BASICS OF "GRIFFITH'S FRACTURE MECHANICS"

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ABSTRACT

This article studies the basics of fracture mechanics. The theory proposed by Griffith is derived starting from the examination of the law of kinetic energy. A condition for crack growth is formulated. This equation also contains the influence of kinetic energy and continuum dissipation. The condition is expressed in the rate form i.e. it describes the response of a continuum in the case of a running crack. In the special case, the derived condition reduces to that proposed by Griffith.

INTRODUCTION

The aim of the present investigation is to study the basics of "Griffith's fracture mechanics" and to derive the above–mentioned theory. A keen reader is recommended to study the work of Santaoja (1992).

Because the behaviour of solids under loads is investigated in this work, it is convenient to define the term 'system' as a group of material points containing the whole body under consideration. Furthermore, the concept of a subsystem is defined as an arbitrary collection of adjacent material points. The volume of the system is denoted by V^b, in which the superscript b refers to the well–defined volume, i.e. to the volume of the body. On the other hand, the volume of the subsystem is denoted by V being without any superscript, which shows the arbitrariness of the subsystem.

In the subsequent study the investigation is restricted to problems involving small displacements, deformations and rotations. Therefore, with sufficient accuracy the values of the variables can be replaced by their counterparts in the initial configuration. Thus, it is obtained: $V \rightarrow V^0$, $V^b \rightarrow V^{b0}$ etc.

LAW OF KINETIC ENERGY

In the present chapter the theorem of stress means is derived. The final result is the law of kinetic energy.

First the dot product of the stress tensor $\boldsymbol{\sigma}$ and an arbitrary vector field $\bar{g}$ is formed. Next the divergence of the previously obtained dot product is studied. Thus, the scalar of interest has the following appearance

$$\bar{\nabla} \cdot (\boldsymbol{\sigma} \cdot \bar{g}) \qquad \text{in open } V^{b0} \tag{1}$$

Scalar (1) is defined in the system V^{b0} except for the singularity at the crack tip Ct, because the stress tensor $\boldsymbol{\sigma}$ is unbounded at Ct. The vector field $\bar{g}$ is assumed to be continuously differentiable in open V^{b0}. Santaoja (1992. app. B) derives the following equality.

$$\bar{\nabla} \cdot (\mathbf{h} \cdot \bar{e}) = (\bar{\nabla} \cdot \mathbf{h}) \cdot \bar{e} + \mathbf{h} : \bar{\nabla} \bar{e} \tag{2}$$

where $\mathbf{h}$ is a bounded second–order tensor and $\bar{e}$ is a bounded vector. By replacing the second–order tensor $\mathbf{h}$ by the stress tensor $\boldsymbol{\sigma}$ and the vector $\bar{e}$ by the vector $\bar{g}$ Equation (2) leads to

$$\bar{\nabla} \cdot (\boldsymbol{\sigma} \cdot \bar{g}) = (\bar{\nabla} \cdot \boldsymbol{\sigma}) \cdot \bar{g} + \boldsymbol{\sigma} : \bar{\nabla} \bar{g} \qquad \text{in open } V^{b0} \tag{3}$$

where the notation $\bar{\nabla} \bar{g}$ stands for the open product of the vectors $\bar{\nabla}$ and $\bar{g}$ [see e.g. Malvern (1969, pp. 36 and 590 – 591)].

The material derivative of the vector $\bar{v}$ is denoted by $\dot{\bar{v}}$.

The Cauchy's equation of motion {see e.g. Malvern [1969, Eq. (5.3.4)]}

$$\bar{\nabla}\cdot\boldsymbol{\sigma} + \rho_0\bar{b} = \rho_0\dot{\bar{v}} \qquad \text{in open } V^{b0} \tag{4}$$

allows Equation (3) to be casted into the form

$$\bar{\nabla}\cdot(\boldsymbol{\sigma}\cdot\bar{g}) = \rho_0(\dot{\bar{v}}-\bar{b})\cdot\bar{g} + \boldsymbol{\sigma}:\bar{\nabla}\,\bar{g} \qquad \text{in open } V^{b0} \tag{5}$$

The integration is performed over the volume of the subsystem V^0 on both sides of Expression (5). This procedure gives

$$\int_{V^0} \bar{\nabla}\cdot(\boldsymbol{\sigma}\cdot\bar{g})\, dV = \int_{V^0} [\, \rho_0(\dot{\bar{v}}-\bar{b})\cdot\bar{g} + \boldsymbol{\sigma}:\bar{\nabla}\,\bar{g}\,]\, dV \tag{6}$$

According to Santaoja [1992, Eq. (A.13)] "the modified generalized Gauss's theorem" for the unbounded generalized tensor ø can be written as

$$\int_{V^0} \bar{\nabla} * ø\, dV = \oint_{S^0} \bar{n} * ø\, dS \tag{7}$$

By replacing the star product $*$ by the dot product $\cdot$ and denoting $ø = \boldsymbol{\sigma}\cdot\bar{g}$ the modified generalized Gauss's Theorem (7) gives the following equality

$$\int_{V^0} \bar{\nabla}\cdot(\boldsymbol{\sigma}\cdot\bar{g})\, dV = \oint_{S^0} \bar{n}\cdot(\boldsymbol{\sigma}\cdot\bar{g})\, dS \tag{8}$$

Substitution of Equality (8) into the left-hand side of Equation (6) gives

$$\oint_{S^0} \bar{n}\cdot(\boldsymbol{\sigma}\cdot\bar{g})\, dS = \int_{V^0} [\, \rho_0(\dot{\bar{v}}-\bar{b})\cdot\bar{g} + \boldsymbol{\sigma}:\nabla\bar{g}\,]\, dV \tag{9}$$

The theorem of stress means, Equation (9), was derived by Signorini (1933, p. 232).

An arbitrary vector field $\bar{g}$ in Equation (9) is replaced by a virtual displacement vector $\delta\bar{u}$. By changing the order of the terms in the obtained equation one can write

$$\int_{V^0} \rho_0 \bar{\dot{v}} \cdot \delta\bar{u} \, dV = \oint_{S^0} \bar{n} \cdot (\boldsymbol{\sigma} \cdot \delta\bar{u}) \, dS + \int_{V^0} \rho_0 \bar{b} \cdot \delta\bar{u} \, dV - \int_{V^0} \boldsymbol{\sigma} : \bar{\nabla}(\delta\bar{u}) \, dV \tag{10}$$

Result (10) can be found in Malvern (1969, p. 242). Virtual strain tensor $\delta\varepsilon$ and virtual rotation tensor $\delta\omega$ associated with the infinitesimal virtual displacement vector $\delta\bar{u}$ are [see e.g. Malvern (1969, pp. 132, 161 and 239]

$$\delta\boldsymbol{\varepsilon} = \frac{1}{2}[(\delta\bar{u})\hat{\nabla} + \bar{\nabla}(\delta\bar{u})] \quad \text{and} \quad \delta\boldsymbol{\omega} = \frac{1}{2}[(\delta\bar{u})\hat{\nabla} - \bar{\nabla}(\delta\bar{u})] \quad \text{in open } V^{b0} \tag{11}$$

Subtraction of Equation (11b) from Equation (11a) gives the following relationship

$$\bar{\nabla}(\delta\bar{u}) = \delta\boldsymbol{\varepsilon} - \delta\boldsymbol{\omega} \qquad \text{in open } V^{b0} \tag{12}$$

Substituting Equation (12) into the last integral on the right–hand side of Equation (10) yields

$$\int_{V^0} \rho_0 \bar{\dot{v}} \cdot \delta\bar{u} \, dV = \oint_{S^0} \bar{n} \cdot (\boldsymbol{\sigma} \cdot \delta\bar{u}) \, dS + \int_{V^0} \rho_0 \bar{b} \cdot \delta\bar{u} \, dV - \int_{V^0} \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} \, dV \tag{13}$$

Because the virtual rotation tensor $\delta\omega$ is skew–symmetric [i.e. $\delta\omega^T = -\delta\omega$] and the stress tensor $\boldsymbol{\sigma}$ is symmetric, the following equality holds [see e.g. Flügge (1972, pp. 18 and 19)]

$$\boldsymbol{\sigma} : \delta\boldsymbol{\omega} = 0 \qquad \text{in open } V^{b0} \tag{14}$$

Result (14) has been exploited during the derivation of Equation (13)

The above investigation can be repeated by replacing virtual displacement vector $\delta\bar{u}$ by the differential displacement vector $d\bar{u}$. Instead of Equation (13) the following expression is found

$$\int_{V^0} \rho_0 \bar{\dot{v}} \cdot d\bar{u} \, dV = \oint_{S^0} \bar{n} \cdot (\boldsymbol{\sigma} \cdot d\bar{u}) \, dS + \int_{V^0} \rho_0 \bar{b} \cdot d\bar{u} \, dV - \int_{V^0} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} \, dV \tag{15}$$

The integrand on the left–hand side of Expression (15) can be manipulated formally as follows

$$\bar{v}\cdot d\bar{u} = \frac{d\bar{u}}{dt}\cdot\frac{d\bar{u}}{dt}\,dt = \left[\frac{d\bar{u}}{dt}\cdot\frac{d\bar{u}}{dt}\right]dt$$

$$= \frac{1}{2}\left[\frac{d\bar{u}}{dt}\cdot\frac{d\bar{u}}{dt} + \frac{d\bar{u}}{dt}\cdot\frac{d\bar{u}}{dt}\right]dt \qquad \text{in open } V^{b0} \tag{16}$$

$$= \frac{1}{2}\frac{d}{dt}\left[\frac{d\bar{u}}{dt}\cdot\frac{d\bar{u}}{dt}\right]dt = \frac{1}{2}d\left[\frac{d\bar{u}}{dt}\cdot\frac{d\bar{u}}{dt}\right] = \frac{1}{2}d[\bar{u}\cdot\bar{u}]$$

The term on the left–hand side of Equation (15) is the differential kinetic energy dK which according to Equations (15) and (16) is defined by

$$dK := \int_{V^0} \rho_0 \bar{v}\cdot d\bar{u}\, dV = d\int_{V^0} \frac{1}{2}\rho_0 \bar{u}\cdot\bar{u}\, dV \tag{17}$$

The first two terms on the right–hand side of Equation (15) form the differential external work dW^{ex} i.e.

$$dW^{ex} := \oint_{S^0} \bar{t}\cdot d\bar{u}\, dS + \int_{V^0} \rho_0 \bar{b}\cdot d\bar{u}\, dV \tag{18}$$

where the definition, $\bar{t} := \bar{n}\cdot\boldsymbol{\sigma}$ {see e.g. Malvern [1969, Eq. (3.2.8)], of the stress tensor $\boldsymbol{\sigma}$ has been exploited.

The last term in Expression (15) is the differential internal work dW^i defined by

$$dW^i := -\int_{V^0} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}\, dV \tag{19}$$

Based on Definitions (17)...(19) Expression (15) can be written as follows

$$dK = dW^{ex} + dW^i \tag{20}$$

Performing the integration from the initial configuration to the current configuration on the both sides of Equation (20) gives

$$\Delta K = W^{ex} + W^{i} \tag{21}$$

Where ΔK is the increase of the kinetic energy, W^{ex} is the work performed on the subsystem by the external forces and W^{i} is the work performed by the internal forces in the subsystem. Equation (21) is the law of kinetic energy and it can be expressed as follows : 'The work of all the forces (internal and external) that act on a mechanical subsystem equals the increase of the kinetic energy of the subsystem' (Langhaar 1962, pp. 10 – 13).

CONDITION THAT THE CRACK MAY EXTEND

This chapter derives 'the condition that the crack may extend' proposed by Griffith in 1920. The approach utilized here is different from to that applied by Griffith. Therefore the reader is recommended to investigate Griffith's work (1920).

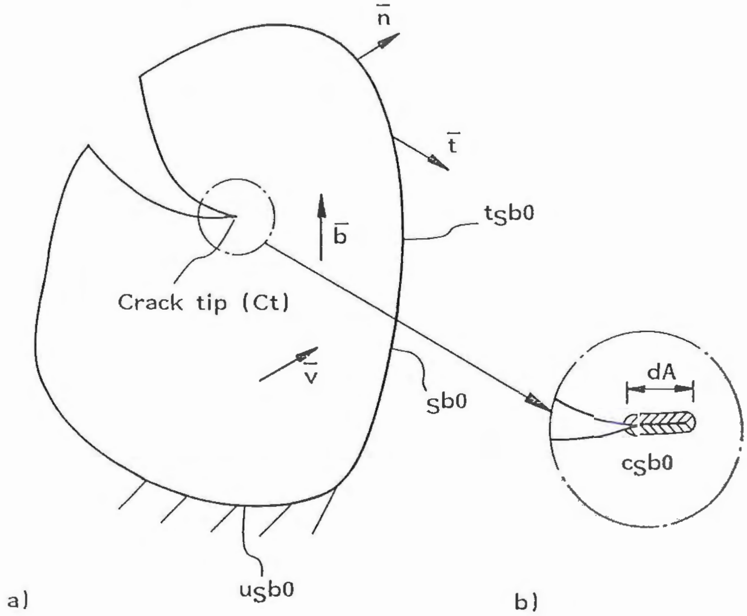

Figure 1. (a) A 3–dimensional system with a crack and (b) a detailed illustration of the fracture process zone.

Figure 1 shows a 3–dimensional system (body) with an arbitrary–shaped crack in the initial configuration. The crack tip Ct forms a curve in the 3–dimensional system. The total surface S^{b0} of the system V^{b0} is divided into two different parts: ${}^{t}S^{b0}$ on which the surface tractions $\bar{t}$ are prescribed and ${}^{u}S^{b0}$ on which the displacements $\bar{u}$ are prescribed. In front of the crack tip there is a zone in which the process to break the bonds holding atoms together is in progress. This zone is called 'fracture process zone' and it is illustrated in Figure 1(b).

Equation (15) is rewritten for the whole system, viz.

$$\int_{V^{b0}} \rho_0 \bar{v} \cdot d\bar{u} \, dV = \int_{S^{b0}} \bar{t} \cdot d\bar{u} \, dS + \int_{{}^{c}S^{b0}} \bar{t} \cdot d\bar{u} \, dS + \int_{V^{b0}} \rho_0 \bar{b} \cdot d\bar{u} \, dV - \int_{V^{b0}} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} \, dV \qquad (22)$$

where the definition, $\bar{t} := \bar{n} \cdot \boldsymbol{\sigma}$, of the stress tensor $\boldsymbol{\sigma}$ has been exploited. In Equation (22) a surface integral over ${}^{c}S^{bo}$ has been added in order to take into account the influence of the fracture process zone.

Next the system is assumed to undergo a process which implies a differential displacement vector $d\bar{u}$, a differential strain tensor $d\varepsilon$ and a differential crack growth dA. The following equalities can be written

$$d\bar{u} = \frac{d\bar{u}}{dt} dt = \dot{\bar{u}} \, dt \qquad \text{and} \qquad d\boldsymbol{\varepsilon} = \dot{\boldsymbol{\varepsilon}} \, dt \qquad (23)$$

Equalities (23) are substituted into Equation (22) and the obtained result is integrated over an arbitrary time interval; $\Delta t = t_b - t_a$. Although the time interval Δt is an arbitrary one, the crack growth is assumed to start at the moment t_a. This procedure gives

$$\int_{t_a}^{t_b} [\int_{V^{bo}} \rho_0 \bar{\dot{v}} \cdot \bar{\dot{u}} \, dV] \, dt = \int_{t_a}^{t_b} [\int_{S^{bo}} \bar{t} \cdot \bar{\dot{u}} \, dS] \, dt + \int_{t_a}^{t_b} [\int_{{}^cS^{bo}} \bar{t} \cdot \bar{\dot{u}} \, dS] \, dt$$

$$+ \int_{t_a}^{t_b} [\int_{V^{bo}} \rho_0 \bar{b} \cdot \bar{\dot{u}} \, dV] \, dt - \int_{t_a}^{t_b} [\int_{V^{bo}} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \, dV] \, dt \tag{24}$$

The term to describe the influence of the fracture process zone, i.e. the second term on the right–hand side of Equation (24), is studied in more detail. This term is

$$\int_{t_a}^{t_b} [\int_{{}^cS^{bo}} \bar{t} \cdot \bar{\dot{u}} \, dS] \, dt \tag{25}$$

It is noteworthy that the investigation of the above term is not included in the (classical) continuum mechanics, which makes the derivation complicated. This is due to the fact that the study of the above mentioned term demands an investigation of the events on the atomic lattice. This lies beyond the scope of the (macroscopic) continuum mechanics.

The term inside the brackets of Term (25) is the rate of the crack separation work denoted by $\dot{C}$ and defined by

$$\dot{C} := \int_{{}^cS^{b0}} \bar{t} \cdot \bar{\dot{u}} \, dS \tag{26}$$

Figure 2(a) gives an ideal atomic–level view of the fracture process. During the crack growth two new surfaces are created in the material. The upper surface is denoted by the superscript + whereas the superscript – stands for the lower surface. The attractive forces (per unit area) in the atomic lattice are denoted by $\bar{t}^+$ and $\bar{t}^-$. Figure 2(b) shows the 'surface' tractions $\bar{t}^+$ and $\bar{t}^-$ as well as the outward unit normals $\bar{n}^+$ and $\bar{n}^-$ of the forming crack surfaces. The positive integration path is shown by the small arrows in Figure 2(b) (see e.g. Apostol 1962, II, Sect. 5.2.7).

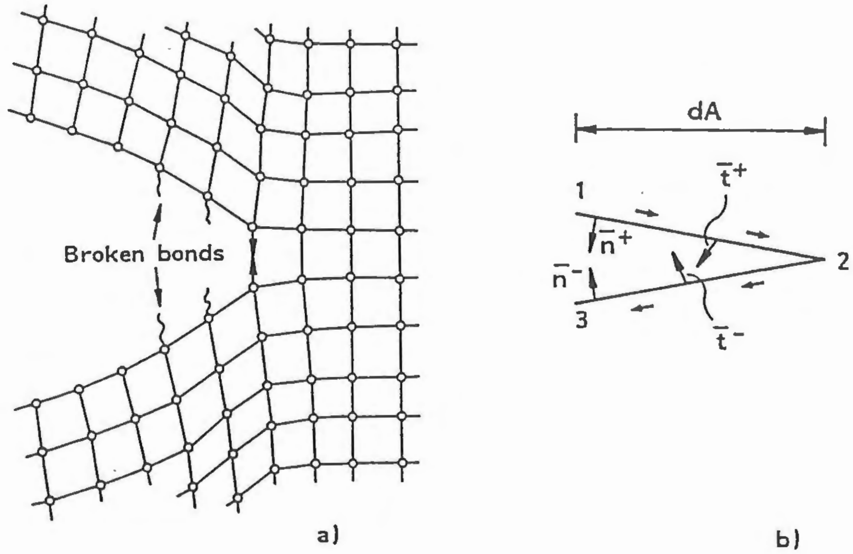

Figure 2. (a) An ideal model of fracture and (b) the 'surface' tractions as well as outward unit normals in fracture process zone.

Based on the notations introduced in Figure 2 Definition (26) takes the following form

$$\dot{C} = \int_0^{dA} \bar{t}^+ \cdot \dot{\bar{u}}^+ \, dS + \int_{dA}^{0} \bar{t}^- \cdot \dot{\bar{u}}^- \, dS \tag{27}$$

where the notation dA stands for the new crack area formed during the time interval $t - t_a$.

The first integral on the right–hand side of Equation (27) represents the power of the tractions on the upper surface of the fracture process zone. The integration is performed from 1 to 2 [see Figure 2(b)]. The second term is the power of the tractions on the lower surface. In this latter case the integration path is from 2 to 3. Because the tractions $\bar{t}^+$ and $\bar{t}^-$ are the forces (per unit area) describing the same bonds between atoms in the crystal lattice, their magnitudes are equal.

With the change in the integration direction the integral changes its sign (see Apostol 1957, Definition 9–5). In the light of the above Equation (27) yields

$$\dot{C} = \int_0^{dA} \bar{t}^+ \cdot \bar{u}^+ \, dS - \int_0^{dA} \bar{t}^+ \cdot \bar{u}^- \, dS \tag{28}$$

which leads to

$$\dot{C} = \int_0^{dA} \bar{t}^+ \cdot (\bar{u}^+ - \bar{u}^-) \, dS \tag{29}$$

According to Equations (25), (26) and (29) the crack separation work C carried out during the investigated period $\Delta t = t_b - t_a$ is found to be

$$C = \int_{t_a}^{t_b} [\int_0^{dA} \bar{t}^+ \cdot (\bar{u}^+ - \bar{u}^-) \, dS] \, dt \tag{30}$$

The crack separation work C depends on the following three variables: the crack growth dA; the material and (if the material is not homogeneous) the location of the crack tip. The above can be written as $C = C[A(x_1, x_2, x_3)]$. Therefore, the rate of the crack separation work $\dot{C}$ can be written in the following form

$$\dot{C}(A) = \frac{dC}{dt} = \frac{\partial C}{\partial A}\frac{dA}{dt} = \frac{\partial C}{\partial A}\dot{A} = \frac{dC}{dA}\dot{A} \tag{31}$$

Because the crack separation work C depends only on the crack area A and on the material under consideration, the partial derivative $\partial C / \partial A$ equals the total derivative dC / dA. This feature has been exploited in Equation (31). The problem in formulation of derivatives with respect to the crack area A will be discussed in the chapter headed 'Discussion and Conclusions'.

In a homogeneous material the derivative dC / dA is a (material dependent) constant and it is traditionally denoted by

$$2\gamma := \frac{dC}{dA} \tag{32}$$

In Equation (30) the 'surface' traction $\bar{t}^+$ is the force (per unit area) to overcome the attractive forces in the atomic lattice and ($\dot{\bar{u}}^+ - \dot{\bar{u}}^-$) is the corresponding velocity. The atom bond strength depends on the material. Also the velocity field on the crack surfaces during the bond breakage may be characteristic of different materials. Thus, most of the evidence supports a view that the crack separation work C is a material parameter. According to Equation (30) the unit of C is work. Therefore, based on Definition (32) 2γ is called the rate of the crack separation work. However, in the present case the word 'rate' does not refer to the material derivative, but indicates a derivative with respect to the crack area A.

The last term on the right–hand side of Equation (24) is

$$-\int_{t_a}^{t_b} [\int_{V^{bo}} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \, dV] \, dt \tag{33}$$

The strain rate tensor $\dot{\boldsymbol{\varepsilon}}$ is assumed to be separable into two different terms as follows

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^d \tag{34}$$

where $\dot{\boldsymbol{\varepsilon}}^e$ is the elastic (reversible) strain rate tensor and $\dot{\boldsymbol{\varepsilon}}^d$ is the dissipative (irreversible) strain rate tensor. It is noteworthy that the elastic strain rate tensor $\dot{\boldsymbol{\varepsilon}}^e$ also contains the effect of thermal expansion. The dissipative strain rate tensor $\dot{\boldsymbol{\varepsilon}}^d$ can be used in the description of a viscoelastic, viscoplastic etc. deformation. Separation (34) allows the term inside the brackets of Expression (33) to be written in the following form

$$\int_{V^{bo}} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \, dV = \int_{V^{bo}} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^e \, dV + \int_{V^{bo}} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^d \, dV \tag{35}$$

Strain–energy density denoted by w is defined by

$$w := \int_0^{\boldsymbol{\varepsilon}^e} \boldsymbol{\sigma}(\boldsymbol{\varepsilon}^e) : d\boldsymbol{\varepsilon}^e \tag{36}$$

The strain–energy W is

$$W := \int_{V^{bo}} w \, dV \tag{37}$$

Based on Equations (36) and (37) the first term on the right–hand side of Equation (35) is found to be

$$\int_{V^{bo}} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^e \, dV = \int_{V^{bo}} \dot{w} \, dV = \dot{W} \tag{38}$$

In the system shown in Figure 1(a) the value of the strain–energy density w varies within the system i.e. w is a function of the coordinates (x_1, x_2, x_3). Furthermore, the crack area A has a certain influence on the magnitude of the strain–energy density. Thus, it can be written

$$W(A) = \int_{V^{bo}} w(x_1, x_2, x_3; A) \, dV \tag{39}$$

where the crack area A is interpreted as a parameter.

Equations (36) and (39) give

$$\frac{dW(A)}{dA} = \frac{\partial W(A)}{\partial A} = \int_{V^{bo}} \frac{\partial w(x_1, x_2, x_3; A)}{\partial A} \, dV = \int_{V^{bo}} \boldsymbol{\sigma} : \frac{\partial \boldsymbol{\varepsilon}^e}{\partial A} \, dV \tag{40}$$

The first equality in Expression (40) shows the fact that the partial derivative of W(A) equals the (total) derivative of W(A), because W(A) is a function of one variable (see e.g. Piskunov 1974, I, p. 269). The second equality can be found in Korn and Korn (1968, p. 103).

The second term on the right–hand side of Equation (35) is called the power of continuum dissipation denoted by $\dot{}D$ and defined by

$$\dot{D}(A) = \int_{V^{b0}} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{d} \, dV \tag{41}$$

Application of Definition (17) shows that the term inside the brackets on the left–hand side of Equation (24) is the material derivative of the kinetic energy $\dot{K}$ (of the whole system). It is

$$\dot{K}(A) = \int_{V^{b0}} \rho_0 \bar{v} \cdot \bar{\dot{u}} \, dV \tag{42}$$

Based on Equation (18) the rate of the external work $\dot{W}^{ex}$ (of the whole system) is

$$\dot{W}^{ex} = \oint_{S^{b0}} \bar{t} \cdot \bar{\dot{u}} \, dS + \int_{V^{b0}} \rho_0 \bar{b} \cdot \bar{\dot{u}} \, dV \tag{43}$$

Equations (26), (31), (35), (36), (41), (42) and (43) allow Expression (24) to be casted into the form

$$\int_{t_a}^{t_b} [\dot{K}(A) - \dot{W}^{ex}(A) + \dot{C}(A) + \dot{W}(A) + \dot{D}(A)] \, dt = 0 \tag{44}$$

Because the time interval $\Delta t = t_b - t_a$ is an arbitrary interval, the integrand of Integral (44) has to vanish at any moment, viz.

$$\dot{K}(A) - \dot{W}^{ex}(A) + \dot{C}(A) + \dot{W}(A) + \dot{D}(A) = 0 \tag{45}$$

which can be written in the following form

$$\left(\frac{dK}{dA} - \frac{dW^{ex}}{dA} + \frac{dC}{dA} + \frac{dW}{dA} + \frac{dD}{dA} \right) \dot{A} = 0 \tag{46}$$

In the case of a growing crack, i.e. when $\dot{A} \neq 0$, the term inside the parentheses has to vanish. This is

$$\frac{dK}{dA} - \frac{dW^{ex}}{dA} + \frac{dC}{dA} + \frac{dW}{dA} + \frac{dD}{dA} = 0 \qquad \text{if } \dot{A} \neq 0 \tag{47}$$

The above equation is here called the condition for crack growth.

In the subsequent part of the present chapter deformation is assumed to be pure elastic. This can be argued by the following remark given by Barenblatt.

The papers by G. R. Irwing (1947) and E. O. Orowan (1950), in which the concept of quasi–brittle fracture was developed, represent an important stage in the theory of cracks. Irwin and Orowan noticed that a number of materials, which behave as highly ductile in standard tensile tests, fracture by a quasi–brittle mechanism when cracks are forming. This means that the arising plastic deformations are concentrated in a very narrow layer near the surface of a crack. As was shown by Irwin and Orowan, it is possible in such cases to employ Griffith's theory of brittle fracture, introducing instead of surface tension the effective density of surface energy. This quantity, in addition to the specific work required to produce rupture of internal bonds (= surface tension), includes the specific work required to produce plastic deformations in the surface layer of a crack; it is sometimes several orders of magnitude larger than the surface tension. (Barenblatt 1962, p. 65).

Based on the above the following is obtained

$$\dot{\boldsymbol{\varepsilon}}^{d} \equiv 0 \qquad \text{which yields that} \qquad D \equiv 0 \tag{48}$$

Equation (43) gives

$$\frac{dW^{ex}}{dA}\frac{dA}{dt} = \oint_{S^{bo}} \bar{t} \cdot \frac{\partial \bar{u}}{\partial A}\frac{dA}{dt}\, dS + \int_{V^{bo}} \rho_0 \bar{b} \cdot \frac{\partial \bar{u}}{\partial A}\frac{dA}{dt}\, dV \tag{49}$$

which yields

$$\frac{dW^{ex}}{dA} = \int_{{}^tS^{bo}} \bar{t} \cdot \frac{\partial \bar{u}}{\partial A} \, dS + \int_{V^{bo}} \rho_0 \bar{b} \cdot \frac{\partial \bar{u}}{\partial A} \, dV \tag{50}$$

The first term on the right–hand side of Equation (50) exploits the following fact: On the surface ${}^uS^{b0}$ on which the displacements $\bar{u}$ are prescribed (but can change within time t) the displacements $\bar{u}$ are independent of the crack growth (or crack area A) and therefore the partial derivative $\partial\bar{u} / \partial A$ vanishes on ${}^uS^{b0}$.

Hellan [1985, Eq. (3.23)] gives the expression for the potential energy Π, viz.

$$\Pi(A) = \int_{V^{bo}} w \, dV - \int_{{}^tS^{bo}} \bar{t} \cdot \bar{u} \, dS - \int_{V^{bo}} \rho_0 \bar{b} \cdot \bar{u} \, dV \tag{51}$$

which gives

$$\frac{d\Pi(A)}{dA} = \int_{V^{bo}} \frac{\partial w}{\partial A} \, dV - \int_{{}^tS^{bo}} \bar{t} \cdot \frac{\partial \bar{u}}{\partial A} \, dS - \int_{V^{bo}} \rho_0 \bar{b} \cdot \frac{\partial \bar{u}}{\partial A} \, dV \tag{52}$$

Substitution of Equation (40) into Expression (52) yields

$$\frac{d\Pi(A)}{dA} = \int_{V^{bo}} \boldsymbol{\sigma} : \frac{\partial \boldsymbol{\varepsilon}^e}{\partial A} \, dV - \int_{{}^tS^{bo}} \bar{t} \cdot \frac{\partial \bar{u}}{\partial A} \, dS - \int_{V^{bo}} \rho_0 \bar{b} \cdot \frac{\partial \bar{u}}{\partial A} \, dV \tag{53}$$

Equations (40), (48), (50) and (52) allow Equation (47) to be written in the following form

$$\frac{dK(A)}{dA} + \frac{d\Pi(A)}{dA} + \frac{dC(A)}{dA} = 0 \qquad \text{if } \dot{A} \neq 0 \tag{54}$$

which is here called the condition for crack growth for pure elastic deformation.

Griffith (1920, p. 166) assumed that the strains $\boldsymbol{\varepsilon}$ are elastic, which is assumed also in Equation (54). Although he did not mention it explicitly, he also assumed that the deformation is quasi–static, i.e. $K = 0$, and that the body forces $\bar{b}$ can be neglected. Based on the above discussion Condition (54) reduces into

$$\frac{d\Pi(A)}{dA} + \frac{dC(A)}{dA} = 0 \qquad (\text{if } \dot{A} \neq 0) \tag{55}$$

According to Griffith [1920, Eq. (11)] Equation (55) is 'the condition that the crack may extend'. Substitution of Notation (32) into Condition (55) gives the final result

$$-\frac{d\Pi(A)}{dA} = 2\gamma \qquad (\text{if } \dot{A} \neq 0) \tag{56}$$

The left-hand side of Equation (56) is called the potential energy release rate.

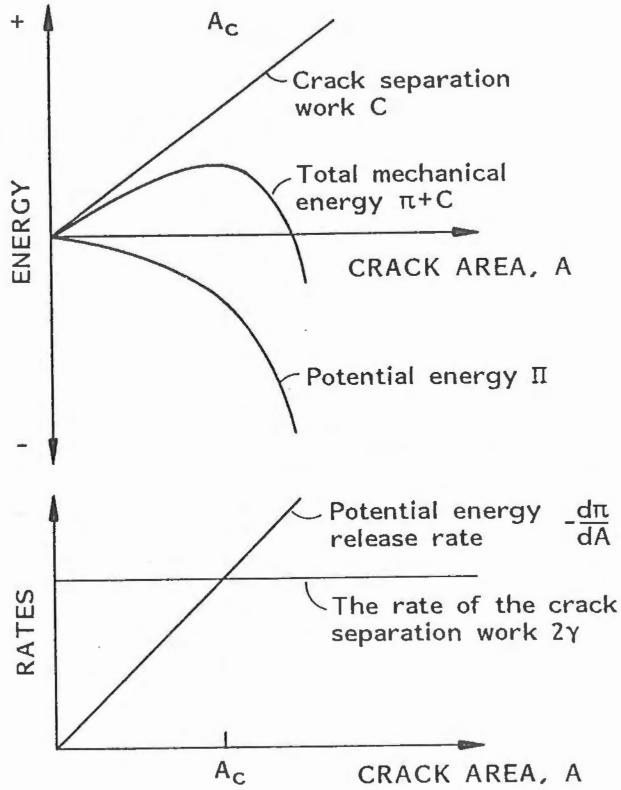

Figure 3. Variation of energy and rates of energy with crack area. The Condition (55) is satisfied when the crack area A has a critical value A_c. (Billington & Tate 1981, Fig. 14–8)

Figure 3 illustrates Condition (55) by showing that in case the crack area A equals its critical value A_c the total mechanical work $\Pi + C$ has a stationary value i.e. $d(\Pi + C) / dA = 0$.

DISCUSSION AND CONCLUSIONS

The focus of the present work was to investigate the basics of fracture mechanics.

The mathematical derivation was begun by examining of the Cauchy's equation of motion (the momentum principle). This led to the introduction of the theorem of stress means. By utilizing the symmetry of the stress tensor (the moment of momentum principle) the theorem of stress means provided the law of kinetic energy. The condition for crack growth was derived by applying the above mentioned law. This expression also includes the influence of kinetic energy as well as that of the continuum dissipation on the crack growth process. If a pure elastic deformation, a quasi–static process and the absence of body forces is assumed the derived law reduces to that proposed by Griffith.

The author is willing to open the discussion about the physical interpretation of the derivatives with respect to the crack area A (or crack length). As an introduction to the investigation of this problem the following is expressed. There are two different kinds of variables: Independent and dependent. The role of these variables is clarified by Myškis (1975) as follows.

Usually it is possible to pick out certain variables from a number of interrelated quantities such that the values of the variables can be taken arbitrarily whereas the values of the other quantities are determined by the values of the variables entering into the first group. The variables of the first type are called independent variables (or arguments) and the variables of the second type are called dependent variables (or functions). As an example let us consider the relationship between the area S of a circle and the length R of its radius. It is natural to regard R as an independent variable and choose its values arbitrarily; then the area computed by the formula $S = \pi R^2$ is a dependent variable in this functional relation. It should be noted that when we have a functional relation between variables the distinction between the independent variables and the dependent ones is sometimes conditional. For instance, in the previous example the area of the circle S would have been taken as an independent variable and the radius R as the dependent variable. (Myškis 1975, pp. 39 and 40) (the last sentence was written by the author but it is faithful the idea of the example given by Myškis).

The above description given by Myškis can also be stated by using the law of cause and effect: The independent variable is a cause and the dependent variable is an effect.

The problem arises from the fact that from the physics point of view the derivatives with respect to the crack area A are confusing. The following discusses about this question.

The (partial) derivatives are formulated with respect to the independent variables. When fracturing of continua is considered the loading is the cause and the crack growth (the crack area) A (or the crack length a in the 2–dimensional case) is the effect. Thus, from the physics' point of view loading is the cause and the crack growth is the effect. Therefore the derivatives $\partial/\partial a$ and $\partial/\partial A$ are inconsistent with the physics of a fracturing process. From the pure mathematics point of view the roles of the dependent and independent variables can be exchanged, as discussed by Myškiš. But in real nature when fracturing of materials is examined the exchange is difficult to imagine – at least a saw is needed.

A way out of the previous problem can be found: A new theory has to be prepared. This theory should be formulated in a manner that excludes the crack area A (or the crack length in the 2–dimensional case) from the list of independent variables.

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