CONCEPTION OF THE RIGID-PLASTIC BODY IN THE THEORY OF STRUCTURES

Ülo Lepik

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SUMMARY: This is an extended version of the lectures, which were delivered by the author in the Technical Universities of Helsinki and Tampere in spring 1991. The paper does not pretend to give a complete presentation of the discussed problems, rather it expresses some aspects from the sphere of the author's interests. The number of references has been kept as possible short: mainly only such papers and textbooks, which are directly connected with the methods and solutions, discussed in the paper, are referred.

INTRODUCTION

The problem about the dynamic response of elastic-plastic structures is complicated and it mostly can be solved only numerically (e.g. by the FEM). In several problems the plastic deformations dominate and it is possible to neglect the elastic deformations. In such a case we have the model of a rigid-plastic body. Here only the two following states are possible: if the stresses are below the yield stress, the material remains rigid; in the opposite case the material yields. According to this conception many problems about the response of structures can be solved analytically and we get close-form solutions, which usually are in good accordance with the experimental data. It has been assumed, that the conception of a rigid-plastic body is applicable, if the plastic energy exceeds maximal elastic energy more than ten times. This requirement can complicate the solution of the dynamic problems, because the external forces may be so high that geometrical nonlinearity of the structures must be taken into account. In static problems, where we have to calculate the carrying capacity of the structures, the elastic and plastic energies have the same order of magnitude, therefore it can be assumed that the model of the rigid-plastic body is more favorable for dynamic problems. The gap between theoretical and experimental results can be still reduced if we shall consider the material strain-hardening and the strain rate sensitivity, but in the other hand it highly complicates the solution of the problems in question.

In the following analysis we shall not consider the stress wave propagation problems ("early time response"); we shall assume that the time duration is so great (typically of the order of milliseconds), that the stress-waves have reflected several times from the supports.

For simplicity sake we shall confine us in following with beams and axisymmetric shells. The conception of a rigid-plastic body can be successfully used also for the response of more complicated structures (circular and annular plates, axisymmetric shells e.t.c.). Solutions for such problems can be found in some textbook (e.g. N.Jones, 1989).

EQUATIONS OF MOTION

We shall consider a beam with a rectangular cross-section; B, h and l are the width, thickness and length of the beam. The beam is subjected to a transverse load. If we shall take into account also the axial forces in the beam, the equations of motion are

$$\frac{\partial T^*}{\partial x^{*2}} = Bh \frac{\partial^2 u^*}{\partial t^{*2}} \tag{1}$$

$$\frac{\partial^2 M^*}{\partial x^{*2}} + \frac{\partial}{\partial x^*} \left(T^* \frac{w^*}{x^*} \right) = q^* + \rho Bh \frac{\partial^2 w^*}{\partial t^{*2}} \quad . \tag{2}$$

Here x^* is the axial coordinate, u^* - axial displacement, w^* - deflection, q^* - lateral pressure, ρ - density, t^* - time; T^* and M^* denote the axial force and the bending moment, respectively.

Let us introduce the following dimensionless quantities

$$x = \frac{x^*}{l}, \quad c = \left(\frac{s}{\rho}\right)^{1/2}, \quad t = \frac{c}{l} t^*, \quad u = \frac{u^*}{l}, \quad w = \frac{w^*}{h},$$

$$v = \frac{lv^*}{hc}, \quad q = \frac{4q^* l^2}{\sigma_s Bh^2}, \quad T = \frac{T^*}{\sigma_s Bh}, \quad M = \frac{4M^*}{\sigma_s Bh^2} . \tag{3}$$

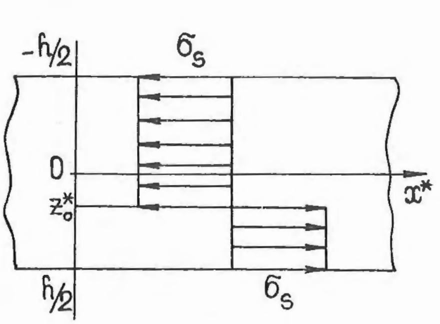
In these formulae σ_s is the yield stress, v^* - denotes the initial velocity (for the impulsive loading).

The equations (1)-(2) take now the form

$$T' = \ddot{u} , \quad M'' + 4 (Tw')' + q = 4\ddot{w} . \tag{4}$$

From now on primes and dots will denote differenciation with respect to x and t.

For the rigid-plastic material hinges appear in some cross-sections of the beam (Fig.1).



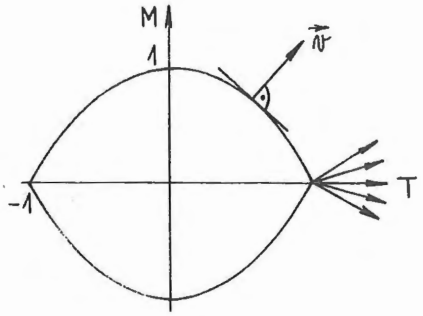


Fig.1. Distribution of stress in a cross- -section with a plastic hinge.

Fig.2. Yield curve for the beam.

It follows from this Figure that

$$T^* = -2\,\sigma_s B z_0^* , \quad M^* = \sigma_s B\left(\frac{h^2}{4} - z_0^{*2}\right). \tag{5}$$

If we shall eliminate the quantity z_0^* from these equations and go over to the dimensionless quantities (3), we get the yield curve in the form (Fig.2):

$$M = \pm\,(1 - T^2)\ . \tag{6}$$

Plastic deformations take place only then, when the point (T,M) lies on the yield curve (if we are inside this curve, the corresponding cross-section remains rigid).

According to the flow rule the strain rate vector $\mathbf{E} = (\dot\varepsilon^*, \dot\varkappa^*)$ must be normal to the yield curve. This condition gives

$$\dot\varepsilon^* = \frac{2\mu}{\sigma_s Bh}\, T = \dot u' + \delta w'\dot w' \tag{7}$$

$$\dot\varkappa^* = \frac{4\mu}{\sigma_s Bh^2}\, M = -\frac{h}{l^2}\, w'' \quad , \quad \delta = \left(\frac{h}{l}\right)^2 \tag{8}$$

Here $\mu > 0$ is the coefficient of plasticity. By eliminating μ from the equations (7)-(8), we obtain

$$\dot u' = -\frac{\delta}{2}\left(Tw'' + 2w'\dot w'\right)\ . \tag{9}$$

At the vertexes $T = \pm 1$ the vector **E** is nonunic - it may have any direction between the normals at the vertex.
This requirement can be put into the form

$$u' + \delta w' \dot{w}' < \frac{\delta}{2} |\dot{w}''| . \tag{10}$$

In the cross-section, where plastic hinges appear, the jump conditions must be fulfilled. If the symbol] denotes jump on the cross-section $x = \xi$, then these conditions are as follows:

(i) Since w and $\dot{w}$ must be continuous, we have

$$\dot{w}] + \dot{\xi}\dot{w}'] = 0 \tag{11}$$

$$\ddot{w}] + \dot{\xi}\dot{w}'] = 0 . \tag{12}$$

(ii) In the case of a moving hinge we have $w'] = 0$ and followingly

$$\dot{w}'] + \dot{\xi} w''] = 0 \tag{13}$$

(iii) By integrating the equation (10) and second equation of (4) over the cross-section $x = \xi$ we get

$$\dot{u}] = \mp 0,5 \ T(\xi)\dot{w}'] \tag{14}$$

$$M'] = - 4T(\xi)w'] . \tag{15}$$

To the equations (4) belong still the initial and boundary conditions, but these we shall write out when we are going to solve definite problems.

DYNAMIC RESPONSE OF RIGID-PLASTIC BEAMS

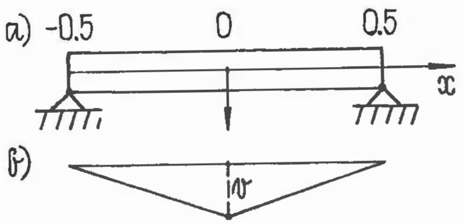
As the first problem let us consider a beam with simply supported ends. A uniform pressure q^*, which varies in time according to the rectangular law

$$q^*(t^*) = \begin{cases} q^* & \text{for } t^* \in (0, t_1^*) \\ 0 & \text{for } t^* \in (t_1^*, \infty) , \end{cases} \tag{16}$$

is applied. In this chapter we shall confine us only to the case of small deflections; besides we shall neglect the axial displacement. Therefore $u = T \equiv 0$ and the equations (4) take the form

$$M'' + q = 4 \ddot{w} . \tag{17}$$

Due to symmetry we can examine only one half of the beam. The origin of the coordinate x we shall put into the midpoint of the beam (Fig.3a).



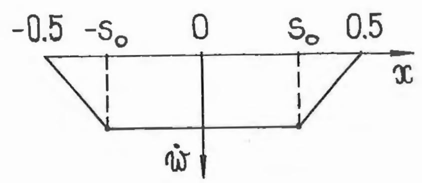


Fig.3a).Transversally loaded beam,
b).Deflection rate profile.

Fig.4.Deflection rate profile for high loads.

We shall assume that in the section $x = 0$ a plastic hinge appears. The deflection rate profile has now the form

$$\dot{w}(x,t) = v\ (1 - 2x)\ . \tag{18}$$

Differentiating this equation with respect to the time we get $\ddot{w}(x,t) = \dot{v}\ (1 - 2x)$. This result we shall put into the equation of motion (17). By integrating this result twice in regard of the coordinate x and satisfying the boundary conditions $M(0.5,t) = M'(0,t) = 0$ we obtain

$$M = \frac{q}{2}\left(\frac{1}{4} - x^2\right) + \frac{\dot{v}}{3}\left(-1 + 6x^2 - 4x^3\right) \tag{19}$$

Since we have a plastic hinge at $x = 0$, it must be $M(0,t) \equiv 1$ and the equation (19) gives

$$\dot{v} = \frac{3}{8}\ (q - 8)\ . \tag{20}$$

For static loading we have $\dot{v} = 0$ and so the static limit load is $q_{st} = 8$. It follows from equation (20) that the quantity $\dot{v}$ does not depend upon time: this allows us to integrate this equation twice with regard of time. Considering the equation (18) we obtain

$$\begin{aligned} \dot{w}(x,t) &= \frac{3}{8}\ (q - 8)\ t\ (1 - 2x) \\ w(x,t) &= \frac{3}{16}\ (q - 8)\ t^2(1 - 2x). \end{aligned} \tag{21}$$

These results are valid for the loading stage $t < t_1$. In the instant

$t = t_1$ the load is taken off and the following motion takes place by inertia. For this stage the equations (17)-(18) remain valid, if we shall take $q \equiv 0$. Satisfying the continuity conditions for w and $\dot{w}$ at $t = t_1$ we get

$$\dot{w}(x,t) = \dot{w}(x,t_1-) - 3(t - t_1)(1 - 2x) \tag{22}$$

$$w(x,t) = w(x,t_1-) + \dot{w}(x,t_1-)(t - t_1) - \frac{3}{2}(t - t_1)^2(1 - 2x) \tag{23}$$

The motion stops at an instant $t = t_f$, where $\dot{w}(x,t_f)=0$. It follows from equations (21)-(22) that $t_f = (q/8)t_1$. Putting this value into (23) we find the residual deflections in the form

$$w(x,t_f) = \frac{3}{128} q (q - 8) t_1^2 (1 - 2x) . \tag{24}$$

This result is valid only if the inequality $|M)(x,t)|\leq 1$ is fulfilled for $\forall\ x \in (0,0.5)$ and for $\forall\ t \in (0,t_f)$. It follows from (19) that the bending moment has an extremum at $x = 0$: this will be a maximum if $M''(0,t)<0$. Making use of the equations (19)-(20), this condition can be put into the form $q<24$. Followingly our solution holds if the load does not exceed the static limit load more than three times (the case of medium loads).

For high loads (i.e. $q>24$) the solution is more complicated. Therefore we shall not go into the details and shall present here only the main results. Now we must assume, that the plastic hinge does not appear in the centre of the beam, but in some cross-section $x = s_0$. The deflection rate profile has the form as shown in Fig.4. It follows from the equation (17) that s_0 = const for loading stage $t < t_1$ and it can be calculated from the formula

$$s_0 = \frac{1}{2} - \left(\frac{6}{q}\right)^{1/2} . \tag{25}$$

The deflection rate and deflection in the midpoint of the beam $x = 0$ in the loading stage are

$$\dot{w}(0,t_1) = \frac{1}{4} qt_1 \quad , \quad w(0,t_1) = \frac{1}{8} qt_1^2 . \tag{26}$$

The unloading stage $t > t_1$ falls into two substages. In the first substage the hinge $x = s_0$ begins to travel and continuously moves to the midpoint $x = 0$. After that the second substage begins. Now the deflection rate can be found from the equation (18). This substage goes on to the instant $t = t_f$ where the motion is terminated. The residual deflection of the centre for this instant is equal to

$$w(0,t_f) = \frac{1}{48} q t_1^2 (q - 6) . \tag{27}$$

The inequality $|M| \le 1$ holds for arbitrary $q > 24$.

In an analogical way the beam response problem can be solved for other boundary conditions. The assumption of a rigid-plastic body has approved its advantage also in solving several problems of dynamic response of circular plates and axisymmetric shells (see e.g. Jones, 1990).

APPROXIMATE METHODS OF CALCULATION

The method of solution, which was presented in the previous chapter, brings us in the case of more complicated problems to troublesome calculations. For such cases approximate methods of solution have been worked out. From these most known is the method of mode form solutions, which was presented by Symonds and Martin (1966). Although this method is applicable for arbitrary structures, we for conciseness sake shall consider only the case of beams, making use of the dimensionless quantities (3).

According to the method of mode form solutions we shall seek the deflection rates in the form

$$\dot{w}_m(x,t) = \Phi(t) V_m(x) , \tag{28}$$

where $\Phi(t)$ is the amplitude function, $V_m(x)$ - the spatial mode (which is prescribed). We shall choose the function $\Phi(t)$ so, that the modal velocity $\dot{w}_m$ would be as possible near to the actual velocity. This requirement can be put into the form

$$\Delta(t) = \int_0^1 [\dot{w}_m(x,t) - \dot{w}(x,t)]^2 dx = \min . \tag{29}$$

Martin and Symonds have shown, that $\dot{\Delta}(t) \le 0$; followingly the difference between the two solutions is biggest in the initial instant $t = 0$ and so we have to minimize the quantity

$$\Delta(0) = \int_0^{0.5} [\Phi(0) V_m(x) - \dot{w}(x,0)]^2 dx .$$

The extremum condition $d\Delta(0)/d\Phi(0) = 0$ gives

$$\Phi(0) = \frac{\int_0^{0.5} w(x,o)V_m(x)\ dx}{\int_0^{0,5} V_m^2(x)\ dx} \ . \qquad (30)$$

So we have found the initial value for $\Phi(t)$; the function $\Phi(t)$ itself we shall calculate by integrating the equation of motion (17).

Martin and Symonds presented their method for the case of impulsive loading (i.e. the initial velocity is prescribed). But it can be used also for pressure loading, if the load varies in time according to the law (16). Now in the loading stage $t < t_1$ we shall use the exact solution and apply the mode form solution only for the unloading stage (here instead the initial instant $t = 0$ we must take the value $t = t_1$). In following we shall demonstrate the method for the case of high loads from the previous chapter.

For the loading stage we shall use the equations (26). As to the unloading stage $t > t_1$, we shall take the spatial mode $V_m(x)$ in the form $V_m(x) = 1 - 2x$, corresponding to the velocity profill of Fig.3b. The actual velocity field in the instant $t = t_1$ is

$$\dot{w}(x,t_1) = \begin{cases} 0.25qt_1 & \text{for } x \in (0,s_0) \\ 0.25qt(1-2x)/(1-2s_0) & \text{for } x \in (s_0,0.5) \end{cases}$$

Making use of the equations and calculating the integrals in equation (30) we get

$$\Phi(t_1) = \frac{q}{4}\ t_1(1 + 2s_0 - 2s_0^2) \ . \qquad (31)$$

Now we shall integrate the equation of motion (17), which for the mode form solution obtains the form $M'' = = \dot{\Phi}(t)(1 -2x)$. Considering the boundary conditions $M'(0) = M(0.5) = 0$, we find that $M(x) = \dot{\Phi}(t)(6x^2 - 4x^3 - 1)/3$. Since in $x = 0$ must be a plastic hinge, we have $M(0) = 1$ and followingly $\dot{\Phi}(t) = -3$. By integrating this equation we get

$$\Phi(t) = \Phi(t_1) - 3(t - t_1) \ . \qquad (32)$$

The motion is terminated at $t = t_f$: it follows from the equation (32) that

$$t_f - t_1 = \frac{1}{3}\,\Phi(t_1)\ . \tag{33}$$

Now we have to integrate the equation (28) with regard of time, by taking $x = 0$ and making use of the equations (31), (33) we find

$$w(0,t_f) = w(0,t_1) + \frac{1}{6}\,\Phi^2(t_1)\ . \tag{34}$$

As an example let us carry out the calculations for $q = 40$. It follows from the equation (25) that $s_0 = 0.1127$. The formula (31) gives $\Phi(t_1) = 12.00$. Calculating the quantity $w(0,t_1)$ from the equation (26) we finally find $w(0,t_f) = 29.00t_1^2$. For the exact solution we get from (27) the value $w(0,t_f) = 28.33t_1^2$, so the error of the mode form solution is only 2.4%.

In most problems of the dynamic response of structures the method of mode form solutions guarantees exactness, which is sufficient for practical purposes. Nevertheless sometimes more exact solutions would be needed. There are several methods for increasing the exactness achieved by the mode form solutions. We shall describe here only a method, which we have called the "method of quasimode solutions".

According this method we shall seek the solution of the problem in the form (Lepik, 1979)

$$\dot{w}(x,t) = \sum_{i=1}^{n} \Phi_i(t)V_i[x,s_1(t),s_2(t),\ldots,s_q(t)]\ , \tag{35}$$

where $s_1(t),s_2(t),\ldots,s_q(t)$ are some parameter functions (e.g. the locations of the plastic hinges). The functions, which differently from the usual modes contain free parameters, we shall call quasimodes. Each quasimode must have the potentiality to tend towards a permanent mode in the beam motion.

As an example let us examine again the dynamic response of a beam under high loads. Here we shall confine us to two terms in the series (35) and take

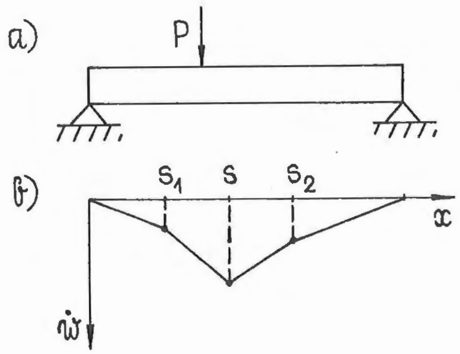
$$V_1 = 1 - 2x\ , \quad V_2 = \begin{cases} -1 + 2x/s & \text{for} \quad x \in (0,s) \\ (1 - 2x)/(1 - 2s) & \text{for} \quad x \in (s,0.5)\ . \end{cases}$$

Since $\dot{w}'(0,t) = 0$, we have $\Phi_2 = s\Phi_1$ and the equation (35) obtains the form

$$\dot{w}(x,t) = \Phi_1(t) \begin{cases} 1 - s & \text{for } x \in (0,s) \\ (1 - s)(1 - 2x)/(1 - 2s) & \text{for } x \in (s,0.5), \end{cases}$$

which corresponds to the velocity profile from Fig.4. The following solution does not differ from the exact solution that was considered before; so this example is only of methodical value.

For more complicated problems the method of quasimode form solutions may be valuable. Making use of this method dynamic response of a beam under a concentrated load was solved (Lepik, 1981).



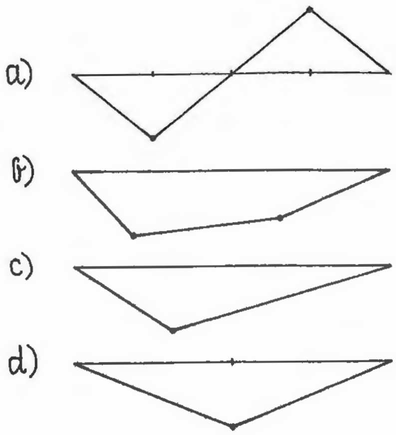


Fig.5a).Beam under a concentrated load,
b).Deflection rate profile.

Fig.6.Evolution of the deflection rate profiles.

For high loads three plastic hinges appear (Fig.5), in the unloading zone two of these hinges get lost and the third hinge begins to move towards the centre of the beam. When this hinge reaches the centre, the motion is terminated.

Another problem was solved by Lepik (1982). The inducement to this problem was the following matter: in some papers has been expressed the opinion that higher modal response of a beam is more efficient in practical design, since the absorbing of kinetic energy is greater as for the first mode. To verify this statement, we have taken for the initial velocity field a profile corresponding to the second mode (Fig.6a) and followed the evolution of this profile in time (Fig.6b-d). It follows from the calculations that the second mode is unstable in the sense of Lyapunoff: for slight deviations this mode steadily goes over to the first mode (Fig.6d). This obstacle makes difficult to use the higher modes in the energy-absorbing devices.

Method of quasi-mode form solutions was applied also for optimal design of two-stepped beams under impulsive loading (Lepik, 1983).

COMPUTER AIDED DESIGN OF STEPPED STRUCTURES

Let us consider beams for which the thickness is a piecewise-constant function of the beam coordinate. Such stepped beams are quite relevant in engineering practice, since the technology of manufacturing such beams is simpler than in the case of beams with continuously varying thickness. If the material of the beam is rigid-plastic, hinges appear in some cross-sections. Generally the number of hinges and their locations are not known, besides during the motion of the beam new hinges can appear or some of the existing hinges can disappear. All this makes the solution of the problem very complicated: we must look through many possible variants, from which only one holds. If we would deliver the analysis of all conceivable variants to a computer, much time and efforts in solving the problem could be saved.

First this idea has been realized by Lepik and Just in 1981 (the English version was published in 1983). In this paper the method of mode form solutions has been used. The case of moving hinges has been examined by Lepik and Lepikult (1987), where an algorithm for calculating the residual deflections has been proposed. This algorithm seeks the proper plastic regimes and locations of the plastic hinges, which are found automatically on a computer with the aid of Fortran-programs. According to these programs a large class of problems can be solved (beams with various number of steps and parameters of beam segments, the cases of arbitrary loading and boundary conditions, optimization problems). The relevance and efficiency of these programs will be shown by the following example.

A simply supported four-stepped beam of Fig. 7 is loaded by continuous pressure load and by a single load $P_1 = 30$. Dependence of the loads upon time is taken in the form

$$p(x,t) = p_0(x)\exp\left[\frac{\pi(t_* - t)}{\tan(\pi t_*)}\right]\cdot\frac{\sin(\pi t)}{\sin(\pi t_*)} ,$$

where $t_* = 0.2$.

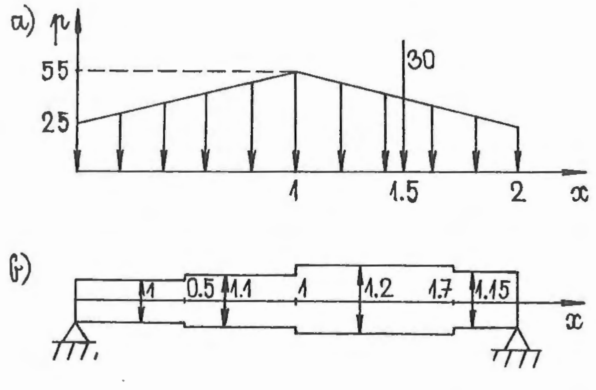


Fig.7.Simply supported beam under lateral loading
a) load distribution, b) beam dimensions.

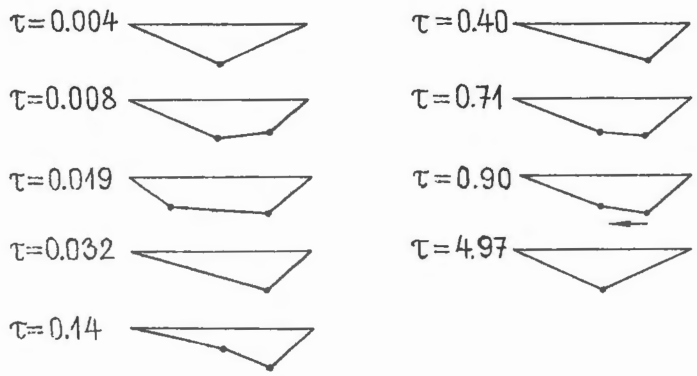


Fig.8.Deflection rate profiles for the beam of Fig.7.

Development and disappearance of the hinges is shown in Fig.8, where for the ordinates the deflection rates $\dot{w}$ have been taken. At the instant t = 0.004 a plastic hinge appears in the midpoint of the beam. Afterwards a second hinge appears at x = 1.5. The first hinge begins travelling and disappears if t = 0.40. At t = 0.14 a new hinge appears in the section x = 1.5 and disappears at t = 0.40. At later times a hinge appears in x = 1.0 and the hinge in x = 1.5 starts moving to the left. In final stage we have only one hinge in the midsection x = 1.0. The motion stops at t_f = 13.79, the maximal residual deflection is 42.77.

The same problem for stepped circular and annular plates for axisymmetric bending was discussed by Salupere (1982).

DYNAMIC RESPONSE OF AXIALLY CONSTRAINED BEAMS

Now let us consider beams, whose ends are prevented from displacing axially and which are subjected to transverse loading. In such beams axial forces arise and we have to consider the geometrical nonlinearity. For rigid-plastic materials this problem was first discussed by Symonds and Mentel (1958). After that many publications were devoted to this problem (Jones 1971, 1973, Symonds and Jones 1972, Symonds 1980, Gürkok and Hopkins 1981 et al.). This interest has held out up to the last time (Vaziri, Olson and Anderson 1987, Schubak, Anderson and Olson 1987 e.t.c.). Several approximate solutions, which are in good accordance with the experimental data, have been obtained. However it seems to us that an exact and mathematically fully correct solution is yet wanting. For assurance of this statement, we shall bring the following arguments.

For getting an exact mathematical solution to the problem we must satisfy the equations of motion (4) with initial and boundary conditions, the yield condition (6) and the inequality (10); besides in the cross-sections, where plastic hinges occur, must be fulfilled the jump-conditions (11)-(15).

Traditional approach for this problem is as follows. The axial displacement u is small and can be neglected; it follows from the first equation of (4) that $T = \text{const}$ (the axial force along the beam is constant). In cross-sections $x = \pm s$ plastic hinges appear, they move towards the centre of the beam. In some instant they get together and for the following motion we shall have a stationary hinge at $x = 0$. The axial forces T increase with the deflections; if $T = 1$ it follows from the yield condition (6) that $M = 0$. It means that the motion goes over to the membrane stage (the beam behaves as a plastic string); if $q = 0$ the equation of motion takes now the form

$$w'' = \ddot{w} \ . \tag{36}$$

Against this solution the following critical remarks can be made.

(i) If $T \neq 0$ there cannot be a stationary hinge at $x = 0$, since the jump condition (15) will be violated (we have $w'] \neq 0$, but $M'] = 0$, since the shear force $Q = M'$ must be zero at $x = 0$). Thereforewe must assume that the hinge at $x = 0$ splits again into two hinges, which travel farther from the centre.

(ii) If the whole beam transfers to membrane stage, then the following motion takes place according to equation (36). This is a hyperbolic equation and has no dissipation; consequently the motion would never stop. Besides it turns out that inequality (10) is not fulfilled in each cross-section. Therefore we must assume that there are some rigid zones in the beam (appearance of such zones has been supposed already in the paper of Symonds and Mentel 1958). In the unloading stage these zones will spread over the whole beam and the motion stops.

(iii) If the loads are not very high then the axial inertia effects may be really omitted and we can take T = const. In the other hand this asumption may turn out to be unsuitable, if we shall investigate the transition of the beam into the membrane stage (if $T \neq$ const various cross-sections of the beam transfer to membrane stage at different instants).

To get a solution of the problem, in which all these remarks are taken into account, is mathematically very complicated and it has not succeeded up to now. Since some quantities for of such a solution are not continuous, it is also difficult to get numerical results making use of FEM or other numerical technics. All this brings to the idea to give up the rigid-plastic model and to solve the problem numerically for an elastic-plastic material.

For elastic-plastic problems several approximate technics have been worked out. From these we shall here describe briefly the method proposed by Symonds (1980). According this method the elastic and plastic response phases are separated; the motion being treated as either wholly elastic or rigid-plastic. In the first stage we shall assume that the motion is purely elastic; this phase is terminated, when the yield condition (6) is satisfied in some cross-section of the beam. Then follows the second stage, where the whole beam is rigid-plastic (with small deflections), a mode form solution is seeked. Third stage is a membrane stage (finite rigid-plastic deflections). After that follows a stage of elastic recovery (an half-amplitude deflection of elastic vibrations is subtracted from the maximal deflection to obtain the final permanent deflection).

This method is simply enough for practical purposes and in good accordance with the results of tests.

COUNTERINTUITIVE BEHAVIOR OF ELASTIC-PLASTIC BEAMS

Let us consider again an elastic-plastic beam attached to fixed pins and subjected to a short pulse of uniform pressure. After maximal deflection the beam moves backward to a deflection on the negative side and after that elastic vibrations between minimum and maximum values of deflection take place. Symonds and Yu (1985) have noticed that for some values of the load both these extreme values may be negative and so the applied pulse results in a permanent final deflection in the opposite direction. This phenomena was called the counter-intuitive behavior of the beam. Such effect was noticed also for circular plates by Galiev and Nechitailo (1986).

For more detailed analysis of the counterintuitive behavior of beams Symonds e. al. have made use of a Shanley type model. This model consists of two rigid bars, which are connected by a small elastic-plastic "cell" (Fig. 9). It has only one degree of freedom, but in spite of it can treat all the essential physical ingredients of a continuous beam. Symonds and his collaborators have shown that permanent negative deflections take place only in a narrow gap of paramaters.

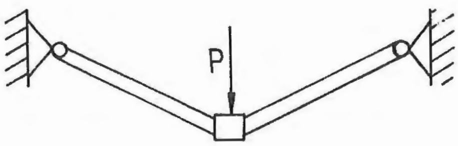


Fig.9.Shanley model for the beam.

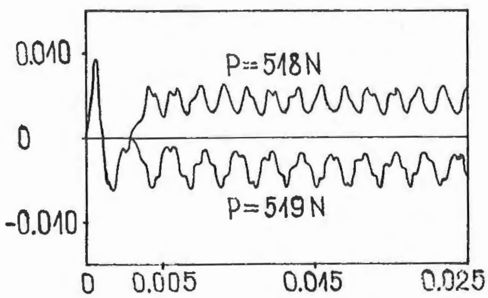


Fig.10.Counterintuitive behavior of the beam

This is demonstrated in Fig.10, where the increase of the load only by 1 N completely changes the behavior of the beam.

The fact that the problem in question is very sensitive to small changes of parameters brings us to a assumption that the behavior of the beam may be chaotic. This problem was discussed by Poddar et al. (1988), who took into account also damping. Their conclusion was that in the phase plane of the initial values some regions exist, where chaos takes place. The boundaries between the chaotic and non-chaotic regions are fractal.

This point of view was criticized by Borino, Perego and Symonds (1989). They marked that Poddar et al. had taken the damping coefficient variable (in the beginning of the motion it was zero), but in a real problem it must be constant. Symonds and his collaborators have also put together the energy diagram and showed that for Shanley model it has two minima (Fig. 11).

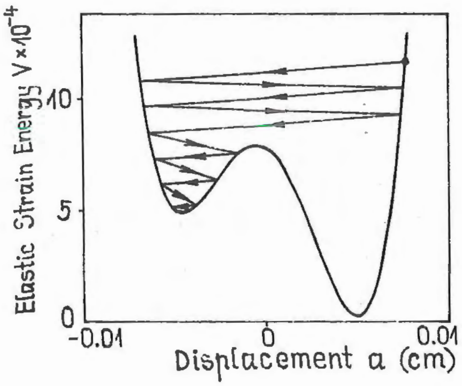


Fig.11.Elastic complementary energy as a function of midpoint deflections.

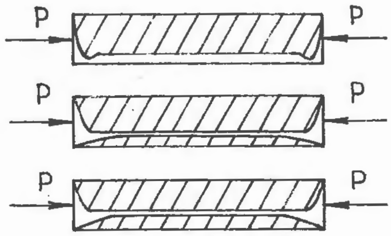


Fig.12.Unloading and secondary loading zones in a compressed beam.

The left minima lays in the range of negative deflections. Due to dissipation the motion stops in one of these minimum points; in which depends from the initial data. The "well" around the left minima is significantly smaller and therefore it is quite difficult to reach the bottom of this "well". So Symonds et al. have shown convincingly that in the case of the Shanley model the motion is fully determined and there cannot be any chaos.

POST-BUCKLING STAGE OF COMPRESSED PLASTIC BEAMS AND CYLINDRICAL SHELLS

Let us consider a structure under compressive forces and assume that the loss of stability has taken place beyond the elastic-limit. If we shall go out of the Shanley conception, then in the beginning of the buckling the whole cross-section of the structure is plastic. In post-buckling stage there appears first an elastic unloading zone, but very soon inside this zone arises a zone of secondary plastic deformations (Fig. 12). The zone of secondary plastic deformations increases rapidly with the growth of deflections. Due to this fact the primary and secondary plastic deformations begin to dominate and between them only a narrow zone of elastic unloading remains.

The analysis of the post-buckling stage of elastic-plastic structures is mathematically complicated, since we must calculate the boundaries of the plastic zones; besides unloading and secondary loading begin in different points of the cross-section at different instants. Therefore it is understandable that we do not have much results in this field at the present time.

The smallness of the elastic zone in the cross-section of the structure brings us to the idea to neglect the elastic deformations at all and to solve the problem for a rigid-plastic material.

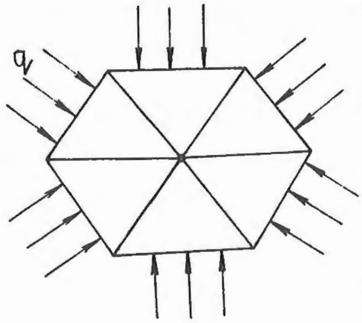


Fig.13.Polygonal plate under uniform compression.

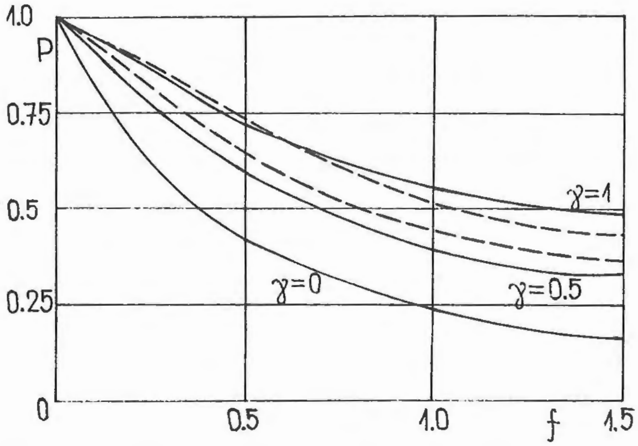


Fig.14.Deflection-load curves for axially compressed beams.

This has been done by Rzanitchyn (1956), who has considered the post-buckling problem of a polygonal plate under uniform edge pressure, it was assumed that the plate deforms along the hinge lines (Fig.13). Rzanitchyn showed that in the post-buckling stage the pressure q monotonously decreases, but approaches asymptotically to a value, which is 0.477 from the critical value of q.

To show the simplicity of the solutions for rigid-plastic materials we shall consider the problem of an axially compressed beam with simply supported ends. Since we have here a static problem and $q = 0$, the equations (4) after integrating with respect to the dimensionless coordinate x get the form (P - is nondimensional compressive load)

$$T = - P = \text{const} , \qquad M(x) = 4Pw(x) .$$

In the midpoint $x = 0$ we have a plastic hinge and there the yield condition (6) holds, followingly $M(0) = 1 - p^2$. If we shall denote $f = w(0)$, we find that

$$f = 0.25(1/P - P) \ . \tag{37}$$

In the bifurcation point we have $f = 0$ and therefore $P = 1$. It follows from equation that if f increases then P monotonously decreases and $P \to 0$ if $f \to \infty$.

In a paper by Lepik (1988) the same problem was solved for rigid-plastic material with linear strain-hardening. The effect of the strain hardening was estimated by the parameter

$$\gamma = \frac{16E'}{\sigma_s} \left(\frac{h}{l} \right)^2 , \tag{38}$$

where E' is the tangent modulus.

The same problem for elastic-plastic materials was solved by Lepik and Sakkov (1976); the solution is approximate since it was assumed that the deflection curves $w(x)$ have the form of a sinusoid. The results are presented in Figs. 12 and 14. In Fig. 12 the distribution of elastic and plastic zones in given, the quantity f denotes the relation of the maximal deflection to the thickness of the beam. In Fig. 14 the load-deflection curves are shown, solid lines correspond to the rigid-plastic material, dotted lines - to elastic-plastic solution. In the case $\gamma = 0$ we have rigid-plastic material without strain hardening. It follows from these results that - in spite of the different assumptions for both solutions - the influence of the elastic deformations is not essential. So we come to the conclusion that at least for beams the conception of rigid-plastic materials gives quite satisfactory results.

Now let us pass to another problem. We shall consider a cylindrical shell under axial compression, the radius, length and thickness of the shell we denote by R, l and h, respectively. We shall assume that the deflections of the shell are axisymmetric (also in post-buckling stage). Let us introduce the nondimensional quantities

$$x = \frac{x^*}{l} , \qquad u = \frac{l}{h^2} u^* , \qquad w = \frac{w^*}{h} , \qquad p = \frac{p^*}{\sigma_s h} ,$$
$$T_i = \frac{T_i^*}{\sigma_s h} \quad (i = 1, 2) , \qquad M = \frac{4M^*}{\sigma_s h^2} , \qquad \alpha = \left(\frac{l}{Rh} \right)^{1/2} . \tag{39}$$

Here u^* is axial displacement, w^* - deflection, p^* - axial compression, T_1^*, T_2^* - radial and circumferential membrane forces, M^* - bending moment; α - is a parameter, which characterizes the length of the shell.

Making use of the formulae (39), the equilibrium equations can be put into the form

$$T_1 = - p \ , \qquad M'' + 4T_1 w'' + 4\alpha^2 T_2 = 0 \ . \tag{40}$$

We shall take the yield condition in the form

$$T_1^2 - T_1 T_2 + T_2^2 + 0.75M^2 = 1 \ . \tag{41}$$

By assuming that the deformation vector $\mathbf{E} = (\varepsilon_1, \varepsilon_2, \varkappa)$ is perpendicular to the yield surface; we obtain

$$\begin{aligned} & u' + \frac{1}{2} w'^2 = \lambda(2T_1 - T_2) \\ & w = - \frac{\lambda}{\alpha^2} (2T_1 - T_1) \ , \qquad w'' = - 6\lambda M \ , \end{aligned} \tag{42}$$

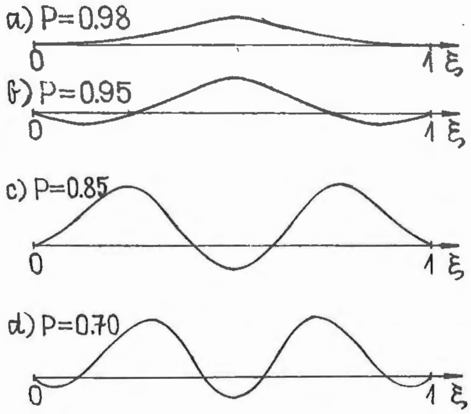
where $\lambda \geq 0$ is the plastic coefficient.

By eliminating λ from equations (42) we get

$$w'' = \frac{6\alpha^2 M}{2T_2 - T_1} w \ , \qquad u' = - \alpha^2 \frac{2T_1 - T_2}{2T_2 - T_1} w - \frac{1}{2} w'^2 \ . \tag{43}$$

The unknown quantities are u, w, T_1, T_2, and M, they can be calculated from the equations (40),(41) and (43). To this system of equations belong also the boundary conditions, which e.g. for a simply supported shell have the form $u(1) = w(0) = M(0) = w(1) = M(1) = 0$. This system of equations was integrated numerically by the method of Runge-Kutta. Let us present here some results from the papers by Lepik (1989 a,1990).

In the instant, when the shell looses its stability, we have $T_2 = M = 0$; it follows from equations (40)-(41) that $T_1 = - p_{kr} = - 1$ and followingly $p_{kr} = 1$. Calculations show, that in the post-buckling stage the parameter p decreases monotonously. Two cases, where $\alpha = 2$ (short shell) and $\alpha = 4$ (long shell) were investigated.



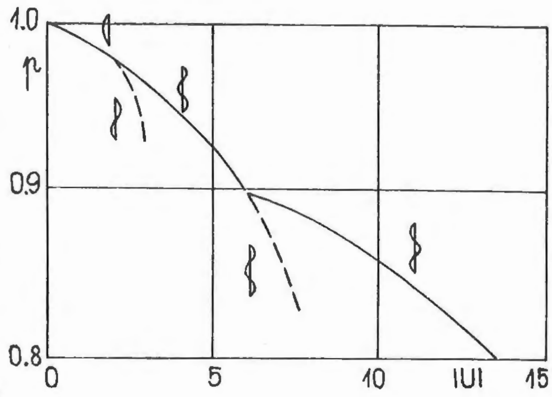


Fig.15.Bending moment distribution in an axially compresssed cylindrical shell.

Fig.16.Nonuniqueness of the solution for the cylindrical shell.

In Fig.15 are given the distributions of the bending moments along the shell for four values of the nondimensional load p. It follows from this Fig., that in the post-buckling stage a redistribution of the bending moments can take place (at $p = 0.98$ we had only one half wave, but for $p = 0.70$ the number of half waves has increased up to 5). It follows from Fig. 16, where the load p versus axial displacement $u(0)$ is presented, that in the post-buckling stage some bifurcation points exist and so the solution is no more unique. In the case of a multiple solution one could ask, which branch of it should be used for practical purposes. It seems to us that we should prefer the solution, for which the axial displacement $u(0)$ is bigger, since here the energy dissipation is higher.

In the case of long shells maximal deflection is no more in the midpoint of the shell; consequently one parameter approximations for the deflection curve (in the form of a parabola or sinusoid) may bring to inaccurate results.

DYNAMIC RESPONSE OF AXIALLY COMPRESSED BEAMS

In this chapter we shall examine two dynamic problems for rigid-plastic axially compressed beams. We shall assume that both ends of the beam are simply supported; the cross-section $x = 0$ may move in the axial direction, but the section $x = 1$ is pinned. For solving these problems we shall go out from the equations (4), (6), (10)-(15); dimensionless quantities of the equations (3) are used. If the axial load is high enough, plastic hinges appear. We shall consider only the case, where we have only one hinge in the midpoint $x = 0.5$. Since the cross-sections, for which $x \neq 0.5$, remain rigid, we have $\dot{\varepsilon}^* = \dot{\varkappa}^* = 0$. Making use of the equations (7)-(8) and from the boundary conditions $w(0) = w(1) = 0$, $u(1) = 0$ we obtain

$$\begin{aligned} &w = 2fx \quad, \qquad u = u_0 - 2\delta f^2 x \quad, \qquad \text{for} \quad x \in (0,0.5) \\ &w = 2f(1 - x) \quad, \qquad u = 2\delta f^2(1 - x) \quad, \qquad \text{for} \quad x \in (0.5,1) \end{aligned} \tag{44}$$

By satisfying the jump condition we find that

$$u_0 = 2\delta f(f - T_1) \quad, \qquad T_1 = T(0.5) \ . \tag{45}$$

Further on we shall integrate the equations of motion, satisfying the boundary conditions and the hinge condition (15); besides in midpoint $x = 0.5$ the yield condition (6) must be fulfilled. Carrying out these calculations we get the following equations

$$\begin{aligned} &1 - T_1^2 = 4Pf - \frac{3}{2} f\ddot{u}_0 + \frac{2}{3} f(\dot{f}^2 + f\ddot{f}) - \frac{1}{3} \ddot{f} \\ &T_1 = - P + \frac{1}{2} \ddot{u}_0 - \frac{1}{2} \delta(\dot{f}^2 + f\ddot{f}) \quad, \end{aligned} \tag{46}$$

where P is the nondimensional compressive load in the end of the beam $x = 0$.

If $\dot{u}_0^*/a < 10^{-3}$, where a is the speed of the elastic wave, the axial inertia effects may be neglected. In this case $\ddot{u} = 0$ and it follows from the first equation of (4) that $T = - P = \text{const}$. The set of equations (45)-(46) gets now the more simple form

$$\ddot{f} = 3(4Pf + P^2 - 1) \quad, \qquad P = \frac{u_0}{2\delta f} - f \ . \tag{47}$$

Now let us go over to the first problem (Lepik, 1989 b). We shall assume that the end of the beam moves with a prescribed axial velocity v. For such kind of problems the velocity is quite moderate and we can use the equations (47), taking for the axial displacement $u_0 = vt$. Now we can integrate the system (47) numerically. The results for the parameters

$h/l = 0.1$ and $v = 0.01$ are presented in Fig.17.

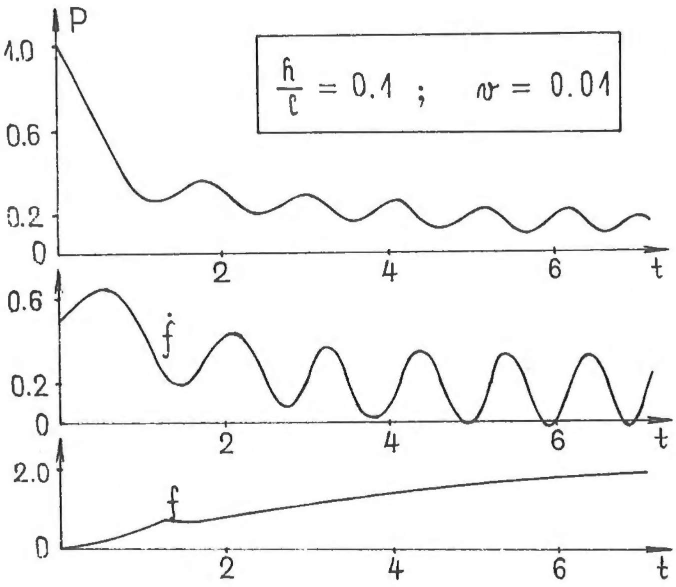


Fig.17. Axially compressed beam, the end section is displaced with a constant speed.

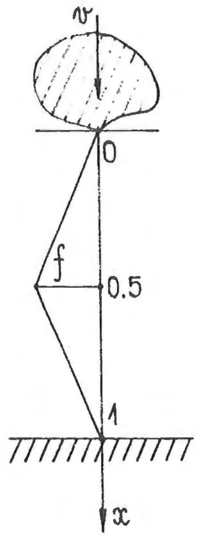


Fig.18. Rigid-plastic beam struck by a mass.

It is interesting to note, that the functions $P = P(t)$ and $f = f(t)$ are oscillating.

The second problem is put up as follows (Lepik, 1990). The end of the beam is struck by a mass G, which is moving with a given velocity v^*. If the striking impulse is great enough the deformations of the beam are plastic with a hinge in the midpoint $x = 0.5$. We have to find the contact force and the midpoint deflection as functions of time (Fig. 18).

The contact force we can calculate from the formula

$$P^*(t^*) = - G \left. \frac{\partial^2 u^*}{\partial t^{*2}} \right|_{x = 0} ,$$

which for nondimensional quantities (3) obtains the form

$$P = - \varkappa \ddot{u}_0 \qquad (\varkappa = G/(\rho Bhl)) . \qquad (47)$$

Now the unknown functions are u_0, T_1, P and f. These we can find by integrating the equations (45)-(47) with regard to time t. This has been done by Lepik (1989 b) for several values of parameters h/l, v and $\varkappa$. It follows from these calculations that the contact force P decreases monotonously in time and at an instant $t = t_1$ if becomes zero. Further on there are two possible cases.

(i) If $\dot{u}_0(t_1) < 0$ the impact is terminated and the mass G moves in the opposite direction with a constant velocity $\dot{u}_0(t_1)$.

(ii) If $u(t)>0$, there can be a multiple impact. Since $P(t_1) = 0$, we shall assume that the hinge gets "freezed". Now the contact force starts to grow again and at an instant $t = t_2$ a second impact takes place. All this can be repeated several times and so we get the multiple impact. This case for parameters $h/l = 0.05$, $v = 0.1$, $\varkappa = 0.5$ is shown in Fig. 19.

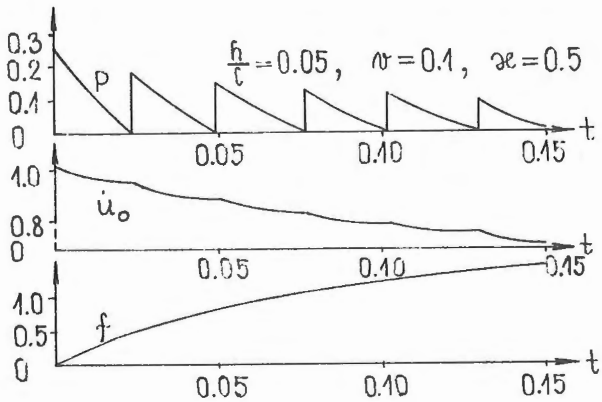


Fig.19. Multiple impact for the beam of Fig.18.

All these solutions are valid if

$$|T(x)| \leq 1 , \qquad |M(x)| \leq 1 - T^2(x) \qquad \text{for} \quad \forall\, x \in [0,1] ,$$

these inequalities must be checked at each instant t.

In our solution we have assumed that the hinge is located in the midpoint of the beam. The calculations show that also other hinge locations are possible, but the location $x = 0.5$ is preferable, since in this case deflections are the biggest.

There are also some limitations, for which the solutions presented in this chapter are valid. First the impulse must be so big that the plastic deformations take place. Secondly the thickness of the beam should not be so small, that the beam would loose its stability in the range of elastic deformations. These limitations have been discussed in the paper by Lepik (1989 b). It has been shown that the initial velocity of the striking mass must fulfill the inequality $v < 2\sqrt{2}(h/l)$; in the opposite case we shall have more than one hinge in the beam. For such problems our method of solution is difficult to use, since the number of hinges and their locations may change during the motion.

It would be necessary to check the theoretical results, obtained in this chapter, also experimentally. Unfortunately all experimental data, which are known to us, belong to more slender beams, for which the loss of stability takes place in the range of elastic deformations. Nevertheless we could quote here the paper of Sugiura et al. (1985). To our opinion experimental and theoretical results of this paper (see e.g. Fig.5) confirm our conclusion about the existence of multiple impact.

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Ülo Lepik
Professor of Tartu University
Department of Theoretical Mechanics