrary parameters δ, ξ, η. Substitute the approximation polynomial (12) and get the algorithm in the form (10) which will depend on six parameters: α, β,γ, δ, ξ, η. The combination

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 1, \quad \delta = -\tfrac{t^3}{36}, \quad \xi = \tfrac{t^2}{6}, \quad \eta = \tfrac{5}{3} t \tag{19}$$

gives to the algorithm the unconditional stability and excludes the approximation viscosity. As one can see it is not necessary to use the artificial damping matrix (α=β =0) and the numerical solution will be obtained at the end point of the time interval (γ =1). So, one can write the algorithm for the polynomial (12) in the following form:

$$\begin{bmatrix} \mathbf{M} + \frac{5t^2}{12}\mathbf{K} & \frac{t}{2}(\mathbf{M} - \frac{t^2}{6}\mathbf{K}) \\ \frac{t}{2}\mathbf{K} & \mathbf{M} - \frac{t^2}{12}\mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{v} \end{Bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{M} - \frac{7t^2}{12}\mathbf{K} & \frac{3t}{2}(\mathbf{M} - \frac{t^2}{18}\mathbf{K}) \\ -\frac{t}{2}\mathbf{K} & \mathbf{M} - \frac{t^2}{12}\mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_0 \\ \mathbf{v}_0 \end{Bmatrix} \tag{20}$$

where $\mathbf{F}_1 = -\frac{1}{36}\dot{\mathbf{F}} + \frac{t}{6}\mathbf{F} + \frac{5t^3}{3}\mathbf{g}^{*}\mathbf{F}$, $\mathbf{F}_2 = \dot{\mathbf{g}}^{*}\mathbf{F}$, $\mathbf{q} = \mathbf{q}_\gamma = \mathbf{q}_1$, $\mathbf{v} = \mathbf{v}_\gamma = \mathbf{v}_1$. The algorithm of this kind was discussed in [9,10].

ANALYSIS OF THE ALGORITHMS

Both algorithms were realized in the FORTRAN -program for 2D elasticity problems and some numerical experiments were carried out to investigate different algorithmic and calculated properties, in particular, the numerical dispersion of algorithms. Triangular elements with the linear and quadratic approximation in space variables were used.

I Because of the construction, algorithm (16) corresponding to the second order approximation polynomial (13) is a linear, one step, implicit, unconditionally stable, without approximation viscosity procedure for integration the problem in time.

In order to estimate the numerical dispersion property of the algorithm propagation of waves of different length in a long beam was considered. All harmonics were expressed in terms of the number of finite elements on wave length. Calculations were done on

different finite element meshes. Based on these calculations the following was concluded: the algorithm possesses the normal numerical dispersion less than 3% for the wave components with more than 6 elements. For the harmonics less than 6 elements the dispersion grows very quickly. One can conclude that the algorithm can be used rather successfully for calculations of wave fields caused by the influences composed with harmonics of more than 8-12 elements.

Let us consider calculation properties of the algorithm. Because the block $\mathbf{A}_{12}$ equals to zero calculation process is devided into two parts: solution of the SLAE concerning the nodal displacement vector $\mathbf{q}_{1/3}$ and calculation of the nodal velocity vector $\mathbf{v}_{1/3}$. The second part doesn't contain the operation of matrix inversion. The first part can be done rather effectively. The matrix of this SLAE is the sum of the matrices **K** and **M** with the positive coefficients. That is why this matrix is the positive definite matrix and has the same structure (symmetry and banding) as **K** and **M.** If the time step t is not changed in the step by step procedure the matrix of the SLAE can be inversed only once before the time-step procedure and after that on each time step it is necessary to do only multiplications of different matrices by vectors and the summation of the received vectors. One can say that this kind of solution procedure is the most effective (from the point of view of the computer resources) for dynamical problems.

As for deficiency of the algorithm one can mention the fact that the velocity $\mathbf{v}_{1/3}$ (for a next time step) is found with the help of the nodal displacement vector $\mathbf{q}_{1/3}$. In fact it means the differentiation of the numerically defined vector and it is not good for the short wave components in the numerical solution. One can come to the conclusion that this algorithm composed with the help of the second order approximation polynomial is fairly good for the solution of dynamical problems with sufficiently smooth solution (without short wave components).

II Like the previous algorithm (16) the algorithm (20) corresponding to the third order polynomial (12) is a linear, one step, implicit, unconditionally stable, without approximation viscosity procedure for the solution of dynamical problems in time, too. It possesses the normal numerical dispersion of the same kind as the first one, but it is even better: the dispersion is less than 2% for wave components with more than 6 elements on a wave length.

Calculated properties of the algorithm (20) are not so good as in (16). Though each matrix block of the matrix of the SLAE (20) is composed with the help of matrices **K** and **M** and consequently has the same structure, some of them in common cases are not positive definite, either. It is necessary to solve the SLAE of the 2N order with the vectors **q** and **v** as unknowns with the non symmetric and non banded matrix which is not positive definite. Of course, it is not effective to use the direct method for solving it, but the special block construction of the matrix and the good properties of blocks allows to hope that one can obtain an effective iterative procedure. In this connection it is possible to use the first algorithm to define the initial approximation and thus it would be obtained a joint algorithm which will be considered further.

THE JOINT ITERATIVE ALGORITHM

I Let us consider first the iterative procedure for solving the SLAE (20). Let the algorithm (20) be written in the form

$$\mathbf{A}\begin{pmatrix}\mathbf{q}\\ \mathbf{v}\end{pmatrix}=\begin{pmatrix}z_1\\ z_2\end{pmatrix}, \tag{21}$$

where

$$\mathbf{A}=\begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{pmatrix}; \qquad \begin{aligned}&\mathbf{A}_{11}=(\mathbf{M}+\tfrac{5t^2}{12}\mathbf{K}); \quad \mathbf{A}_{12}=\tfrac{t}{2}(\mathbf{M}-\tfrac{t^2}{6}\mathbf{K});\\ &\mathbf{A}_{21}=\tfrac{t}{2}\mathbf{K}; \qquad\qquad\quad \mathbf{A}_{22}=(\mathbf{M}-\tfrac{t^2}{12}\mathbf{K});\end{aligned}$$

z_i is the right hand part of the i-th matrix equation in (20) (i=1,2). Matrices $\mathbf{A}_{11}$ and $\mathbf{A}_{21}$ are positive definite matrices [11]. Matrices $\mathbf{A}_{12}$ and $\mathbf{A}_{22}$ shall also be positive definite if one supposes that

$$t<\frac{6}{\omega_1} \tag{22}$$

where ω_1 is the maximum eigenfrequency of the finite element system, i.e. $\omega_1=\lambda_1$, where λ_1 is the maximum eigenvalue of the matrix $(\mathbf{M})^{-1}\mathbf{K}$. It is supposed in the following that the condition (22) is satisfied.

Let us consider the following iterative procedure:

$$\begin{aligned}\mathbf{v}^{(m+1)}&=\mathbf{A}_{22}^{-1}z_2-\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\mathbf{q}^{(m)}\\ \mathbf{q}^{(m+1)}&=\mathbf{A}_{11}^{-1}z_1-\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{v}^{(m+1)}\end{aligned} \tag{23}$$

It is easy to prove that this iterative process is convergent for any initial vector $\mathbf{q}^{(0)}$ if the condition (22) is satisfied. The iterative process (23) is rather effective from the calculation point of view. If the time step t is not changed in the step by step procedure matrices $\mathbf{A}_{22}$ and $\mathbf{A}_{11}$ can be inversed only before the time step integration. Both are positive definite matrices and have same structure as **K** and **M**.

II Though the iterative process (23) is convergent for any initial vector $\mathbf{q}^{(0)}$, a good initial approximation makes the speed of convergence faster. For accelerating the convergence in our case one can take the solution obtained from the algorithm (16) and consider both algorithms together in one joint algorithm. Let the first matrix equation of (16) be written in the form

$$\mathbf{q}^{(0)}=\mathbf{A}_0^{-1}z_0, \tag{24}$$

where $\mathbf{A}_0=\mathbf{M}+\frac{t_1^2}{36}\mathbf{K}$; $z_0=\frac{1}{9}(g*\mathbf{F})+(\mathbf{M}-\frac{t_1^2}{36}\mathbf{K})\mathbf{q}_0+\frac{t_1}{3}\mathbf{v}_0$; $t_1=\frac{3}{2}t$ and consider the following iterative process:

$$\mathbf{q}^{(0)} = \mathbf{A}_0^{-1} z_0$$
$$\mathbf{v}^{(m+1)} = \mathbf{A}_{22}^{-1} z_2 - \mathbf{A}_{22}^{-1} \mathbf{A}_{21} \mathbf{q}^{(m)} \qquad (25)$$
$$\mathbf{q}^{(m+1)} = \mathbf{A}_{11}^{-1} z_1 - \mathbf{A}_{11}^{-1} \mathbf{A}_{12} \mathbf{v}^{(m+1)}$$

Solution of the problem on each time step can be divided into three parts: solving of the problem with the help of the first equation from (25), the solution of the vector $\mathbf{v}$ ($\mathbf{v}^{(0)}$). with the help of the second equation from (16) or the second equation from (25) and making more precise both vectors by means of the last two equations from (25). It is possible to use only one or two or all three parts of (25) on each time step. It depends on the precision we need, ultimately from the mechanical point of view it depends on if any higher frequency harmonics are in the solution or not. The calculated properties of the joint algorithm are rather good and include the best properties of both algorithms.

CONCLUSIONS

Algorithms based on the Gurtin`s convolution functional of the Lagrange type were considered in this article. It means the FEM realization in the form of the displacement method. But algorithms of this kind could be constructed also in the mixed form of FEM on the basis of the mixed Gurtin´s convolution functionals. In particular, one has constructed and realized in the FORTRAN program some of algorithms on the basis of the first form of the Reissner convolution functional and of the Lagrange-Reissner convolution functional.

All these algorithms are linear, one step, implicit, unconditionally stable, without approximation viscosity finite element procedures for solving dynamical elasticity problems in time. Different statical and dynamical elasticity problems were solved successfully with the help of the FORTRAN program based on all these algorithms and their statical analogs. There were model tasks and problems which have practical meaning. Some of these results can be found in [12,13,14,15].

REFERENCES

1. Marchuk, G.I., Methods of computational mathematics. Moscow, Nauka, 1977.

2. Bathe,K.J. and Wilson, E.L., Numerical methods in finite element analysis. Prentice-Hall, Inc., Englewood Clifts, N.J., 1976.

3. Gurtin,M.E., Varitional principles for linear elastodynamics. Archs. Ration.Mech. Anal., 1964, Vol. 16, No 1, pp. 34-50.

4. Gurtin,M.E., Varitional principles for linear initial-value problems. Quart.Appl. Math., 1964, Vol. 22, No 3, pp. 252-256.

5. Richtmyer, R.D. and Morton, K.W., Difference methods for initial-value problems. John Wiley & Sons, N.Y., L.,S., Interscience, 1967.

6. Nickell, R.E. and Sackman, J.L., Approximate solutions in linear coupled thermoelasticity. J. Appl. Mech., 1968, Vol. 35, pp. 255-266.

7. Nickell, R.E., On the stability of approximation operators in problems of structural dynamics. Int.J. Solids & Struct., 1971, Vol. 7, pp. 301-319.

8. Vinokurov, O.A., The application of varitional convolution functionals for the solving dynamical elasticity problems. Leningrad Polytecnical Institute, 1979, 14p. (Article was published in the VINITI, 13th of March, 1979, No. 895).

9. Gellert, M., A new algorithm for integration of dynamic system. Comp. and Struct.,1978, Vol. 9, No. 4, pp. 401-408.

10. Terpugov, V.N., The algorithm of the FEM for solving of nonstationary linear elasticity problems. In "Questions of dynamics and solidity", Riga: Zinatne, 1987 , Vol.48, pp. 34-45.

11. Lancaster,P., Theory of matrices. Academic Press, N.Y., L., 1969.

12. Vinokurov,O.A. and Terpugov, V.N., Solution dynamical elasticity problems on the basis of different variational forms of the finite element method. Leningrad Polytechnical Institute, 1981, 19 p. (Article was published in the VINITI, 16th of July, 1981, No. 3214).

13. Vinokurov,O.A. and Terpugov, V.N., Solution wave elasticity problems by the finite element method. In: "Space construction in the Krasnoyrsk region", Krasnoyrsk Polytechnical Institute, 1982, pp. 94-101.

14. Rozin, L.A., Konstantinov, I.A. and Terpugov, V.N., Solution non-stationary elasticity problems applied to the calculations of constructions under the seismic loadings by the FEM. Leningrad: Izvestiy VNIIG named after B.E. Vedeneev, 1985, Vol. 181, pp. 88-105.

15. Omelchak,I.M. and Terpugov, V.N., The program complex "ELESOIL" for statical and dynamical calculations of the system "foundation-soil". In: " Soil and foundations in geological conditions of the Ural ", Perm Polytechnical Institute, 1989, pp. 22-27.

Victor N. Terpugov, M.Sc.
Perm State University, Department of Applied Mathematics
614600 Perm, GSP-600
RUSSIA