ON THE MECHANICS OF CONTACT AND CRACKING OF SEGMENTAL BEAMS

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Summary The opening of vertical cracks and dry joints may by linearization be reduced to local dilatations and rotations of the centroid axis of the beam. The contact analysis for joints or cracks of beams with finite crack-spacing reveals that the positions of the rotation axis, the compressive force and the centroid of the crack volume comply with the known rules concerning beams with infinitesemal crack-spacing if the tendons are unbonded. This renders possible simple graphical methods to determine the positions mentioned, of beams with various cross-sections and provide information about the relation between the dilatational and rotational parameters. Also the stress distribution and the initiation of longitudinal cracks of the beam block is investigated. Estimates of the location of the longitudinal cracks are deduced. Some of the theoretical results are compared with corresponding tests with post-tensioned segmental beams with dry joints.

1. Introduction

Research concerning the elastic contact problems of cracked beams started in the first half of this century. The first investigation carried out by H. M. Westergaard (1933), was based on the use of stressfunctions in polar coordinates, which led to an approximately parabolic stressdistribution in the compressive zone of the cracked section [1]. Among other investigations worth mentioning is the work led by P. J. Vasiliev 1970-81 where the contact problem of the cracked section has been solved by using Greens functions [2]. For beams with rectangular cross-section and unbonded tendons graphs of the structural quantities where given.

At the chair of structural mechanics of Tampere University of Technology, these problems have been investigated by a different approach using the extremum principles of stiffness [3]. The total state of stress and strain has by superposition been separated into the state, corresponding to the uncracked beam, and the state induced by the edge effect of the cracks. Furthermore this approach makes it possible to incorporate the contact theory into the framework of the classical theory of bending of reinforced concrete beams with zero tensile stresses.

2. The non-monolithic elastic beam

We consider an elastic cylindrical beam with straight tendons, occupying a volume V = LA with span L, and vertical dry joints or cracks in cross-sections A_v, which divide the beam into a sequence of segments (Fig 1a). The load **p*** acting on external surface Γ_e induces then an additional state of stress $(\sigma-\sigma^o)$ and strain ε, the resulting state being $\{p, \sigma, u, \varepsilon, \gamma\}$. Here $\mathbf{p}(x,y,z) = \{\sigma_x, \tau_y, \tau_z\}^T$ stress-vector in cross-section $A(x)$, and $\mathbf{u}(x,y,z) = \{u_x, u_y, u_z\}^T$ displacement vector. The discontinuity $[\mathbf{u}]=\mathbf{u}_v(x,y,z)-\mathbf{u}_{v-1}(x,y,z)$ defines the gap vector $\boldsymbol{\gamma}$ in A_v.

$$\gamma_{v,v-1} = [\mathbf{u}(y,z)] = \gamma_x\mathbf{i} + \gamma_y\mathbf{j} + \gamma_z\mathbf{k} \qquad 1)$$

σ^o is the initial prestress of the beam, **i**,**j**,**k** are unit vectors in principal directions x,y,z of A. If the beam would remain uncracked, the same load would induce a state $\{p_e, \sigma_e, u_e, \varepsilon_e, 0\}$. The superposition of the states of the monolithic beam $\{p_e, \sigma_e, u_e, \varepsilon_e\}$ and the edge-effect of crack $\{p_h, \sigma_h, u_h, \varepsilon_h, \gamma\}$ gives

$$\{p,\sigma,u,\varepsilon,\gamma\} = \{p_e,\sigma_e,u_e,\varepsilon_e\} + \{p_h,\sigma_h,u_h,\varepsilon_h,\gamma\} \qquad 2)$$

If $\{u', \varepsilon', \gamma'\}$ is a possible kinematic state and $\{p'', \sigma''\}$ a state of equilibrium induced by the load **p*** on Γ_e, there holds because of $\mathbf{p}''_{v-1,v}(y,z) = -\mathbf{p}''_{v,v-1}(y,z)$; $[\mathbf{u}'(y,z)] = \gamma'_{v,v-1}$ on A_v and Greens theorem:

$$\int_{\Gamma e}\mathbf{p}^*\cdot\mathbf{u}'d\Gamma = \int_V\sigma''_{ij}\varepsilon_{ij}'dV + \Sigma\int_A\mathbf{p}''\cdot\gamma'dA; \qquad 2')$$

where $\mathbf{p}''\cdot\gamma = \mathbf{p}''_{v-1,v}\gamma_{v,v-1}$. The generalized cross-sectional forces (normal- and shearforce N,Q, moments M) define a vector $\mathbf{R} = \{N, M_y, M_z, Q_y, Q_z, M_x\}$ whose components are

$R_i = \int_A \mathbf{p} \cdot \mathbf{g}_i \cdot dA$ 3)

with: $\mathbf{g}_1 = \mathbf{i}$; $\mathbf{g}_2 = z\mathbf{i}$; $\mathbf{g}_3 = -y\mathbf{i}$; $\mathbf{g}_4 = \mathbf{j}$;   3')

$\mathbf{g}_5 = \mathbf{k}$; $\mathbf{g}_6 = y\mathbf{k} - z\mathbf{j}$

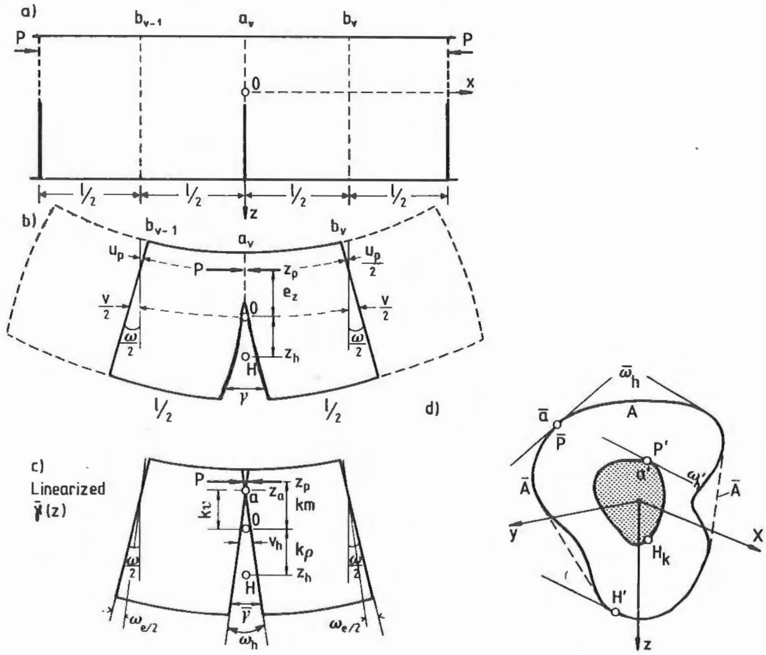
The generalized displacements (translations U, rotations φ)of cross-section $A(x)$ form a vector $\mathbf{U}=\{U_x, \varphi_y, \varphi_z, U_y, U_z, \varphi_x\}$. Using the state of stress $\{p^i{}_e, \sigma^i{}_e\}$ of an infinite uncracked beam subjected in $A(x)$ to only one component $R_i = 1$, denoting $\mathbf{p}^i{}_e = \mathbf{g}^i$, $E_k/E_c = n_k$ we define component U^i by:

$$U^i(x) = \int_A \mathbf{g}^i \cdot \mathbf{u} dA \qquad 4)$$

with: $\mathbf{g}^1 = (n/A)\mathbf{i}$; $\mathbf{g}^2 = (nz/I_y)\mathbf{i}$; $\mathbf{g}^3 = (-ny/I_z)\mathbf{i}$ 4')

$\mathbf{g}^4 = \tau^4{}_y \mathbf{j} + \tau^4{}_z \mathbf{k}$; $\mathbf{g}^5 = \tau^5{}_y \mathbf{j} + \tau^5{}_z \mathbf{k}$; $\mathbf{g}^6 = \tau^6{}_y \mathbf{j} + \tau^6{}_z \mathbf{k}$

Fig 1



We subject the beam to a state of eqilibrium $\sigma^i{}_e$ induced by $R_i = 1$ and the state $\{u, \varepsilon, \gamma\}$ induced by load $\mathbf{p}^*$. Then we get by eq. 2') and using twice Betti's rule due to $\mathbf{p}^o = 0$, and $\mathbf{p}^* = \mathbf{p}^*{}_e$ on Γ_e the mutual generalized displacements of the end faces:

$$\Delta U^i = U^i(L)-U^i(O) = \int_0^L \int_V \mathbf{u}\cdot\mathbf{g}^i dA = \int_V \sigma^i{}_{rs}\varepsilon_{rs}dV + \Sigma \int_{Av}\mathbf{g}^i\cdot\gamma dA \quad 5)$$

and $\int_V \sigma_{ij}{}^k \varepsilon_{ij}dV = \int_V(\sigma_{ij}-\sigma_{ij}{}^o)\varepsilon_{ij}{}^k dV = \int_{\Gamma e}\mathbf{p}^*\cdot\mathbf{u}^k d\Gamma = \int_V \sigma_{eij}\varepsilon_{ij}{}^k dV$

$$= \int_0^L \int_A \mathbf{g}^k\cdot\mathbf{u}_e dA = \Delta U_e{}^k$$

Hence $\quad \Delta U^i = \Delta U^i{}_e + \Delta U^i{}_h; \qquad \Delta U^i{}_h = \Sigma\int_A \mathbf{g}^i\cdot\gamma dA \qquad 6)$

Lemma I. The generalized mutual displacement of the end faces equals the sum of respective displacements of the monolithic beam and the corresponding work at the cracks and joints.

Complementarity [3] requires with $\int \mathbf{g}^i \Delta\mathbf{u}_o dA = 0$; $i = 1,\ldots 6$

$$\mathbf{u}(x,y,z) = \Sigma(\mathbf{g}_i U^i{}_e(x) + \mathbf{g}_i U^i{}_h(x)) + \Delta\mathbf{u}_o \qquad 7)$$

The dilatation $v = \Delta U^1$ and respective mutual rotations $\omega_y = \Delta U^2, \omega_z = \Delta U^3$ of the end faces of the beam are

$$v = \int_A \mathbf{g}^1\cdot\mathbf{u}dA = v_e+v_h = v_e+\Sigma V_h/A; \qquad (V_h = \int_A \gamma_x dA)$$

$$\omega_y = \int_A \mathbf{g}^2\cdot\mathbf{u}dA = \omega_{ey}+\omega_{hy} = \omega_{ey}+\Sigma z_h V_h/I_y; \quad (z_h V_h = \int_A z\cdot\gamma_x dA) \; 8)$$

$$\omega_z = \int_A \mathbf{g}^3\cdot\mathbf{u}dA = \omega_{ez}+\omega_{hz} = \omega_{ez}-\Sigma y_h V_h/I_z; \quad (y_h V_h = \int_A y\cdot\gamma_x dA)$$

where ΣV_h is the gap volume and $\Sigma z_h V_h$, $\Sigma y_h V_h$ its moments with respect to the principal axes. Especially in a beam, where $N = 0$ and $v_e = 0$ there holds, if the endfaces remain plane

$$v = v_h = \Sigma V_h/A; \; \omega_{hy} = \Sigma z_h V_h/I_y; \; \omega_{hz} = -\Sigma y_h V_h/I_z \qquad 9)$$

Hence:

Corollary. The total elongation of the centroid axis of a bent beam is proportional to the crack volume ΣV_h and independent of the depth of the crack. The additional mutual rotations ω_{hy}, ω_{hz} of the end faces caused by the gaps are proportional to the moments of the gap volume $\Sigma z_h V_h$ and $-\Sigma y_h V_h$ respectively.

Fig 2a,b show the linear dependence of v_h on V_h and ω_h on $z_h V_h$ of loaded prestressed granite beams with unbonded tendons and one dry joint in the middle section by measuring the elongation v_h^d and ω_h^d on a length l_d and the corresponding gap volume quantities V_h, z_hV_h of the joint [4].

Considering a beam with constant crack spacing $l = \lambda d$ subjected to a constant compressive force P with eccentricities $y_p = -e_y$, $z_p = -e_z$, the mid sections b_{v-1}, b_v between two cracks remain plane (Fig 1b). Hence extension

$$v_x(y,z) = v_{ex}(y,z) + v_{hx}(y,z) = v + \omega_y z - \omega_z y \qquad 10)$$

$$v_{ex}(yz) = v_e + \omega_{ey}z - \omega_{ez}y; \; v_{hx}(y,z) = v_h - \omega_{hy}z + \omega_{hx}y; \qquad 10')$$

where acc. to the bending theory of beams and conditions 9)

$$v_e = -Pd\lambda/EA; \; \omega_{ey} = Pd\lambda e_z/EI_y; \; \omega_{ez} = -Pd\lambda e_y/EI_z; \qquad 10a)$$

$$v_h = V_h/A; \; \omega_{hy} = z_hV_h/I_y; \; \omega_{hz} = -y_hV_h/I_z \qquad 10b)$$

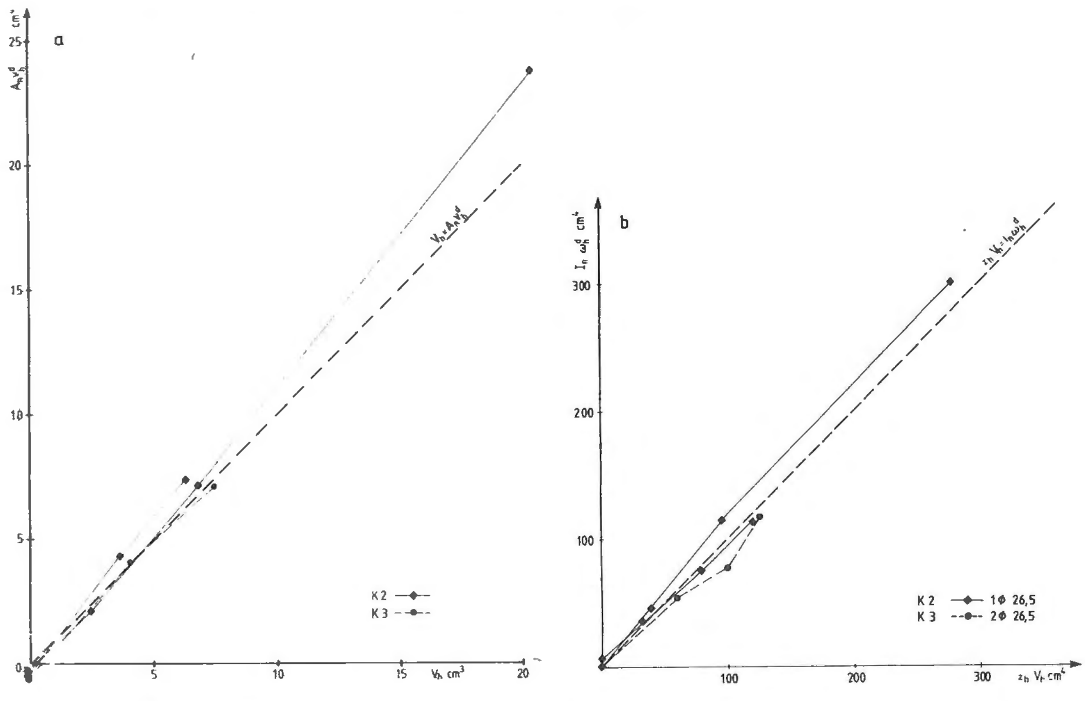


Fig 2. Test of prestressed granite beam with unbonded tendons, net area of section A_h and unbonded tendons

a) Measured extension V_h^d of centroid axis versus gap volume V_h

b) Measured mutual rotation ω_h^d versus gap moment z_hV_h.

The deformation v_x may therefore be separated into the monolithic deformation v_{ex} and the effect of the joint v_{hx}, which actually implies that the two halves of the period are subjected to deformations $v_{ex}/2$ and they are separated by a linear discontinuity $\bar{\gamma}_x = v_{hx}$ at the joint.

The real gap width $\gamma_x(y,z)$ can therefore be substituted by a linear distribution (Fig 1b,c).

$$\bar{\gamma}_x(z,y)=(1+z_h z/i^2{}_y+y_h y/i^2{}_z)V_h/A \qquad 11)$$

where $\bar{\gamma}_x(y'_h,z'_h) = 0$ determines the axis of the additional rotation ω_h caused by the crack. According to this scheme the displacement u_h of the half blocks corresponds to rigid body motions with interpenetration above axis ω_h at z'_h (Fig 1c). This corresponds to an analogous stress distribution in the monolithic beam induced by a compressive force P at $y_p = -e_y$, $z_p = -e_z$

$$\sigma_e(y,z) = - \ (1 + z_p z/i^2{}_y + y_p y/i^2{}_z)P/A = v_{ex} E/l \qquad 12)$$

3. Deformation parameters and stiffness of block

The extensions and rotations depend on P, e_y, e_z and γ

$$v = Pd\beta/EA; \ \omega_y = Pd\alpha_y/EAk_z; \ \omega_z = -Pd\alpha_z/EAk_y \qquad 13)$$

The three parameters β, α_y, α_z depend on the relative kern-distances m_y, m_z ($m_j = e_j/k_j$) and the ratio $\lambda = l/d$. They are not independent because v, ω_y, ω_z are derivatives of the strain energy W which is a second degree homogeneous function of N, M_y, M_z.

$$W = P^2 d\delta(m_y,m_z,\lambda)/2EA = Pu_p/2 \ ; \ u_p = -v_x(y_p,z_p) \geq 0 \qquad 14)$$

From relations $\partial W/\partial N = v$; $\partial W/\partial M_y = \omega_y$; $\partial W/\partial M_z = \omega_z$; there follows

$$\alpha_y = (1/2)\partial\delta/\partial m_z; \ \alpha_z = (1/2)\partial\delta/\partial m_z; \ \beta = m_y\alpha_z+m_z\alpha_y-\delta \qquad 15)$$

Because the state $\{p_h, \sigma_h\}$ corresponds to no external load the strain energies can be separated

$$W(\sigma) = W(\sigma_e + \sigma_h) = W(\sigma_e) + W(\sigma_h) \qquad 16)$$

where $\quad W(\sigma_e) = Pu_{ep}/2 = P^2 d\delta_e/2EA$

$$W(\sigma_h) = Pu_{hp}/2 = P^2 d\delta_h/2EA$$

The relations 15) hold separately for $\alpha_e, \beta_e, \delta_e$ and α_h, β_h, δ_h since $\delta = \delta_e + \delta_h; \ldots \alpha_i = \alpha_{ei} + \alpha_{hi}$. The α_e, β_e, δ_e are determined by eq. 10a) whereas expressions 10b) become

$$v_h = Pd\beta_h/EA; \ \omega_{hy} = Pd\alpha_{hy}/EAk_z; \ \omega_{hz} = Pd\alpha_{hz}/EAk_y \qquad 16')$$

and the interpenetration (Fig 1c) of the rigid blocks at y_p, z_p are according to 10)

$$u_{hp} = Pd\delta_h/EA = -v_h(y_p, z_p) = -(1+z_p z_h/i_y{}^2 + y_p y_h/i_z)V_h/EA \geq 0 \qquad 16'')$$

The parameters $\beta_h, \alpha_h, \delta_h$ can be determined either by solving the contact problem for the resulting state $\{p, \sigma, u, \varepsilon, \gamma\}$ and then obtain $\beta_h \ldots \delta_h$ from the differences $\beta_h = \beta - \beta_e \ldots \delta_h = \delta - \delta_e$, or by confining the problem exclusively to the state $\{p_h, \delta_h, u_h, \varepsilon_h, \gamma\}$. In this case the block halves of the period are loaded at the joint, where $\gamma > 0$ by known forces $p^*_{hx} = -\sigma_e$, the end faces b_{v-1}, b_v remaining plane. The work carried out is according to 10b) 11)

$$\tfrac{1}{2}\int_{Aav} p^*{}_h \cdot \gamma dA = Pu_{hp}/2 = W(\varepsilon_h) \qquad 17)$$

confined to section a_v since $\int p_h \cdot u_h dA = 0$ on A_{bv}, $p_{hx} = \sigma_{hx}$ being an equilibrum distribution.

A more transparent geometrical interpretation provides the concept of stiffness. The stiffness of the period can be defined as the ratio of the load P to the load-displacement u_p. The total stiffness D and the interpenetration stiffness D_h are

$$D = P/u_p = EA/d\delta; \qquad D_h = P/u_{hp} = EA/d\delta_h \qquad 18)$$

These quantities are defined either by an admissible state of stress $\{p'',\delta''\}$, which include the non-tension condition $\sigma_x < 0$ at dry joints and the stress energy $W(\sigma'')$ (eq. 14, 15)

$$D(\sigma'') = P^2/2W(\sigma''); \; D(\sigma_h'') = P^2/2W(\sigma_h'') \qquad 19)$$

or by an addmissible state of strain $\{u',\varepsilon',\gamma'\}$, which includes the impenetrability condition $\gamma_x \geq 0$ at the joint:

$$D(\varepsilon')=2W(\varepsilon')/(u_p')^2; \; D_h(\varepsilon'_h)=2W(\varepsilon'_h)/(u'_{hp})^2 \qquad 19')$$

If in the structure the unloaded state is devoid of initial stress $\delta°$ and gaps then, according to the extremum principles of stiffness $D(\sigma'')$ provide lower bounds to the actual stiffness D and $D(\varepsilon')$ upper bounds to the actual stiffness D [3] Hence:

$$D(\sigma'') \leq D \leq D(\varepsilon'); \qquad D_h(\sigma'') \leq D_h \leq D_h(\varepsilon_h') \qquad 19')$$

$$\delta'' \geq \delta \geq \delta'; \; \delta_h'' \geq \delta_h \geq \delta_h' \qquad 19'')$$

The lower bound solution based on $\{p'',\sigma''\}$ provide an upper bound δ_h'' whereas the upper bound solution based on $\{u',\varepsilon',\gamma'\}$ provide a lower bound δ_h' to δ_h because of eq. 18).

The functions $\beta_e(m_y,m_z,\lambda)\ldots\delta_e(m_y,m_z,\lambda)$ are all proportional to the slenderness λ acc. to eq. 10a)

$$\beta_e=\lambda\beta_e(m_y,m_z); \; \alpha_{es}=\lambda\alpha_{es}(m_y,m_z); \; \delta_e=\lambda\delta_e(m_y,m_z); \; s=y,z \qquad 20)$$

Closed expressions for parameters $\beta_h\ldots\delta_h$ cannot generally be obtained. The dependence on λ is rather complicated with one expection where the compressive stress-distribution in the cross-section is partly linear. This materializes in a beam with very dense crack spacing λ -> 0 in accordance with the classical theory of bending of reinforced concrete beams. In this exceptional case the contact problem may be solved by elementary means. We get if λ -> 0 with $\beta° = \beta_e+\beta°_h$; . . . . , $\delta° = \delta_e+\delta°_h$.

$\beta^\circ = \lambda\dot{\beta}_o(m_y,m_z)$; $\alpha^\circ_s = \lambda\dot{\alpha}_{os}(m_y,m_z)$; $\delta^\circ = \lambda\dot{\delta}_o(m_y,m_z)$ 20a)

$\beta^\circ_h = \lambda\dot{\beta}_h(m_y,m_z)$; $\alpha^\circ_{hs} = \lambda\dot{\alpha}_{hs}(m_y,m_z)$; $\delta^\circ_h = \lambda\dot{\delta}_h(m_y,m_z)$ 20b)

In the following we confine ourselves to segments with unbonded tendons and accordingly to the deformations of the net area A_n of the joint. For a rectangular section the quantities $\beta_h \ldots \delta_h$ are with $m_y = 0$; $m_z = m$

$\dot{\beta}_h=(|m|-1)^2/(3-|m|)^2$;

$\dot{\alpha}_{hz}=(4-|m|)(|m|-1)^2/3(3-|m|)^2$;

$\dot{\delta}_h=(|m|-1)^3/3(3-|m|)$ 20'')

If $\lambda \neq 0$ approximate solutions are sought by variational methods [6]. For a block $a_v - a_{v+1}$ with narrow rectangular cross section a lower bound solution is obtained by the stress-function expansion

$\phi_a = \sum_s g_{sa}(\eta)(1-(-1)^s \cos s\pi\xi)+f(\eta)$; where $z < z_o$, $f_{,zz} < 0$

$\phi_b = \sum_s g_{sb}(\eta)\ [1-(-1)^s)\cos s\pi\xi]$; where $z \geq z_o$, $f_{,zz} = 0$ 21)

with $\eta=2z/d$, $\xi=2x/l$, $s=1\ldots n$, and boundary conditions satisfying the equilibrum conditions at $z = \pm\ d/2$ and z_o, the integral conditions and stress-conditions at $x = 0$, $\pm\ l/2$

$\int f_{,zz}dz=-P$; $\int f_{,zz}zdz=Pkm$; $\tau_{xz} = 0$ 21')

The solution is obtained by minimizing acc. to Kantorovitsh's method the stress energy by variation of functions $f(\eta)$, $g_{sa}(\eta)$, $g_{sb}(\eta)$.

$W(\sigma'') = (\frac{1}{2}E) \int_{-1/2}^{+1/2} \int_{-d/2}^{+d/2} (\partial^2\phi/\partial x^2+\partial^2\phi/\partial z^2)^2 dxdy$

An upper bound solution is obtained by a kinematically admissible deformation-field $\{u_h', \varepsilon_h', \gamma'\}$ of a half-block a_v-b_v clamped without friction to plane $x = 0$ (Fig 3) and

$$u'_{hx}=\Sigma(A_\mu(\gamma_\mu-\theta_\mu\cos(\mu\pi x/l))+G_\mu(\gamma_\mu\pm\theta_\mu\sin(\mu\pi x/l)))/2+$$

$$\Sigma\ \Sigma(B_{ik}\cos(2i\pi z/d)\sin(2k\pi x/l)+D_{ik}(\sin(2i\pi z/d)\cos(2k\pi x/l))$$

$$u'_{hz}=\Sigma(A_\mu(2\gamma/\mu\pi-\gamma_\mu\sin(\mu\pi x/l))+\dot\theta_\mu(\mu\pi x/l-\sin(\mu\pi x/l))-$$

$$G_\mu(\gamma_\mu(\mu\pi x/l-(\mu^2\pi^2+8)/4\mu\pi)\pm\dot\theta_\mu\cos(\mu\pi x/l)))/2\mu\pi)$$

$$+\Sigma\ \Sigma(C_{ik}\cos(2i\pi z/d)\cos(2k\pi x/l)+E_{ik}\sin(2i\pi z/d)\cos(2k\pi x/l))\quad 22)$$

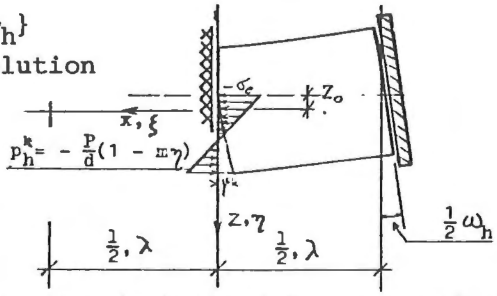
This field satisfies the condition at x=0: $u_{hx}' = \gamma/2 \geq 0$ and at $x = l/2$: $u_{hx}' = \gamma/2$. Here $\gamma = \Sigma A_\mu\gamma_\mu \geq 0$; $\bar\gamma = \Sigma A_\mu\bar\gamma_\mu$; $\theta_\mu = \bar\gamma_\mu - \gamma_\mu$ which represents an equilibrium distribution on b_v. The solution is obtained by the minimum of the potential energy

$$\pi = W(\varepsilon'_h) - \int_{Av} p_{hx}^* u'_h dA \qquad 22')$$

with respect to the constants $A_\mu, G_\mu, B_{ik}, C_{ik}, D_{ik}, E_{ik}$

Another upper bound solution is obtained by the finite element method where the displacement field between neighbouring elements has no discontinuities.

a) Fig 3. Use of $\{u_h, \varepsilon_h, \gamma_h\}$ for an upper bound solution



From these solutions upper bounds δ_h'' and lower bounds δ_h' for parameters δ_h are obtained, with corresponding values for α_h, β_h and for varying values $m = -z_p/k_z$ along the symmetry axis of the cross-section. δ_h and β_h assume only non-negative values whereas α_h:s sign depends on the direction of the bending moment (Fig 4). Of special interest are the parameters dependence on the slenderness λ. For small values we have at given m_z a linear relationship according to formulae 20),20b). Because the edge effect of the joint vanishes at a certain distance from the joint if $\lambda \to \infty$ the parameters approach constant values (Fig 5). For a rectangular section we have approximate values

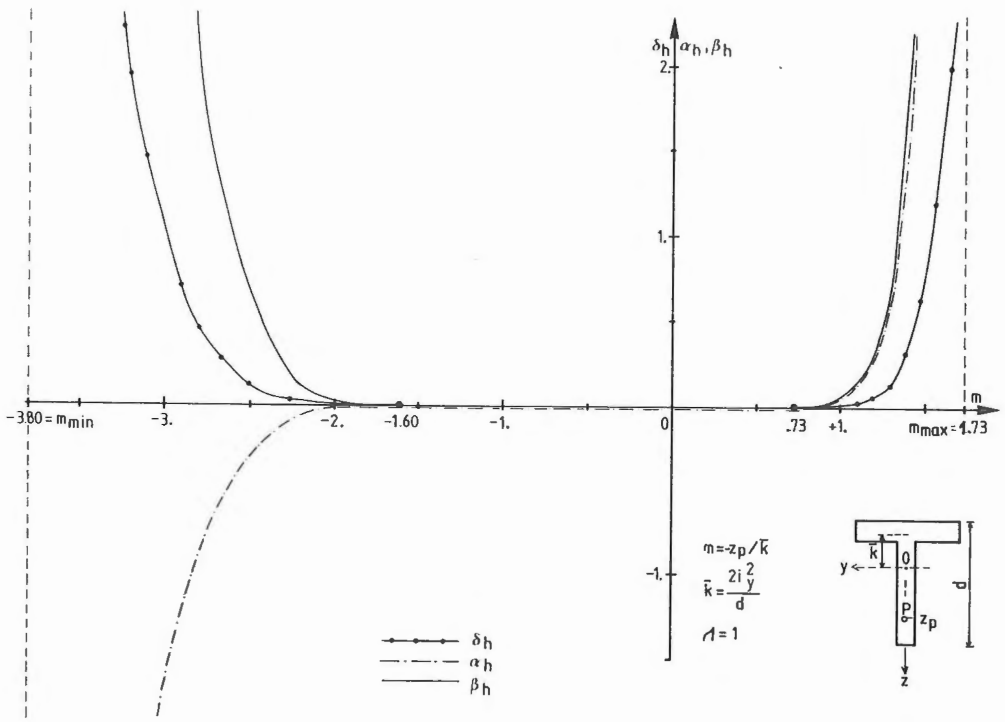


Fig 4. **Deformation parameters of dry joint of T beam.**

$\delta_h(m, \;) = a(1-(|m|-2)^2)^r \delta_h;$

$\beta_h(m, \;) = (3+r(2-|m|)|m|)\delta_h(m, \;)/(1-(2-|m|)^2)$

$\alpha_h \; (4-|m|+r(2-|m|)\delta_h(m, \;)/(1-(|m|-2)^2);$

with a ∴ 0.74; r ∴ 0.55

With increasing eccentricity m the difference between upper bound and lower bound parameter-values increases, which in some cases leads to a decrease of the parameter-values of the upper bound solutions. An estimation of the limit value of δ_h when $\lambda ->$ is possible taking into consideration that δ_h is proportional to $W_h = \int w(\sigma_h)dV$, where $w(\sigma_h) = \sigma_{hij}\varepsilon_{hij}/2$; $\partial w/\partial\sigma_{hij} = \varepsilon_{hij}$. Therefore if the length l of the period $b_{v-1}-b_v$ is increased by dl we get with $\partial\omega/\partial l = \varepsilon_{hij}\partial\sigma_{hij}/\partial l$:

$$dW_h = dl\int_V \partial w_h/\partial l dV + dl(\int_A w dA)_{1/2} = dl(\int_V \partial\sigma_{hij}/\partial l \varepsilon_{hij} dV + (\int_A w dA)_{\frac{1}{2}})$$

but $\int_V (\partial\sigma_{ij}/\partial l)\varepsilon_{hij} dV = \int_{Ab}(\partial p_h/\partial l)\cdot u_{hx} dA - \int_{Av}\partial p_h/\partial l\cdot\gamma dA =$ 0 since on the endface b_v u_{hx} is linear and $\partial p_h/\partial l$ is an equilibrium distribution. On A_v $\partial p_h/\partial l = 0$ only where $\gamma = 0$ because of γ:s continuity. Hence $dW_h = dl\int w dA \geq 0$, $\partial\delta_h/\partial l \geq 0$. Thus the parameter δ_h increases and the stiffness D monotonously decreases with increasing λ at given eccentricity of the load P.

In Fig. 5b the parameters of a T beam are calculated by the finite element method, which provide lower bounds to the parameters δ_h. With increasing length the parameter values are decreasing if $\lambda > 1$ which implies an increasing error. The optimal values are therefore the highest values attained approximately at $\lambda = 1$. The parameters α_h and β_h seem to follow the same pattern as δ_h with $\delta_h'' - \delta'_h \geq 0$. A sufficient condition for this is $\beta''_h - \beta'_h \geq |\alpha''_h| - |\alpha'_h|$. Then from relation $\delta_h'' - \delta_h' \geq 0$ and eq. 15) there follows: $\beta''_h - \beta'_h \geq 0$; $|\alpha''_h| - |\alpha'_h| \geq 0$.

4. Smoothed stress- and gap-distributions at dry joints

Next we investigate the effect of the slenderness λ of the period on the distributions σ_x, γ_x at the joint. The centroid $H(y_h, z_h)$ of the gap volume V_h is determined by the parameters α_h, β_h if the centroid e_y, e_z of P is known (eq. 9,10).

$$z_h = c_z \alpha_{hy}/\beta_h \qquad y_h = c_y \alpha_{hz}/\beta_h \tag{23}$$

The zero-line of $\gamma(z,y)$: $l(H) = \gamma(y'_h, z'_h) = 0$ intersects the principal axes at

$$y'_a = -i^2_z/y_h; \; z'_a = -i^2_y/z_h \tag{23'}$$

In the limiting case $\lambda \to 0$ corresponding to the classical theory of reinforced concrete the continuous compressive stress distribution $\sigma(y,z) \leq 0$ is linear on a part A_- of A. Only the part above the neutral-line $l(\sigma_-) = \sigma_- = 0$ is strained by $\varepsilon = \sigma_-/E$. If we consider a symmetric period b_{v-1}, b_v the mutual displacement of the endfaces is $v_x = \varepsilon_x l = l\sigma_-/E$. Starting from $l(\sigma_-)$ we have a gap-distribution $\gamma°(y,z) \geq 0$ linear on $A_+ = A - A_-$ (Fig 6).

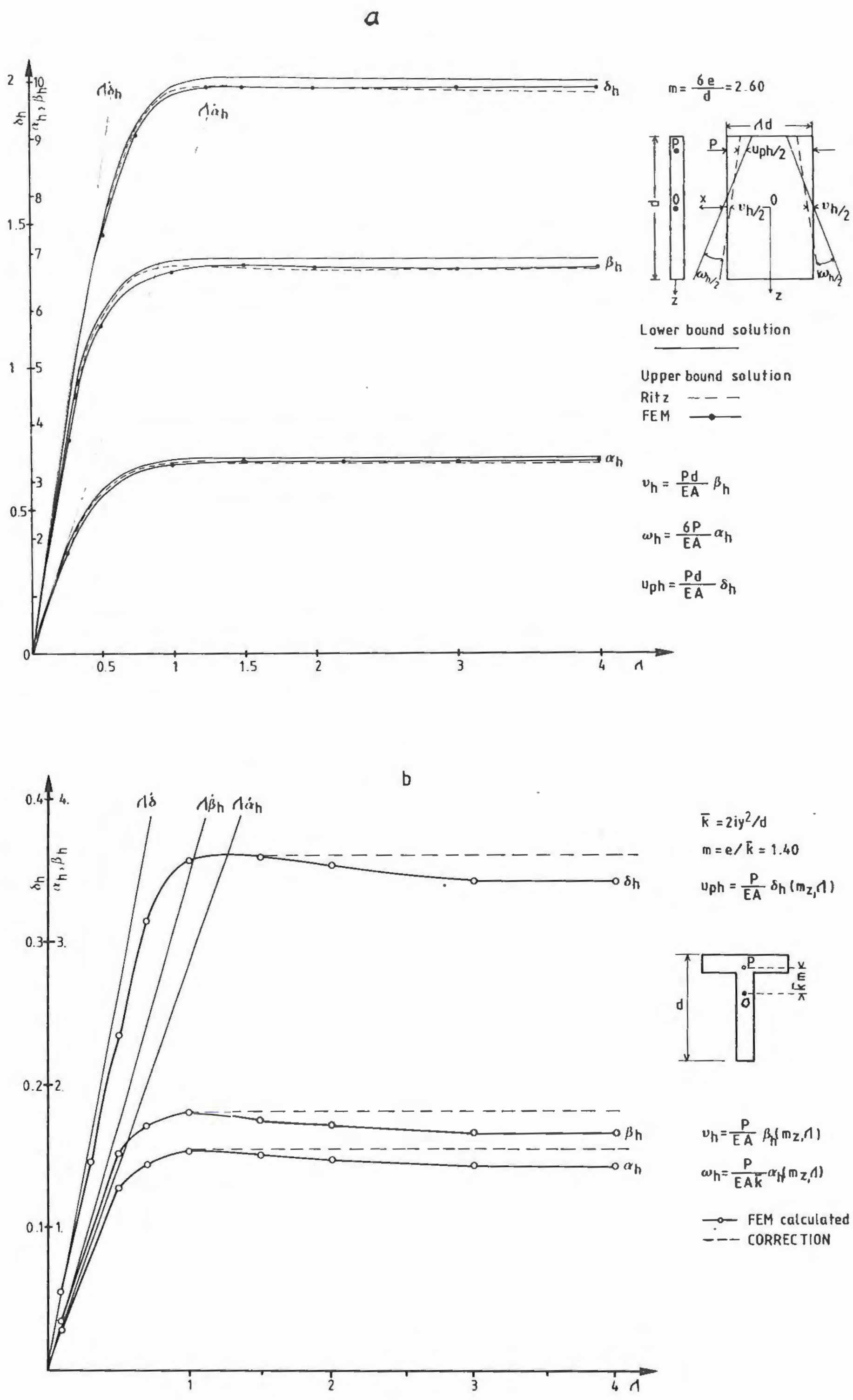


Fig. 5 Dilatation parameters β_h, rotation parameters α_h and interpenetration parameters δ_h versus slenderness ratio λ.

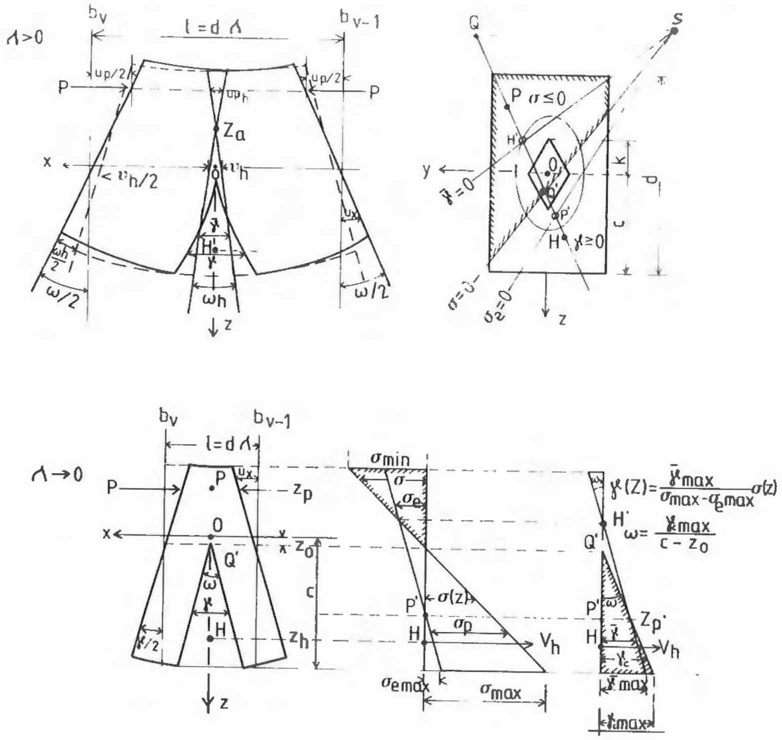


Fig. 6

The symmetry planes b_{v-1}, b_v remain plane. The parts below $l(\gamma^\circ)$ remain unstrained. Hence $v_{x+} = \gamma^\circ$ below $l(\sigma_-)$ is an extension of the plane v_{x-}. The mutual displacement of planes b_{v-1}, b_v is therefore a linear distribution over A

$$v_x(y,z) = l\sigma_-/E + \gamma^\circ \tag{24}$$

to which corresponds a stressdistribution

$$\sigma^\circ_x(y,z) = \sigma^\circ_- + \sigma^\circ_+ \tag{25}$$

where $\sigma^\circ_+ = E\gamma^\circ/l$ is the fictitious stress induced by the gap. σ° represents thus two stressblocks: the real compressive block σ°_- with resultant $\mathbf{P} = -P\mathbf{i}$ $(P \geq 0)$ at point $P = \{y_p, z_p\}$, and the tensile block σ_+ with resultant $\mathbf{H} = H\mathbf{i}$ $(H \geq 0)$, at point $H = \{y_h, z_h\}$. The fictitious force-void couple $(\mathbf{P},\mathbf{H})$ induces linear distributions σ_e and $\sigma^c_\gamma = Ev_x/l$.

$$\sigma_e = -(1+z_p z/i_y + y_p y/i_z)P/A; \quad \sigma^c_\gamma = (1+z_h z/i_y{}^2 + y_h y/i_z{}^2)H/A \tag{26}$$

where $H = EV_h/d\lambda = P\dot{\beta}_h$; $y_h = c_y\dot{\alpha}_{hz}/\dot{\beta}_h$; $z_h = c_z\dot{\alpha}_{hy}/\dot{\beta}_h$. Superposition gives

$$\sigma^\circ = \sigma^\circ_- + \sigma^\circ_+ = \sigma_e + \sigma_\gamma; \quad \sigma^\circ_- - \sigma_e = -(\sigma^\circ_+ - \sigma^\circ_\gamma) \qquad 27)$$

which is induced by a force $Q = (\dot{\beta}_h - 1)Pi$ acting at $Q(y_q z_q)$ on line PH. The zero lines $l(\sigma^\circ), l(\sigma_e), l(\sigma_\gamma)$ define mappings, point to line, $Q \to l(Q)$; $P \to l(P)$; $H \to l(H)$. $l(Q)$ determines the axis of local rotation ω°

$$L(P) = 1 + z_p z/i_y^2 + y_p y/i_z^2 = 0; \quad L(H) = 1 + z_h z/i_y^2 + y_h y/i_z^2 = 0,$$

$$L(Q) = 1 + z_q z/i_y^2 + y_q y/i_z^2 = 0 \qquad 28)$$

Because of the similarity of the expressions for $\sigma_e(y,z)$ and $\tilde{\gamma}(y,z)$ we have the following dual relations known from the theory of eccentrically loaded elastic beams.

i) If P moves along line a_o through the origin O then the neutral line $l(P)$ experiences a translation in the same direction.

ii) If P moves along a line a_p outside the origin O the neutral line $L(P)$ turns round a fixed point $P'(a_p)$ which is the point of action of P whose σ_e-distribution has neutral line $l(P') = a_p$.

i') If centroid H of V_h moves along line a_h through O then axis $\omega_h = l(H)$ experiences a translation in the same direction.

ii') If H moves along a line a_h outside O, the axis $\omega_h = l(H)$ turns round a fixed point $H'(a_h)$ which is the centroid of gap volume V'_h whose $\gamma(y,z)$ distribution has zero line $l(H') = a_h$.

If we consider corresponding collinear points Q,P,H the neutral lines $l(P)$ and $l(H)$ have common point S in the projective plane π. Hence $\sigma_e(S) = 0$; $\sigma_\gamma(S) = 0$. But because of 27) we have also $\sigma^\circ(S) = \sigma_e(S) + \sigma^\circ_\gamma(S) = 0$. Therefore if we denote the points of intersection of the axis PH with $l(P)$, $l(H)$ and $l(Q)$, P', H' and Q' respectively:

iii) The neutral lines $l(P)$, $l(H)$ and $l(Q)$ have common point S which is the point of action of **Q** or **P** or **H** whose neutral line $L(S)$ is PH and contrarely, if Q, P or H are moving along PH, the neutral lines $l(Q)$, $l(P)$ and $l(H)$ respectively are turning round S. Furthermore $l(P') = SP$; $l(H') = SH$; $l(Q') = SQ$ (Fig 6).

Because the point of action of **P** and **H** are independent of the magnitude of the force-void couple (**P**,**H**), there is a one to one mapping H = T(P) of points P $\in$ A to points H $\in$ A in the projective plane. Compressive force **P** at point P induces tensile force **H** = $-\dot{\beta}_h$(P)**P** at point H. Since $\sigma^\circ_- + \sigma^\circ_+$ generates a plane σ°(y,z) induced by force **P** at P and **H** at H, then $(-\sigma^\circ_-)+(-\sigma^\circ_+)$ generates a plane $-\sigma^\circ$(z,y) induced by load **P**" = $-$**H** = β_h(P)**P** at H, and force **H**" = $-\beta_h$(H)**P**" = $-$**P** at P. Hence

$$T(H) = TT(P) = P; \; TT = I; \quad \beta_h(P)\beta_h(H) = 1 \qquad 28)$$

Since V_h = Pdβ_h(P)/E we have

iv) If compressive force **P** with centroid at P induces gap volume V_h with centroid at H then force **P** at H induces gap volume V'_h at point P (Theorem of reciprocity)

From this follows

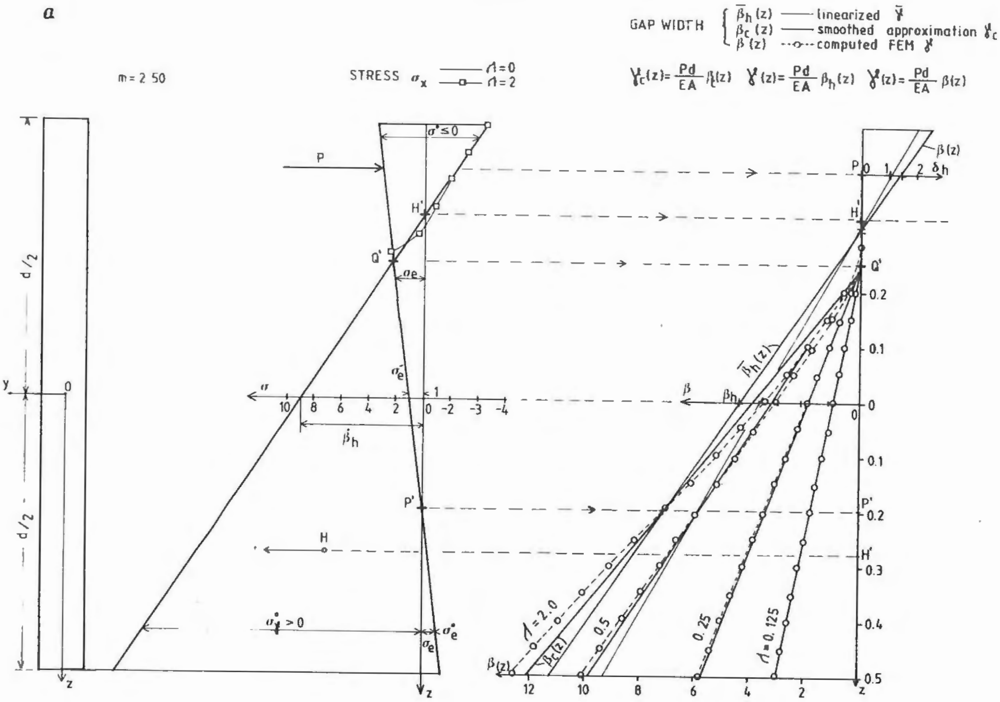
v) If **P** moves from P', a point on ∂K, the conture of the kern, to a point P on the conture $\partial\bar{A}$ of the convex hull $\bar{A}$ of A, Q moves from Q_{k-} = P' through infinity to Q_{k+} on ∂K, and l(Q) moves from the tangent of $\partial\bar{A}$ through H' to a tangent point Q_{k+} on ∂K. L(Q) = ω° moves from l(Q_{k-}) = l(P') to a tangent l(Q_{k+}) of ∂A through P.

v') If **H** moves from point H on ∂A to a point H_k = Q_{k+} on, ∂K, the axis ω_h = l(H) moves from a tangent of ∂K through P' to a tangent through P on ∂A (Fig. 1d).

A numerical comparison shows that the stressblock σ_- for different λ-values doesn't differ much from the linear one on A_- of the classical theory with $\lambda \to 0$. Also the relations between the deformation parameters α_h, β_h, δ_h closely approach the corresponding values for $\lambda \to 0$ with same m_y, m_z. Hence

$$\alpha_h(\lambda)/\beta_h(\lambda)=\dot{\alpha}_h/\dot{\beta}_h; \; \delta_h(\lambda)/\beta_h(\lambda)=\dot{\delta}_h/\dot{\beta}_h \qquad 29)$$

This means that the results concerning σ_x obtained for $\lambda \to 0$ can by induction be extended to values $\lambda \neq 0$. Relations 29) indicate that for given eccentricity of load P the centroid of V_h and the lines l(P), l(Q) and l(H) are approximately independent of λ and can be considered as invariants together with their points of intersections P' Q'



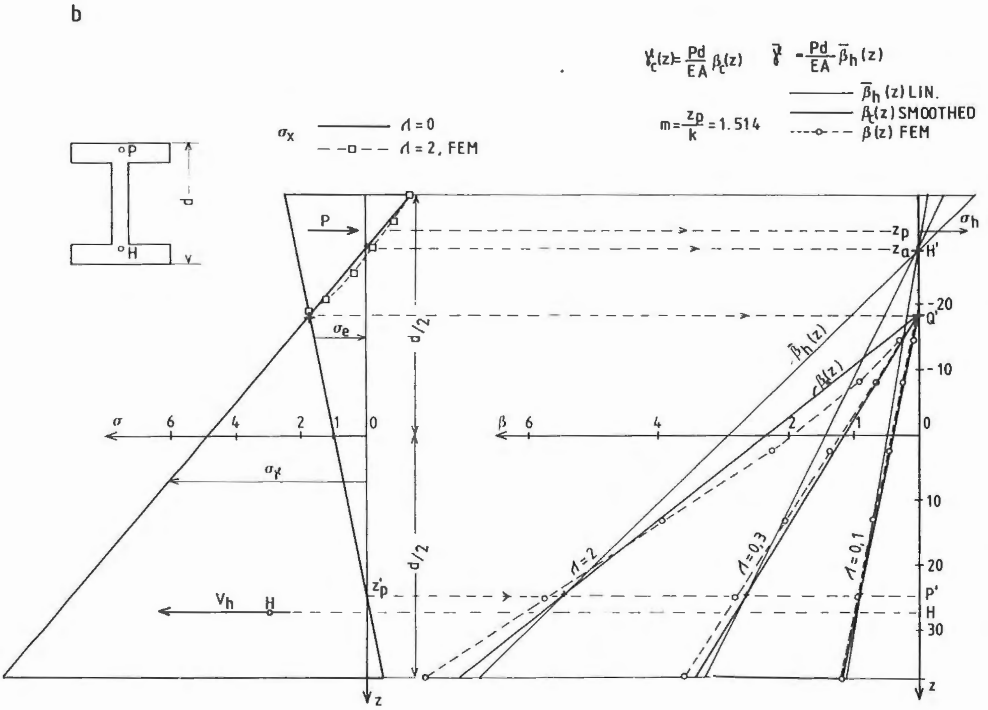


Fig. 7 Compressive stress distribution σ_x and geometric determination of relative gap width $\beta_c(z)$ of dry joint.

and H′ with PH. In order to determine an approximate distribution γ_c affine with σ°_+ of the real one γ we can start from distribution 11).

$$\gamma(y,z) = Pd\beta_h(y,z)/EA \qquad 30)$$

where $\beta_h(y,z)$ is a linear function of y,z

$$\bar{\beta}_h(y,z) = \beta_h(1 + z_h z/i_y{}^2 + y_h y/i_z{}^2 \qquad 31)$$

Especially $\beta_h(0,0) = \beta_h$; $\beta_h(y_p,z_p) = -\delta_h$. The smoothed gap width γ_c at the joint is therefore a linear distribution on A_+

$$\gamma_c(y,z) = Pd\beta_c(y,z)/EA \qquad 32)$$

which is determined by the β_c-plane spanned by the lines l(Q) and β_h(l(P)) [4].
On $\overline{HP}$ γ_c is determined by the β_c line through zeropoint Q′ and the point $\bar{\beta}_h$(P′). $\bar{\gamma}$(PH) is determined by the line through zeropoint H′ and point $\bar{\beta}_h$(O′) through O′ = $\overline{PH} \cap$ SO, (Fig. 6) or by the straight line through $\bar{\beta}_h(P) = -\delta_h$ at point P and $\bar{\beta}_h$(O′) at O′. If the line passes through the origin O then $\bar{\beta}_h(O) = \beta_h$. For some symmetric profiles the γ distribution along the vertical symmetry line y = 0 has been calculated. In Fig 7a the distribution β_c is determined for different λ-values by using for $\bar{\beta}_h$ the line through $\bar{\beta}_h$(P) and $\bar{\beta}_h$(O). $\bar{\gamma}$:s zeropoint is very close to the theoretical point H′. In Fig 7b the theoretical point H′ is used. The diagrams for σ_x and γ calculated, show good agreement with the linear values σ°_- and γ_c although the difference between calculated γ and smoothed γ_c increases with growing λ, but the difference at the maximum gap width does not exceed 7 % when $\lambda = 2$ which actually corresponds to infinite λ.

Also the reciprocity relation between the position vectors of H and P seems well established as is seen from Fig 9. If the cross-section has double symmetry the distance Δz between H and P remains approximately constant.

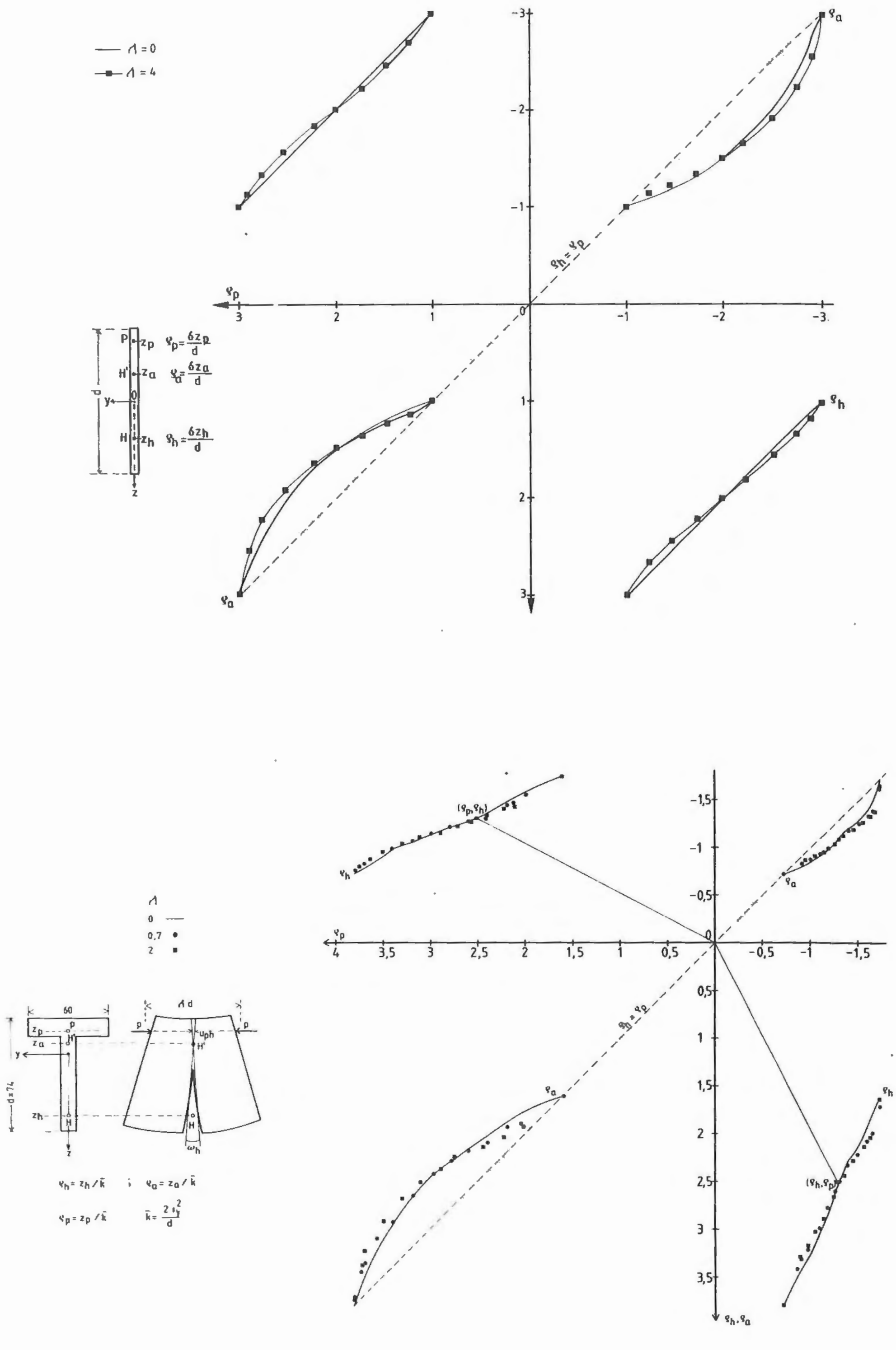


Fig. 9 Reciprocity relation of ratios $\rho_h = z_h/k, \rho_p = z_p/k$ for various slenderness ratios λ.

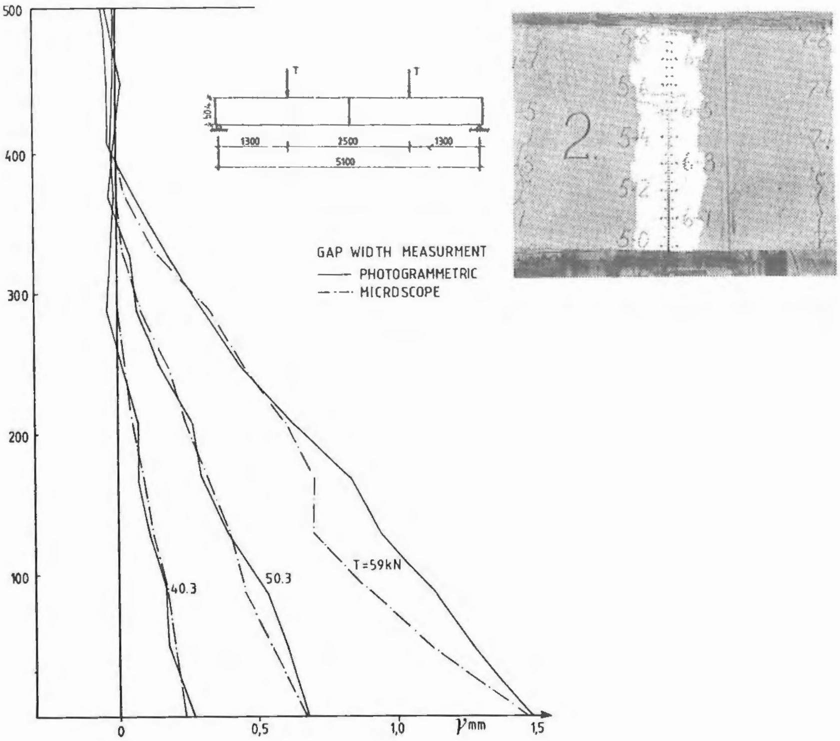


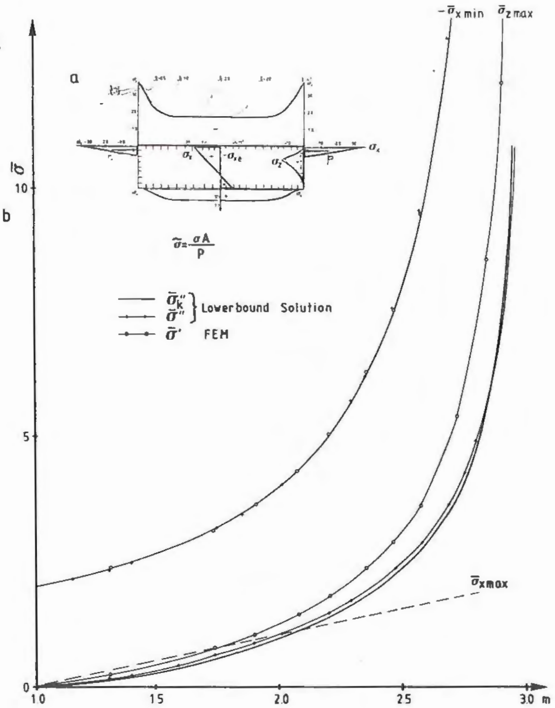
Fig. 8 Measured gaps γ of dry joint of prestressed concrete beam with unbonded tendons. Obs. linearity of γ in A_+.

The numerical comparison carried out, confined to symmetric profiles, covers only in part the general theory developed. Nevertheless they show conclusively that in the most important case of symmetrical profiles with force-void line in the axis of symmetry the asymptotic approach to the classical theory with $\lambda \to 0$ provides a satisfactory base for the evaluation of stress and gap width at the joint.

5. The longitudinal cracking at dry joints

The stress-distribution of a block with rectangular profile between two dry joints loaded by a compressive force P is shown in fig 10a. The σ_x distribution start at the end faces as compressive stress close to the linear distribution $\sigma^\circ_- < 0$ approaching with increasing λ at the middle part of the block the distribution σ_e of the monolithic beam. At the end faces appear tensile stresses σ_z with a maximum below the limit line $\sigma_x = 0$. This maximum increases with λ from 0 at $\lambda = 0$ to a limit value σ_{zmax} when

Fig. 10 a)Stressdistribution in segmental block with rectangular cross-section. b) Dependence of relative- $\sigma_{xmin}, \sigma_{xmax}$, σ_{zmax} on eccentricity m_z.



$\lambda \geq 2$. σ_{zmax} increases with increasing eccentricity m and is considerable greater than the maximum tensile stress σ_{emax} at the lower edge of the block. The maximum values of the compressive stress σ_{xmax} coincide with those corresponding to $\lambda = 0$ independently of the solutions used. A considerable difference occurs between the σ_z values of the upper bound solution and two lower bound solutions (Fig 10b). The σ_z stresses give rise to longitudinal cracks which starts from the vertical joint. In some cases they are initiated earlier than the vertical cracks caused by the bending of beams (Fig 11). The predisposition to longitudinal cracking is very pronounced in prestressed beams with unbonded tendons. In segmental beams with dry joints an important question is the location and extension of possible cracks. This location can be determined by the lemma:

Lemma III: **A possible crack of mode I will be initiated at that location of the body where a minute cut induces the greatest drop in the potential energy $\Delta\pi = -W(\sigma_h)$ and therefore also the greatest drop in the stiffness D of the structure.**

Proof. We consider a non-monolithic uncracked structure with dry joints Γ_k and uniformly distributed microcracks, loaded by external loads $\mathbf{p}^*$, in which we make a minute cut $\Gamma_h(r) = a\cdot\Delta s$ normal to the principal tensile stress at r. The state $\{p,\sigma,u,\varepsilon,\gamma\}$ is separated into the states $\{p'_e\ldots\varepsilon'_e,\gamma'_e\}$ before cracking and $\{p_h\ldots\varepsilon_h,\gamma_h\}$ induced by the cut: $\{p\ldots\gamma\} = \{p'_e\ldots\gamma'_e\} + \{p_h\ldots\gamma_h\}$

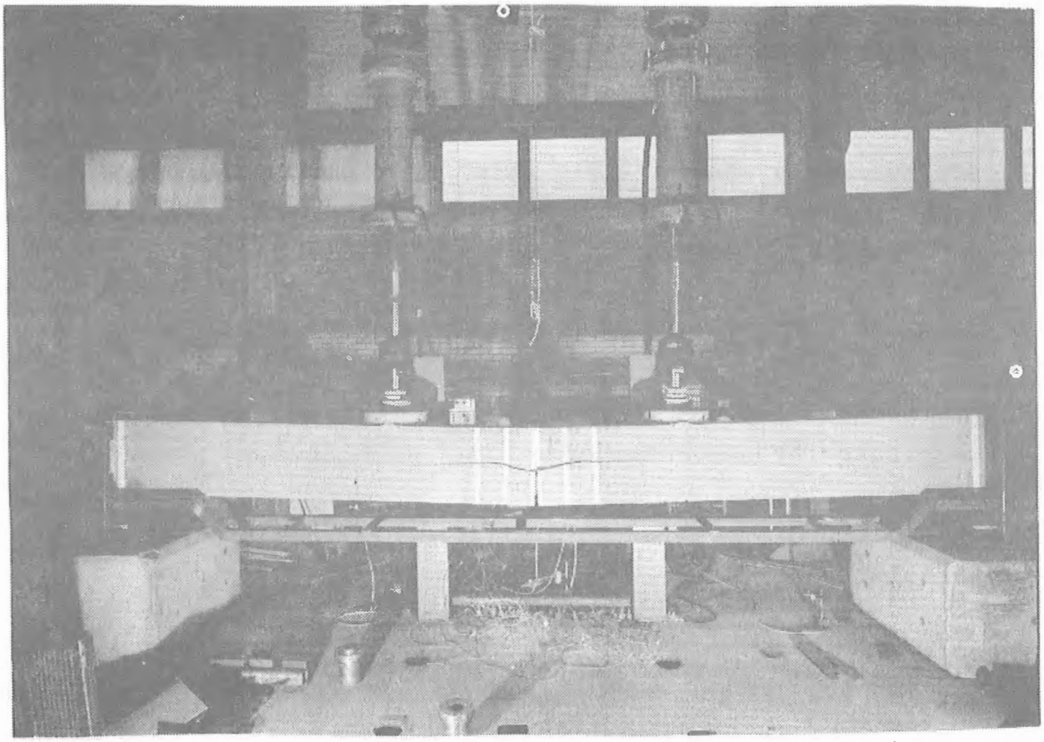


Fig. 11 Longitudinal crack initiated at dry joint of prestressed segmental granite beam.

The potential energy of the uncracked structure is then

$$\pi_o = W(\sigma'_e) - \int \mathbf{p}^*\cdot\mathbf{u}'_e d\Gamma = -(\tfrac{1}{2})\int \mathbf{p}^*\cdot\mathbf{u}'_e d\Gamma \qquad 33)$$

since $W(\sigma_e) = \tfrac{1}{2}\int \mathbf{p}^*\cdot\mathbf{u}'_e d\Gamma$. The potential energy after cutting is

$$\pi_1 = W(\sigma) - \int \mathbf{p}^*\cdot\mathbf{u} d\Gamma = -\tfrac{1}{2}\int \mathbf{p}^*\cdot\mathbf{u} d\Gamma$$

provided the friction law excludes irreversible effects and the work on Γ_k and Γ_h: $\mathbf{p}'_e\boldsymbol{\gamma}'_e, \mathbf{p}\cdot\boldsymbol{\gamma} = 0$. Hence

$$\Delta\pi = \pi_1 - \pi_o = -\tfrac{1}{2}\int \mathbf{p}^* \cdot \mathbf{u}_h d\Gamma = -\Delta W \qquad 33')$$

where $\Delta W = W(\sigma'_e + \sigma_h) - W(\sigma_e') = W(\sigma_h) - \int_{\Gamma h} \mathbf{p}_h \gamma_e' d\Gamma$. Since σ_h, ε_h are permissible states we get using eq. 3)

$$\Delta W = \tfrac{1}{2}(\int_{\Gamma k}(p_h \gamma'_e - p'_e \gamma_h) d\Gamma + \int_{\Gamma h} p_h \gamma_h d\Gamma) \qquad 34)$$

On $\Gamma_h(r)$ the stress $p_{hx}(x,r) = -p'_e(x,r) = -\sigma'_e(x,r)$ induces a singularity at the depth $x = a$. Therefore acc. to linear fracture mechanics [5]

$$\gamma_h(x,r) = 8K_I(a,r)((a-x)/2\pi)^{\frac{1}{2}}/E' \qquad 34)$$

Because of the smallness of the cut, p_h and γ_h differ on Γ_k from zero only where σ'_e, γ'_e are very small. Therefore the first integral can be neglected compared with the second. Hence putting $\sigma'_e(x,r) = \sigma'_e(0,r)g(x)$

$$(-\Delta\pi) = W(\sigma_h) \therefore -\tfrac{1}{2}\int p_h \gamma_h d\Gamma = \tfrac{1}{2}\sigma'_e(o,r) \cdot \Delta s \int g(x,r)\gamma_h(x) dx \qquad 34')$$

The integral depends also on $W(\sigma_h)$ since $\gamma(x,r)$ depends on the stress intensity factor

$$K_I(a,r) = ((E/\Delta s)\partial W_h/\partial a)^{\frac{1}{2}} \qquad 35)$$

Eq.34') therefore leads to a differential equation of $W(\sigma_h)$. Its solution gives

$$W(\sigma_h) = \sigma(0,r)^2 C(g(a)); \; K_I(a,r) = \sigma(0,r)(E(\partial C/\partial a)/\Delta s)^{\frac{1}{2}} \qquad 36)$$

Varying r, keeping Δs,a constant, $W(\sigma_h)$ simultaneously with $K_I(a,r)$ and $\sigma(0,r)$ attains maximum, where the crack is initiated, from which the lemma follows. ∎

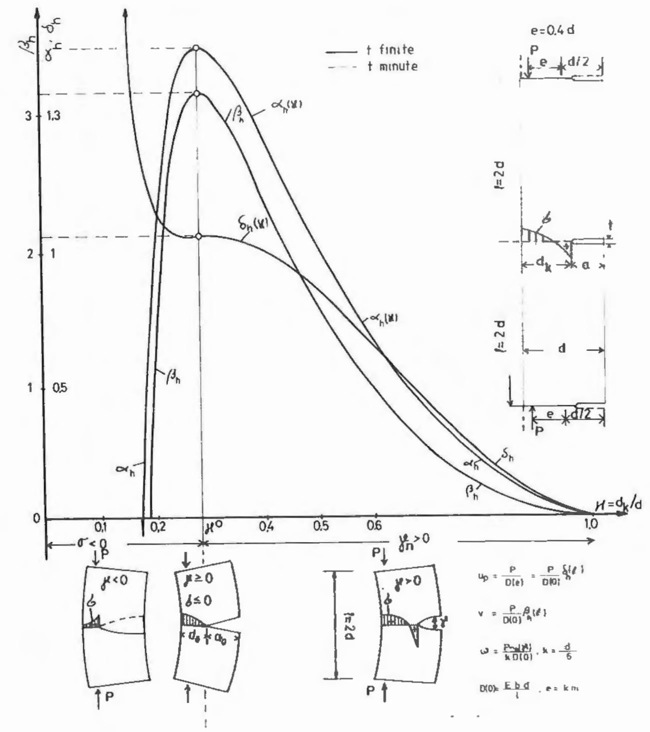
The lemma describes the situation at the beginning of cracking. The cracking continues then leading to unstable growth or it stops when the total energy $U = \pi + \pi_s$, where π_s is the surface energy, attains a minimum with $\Delta\pi = -W(\sigma_h)$

$$\partial U/\partial \alpha = \partial(\pi_s - W(\sigma_h))/\partial a = 0 \qquad 37)$$

In mode I cracks this may happen when in compressed parts a state is attained where all singularities along the crack tip disappear. This case defines the **limit state of free contact**. Since along the crack Δs $K_I(a,s) = 0$, there follows

$$\partial W(\sigma_h)/\partial a = 0 \qquad 37')$$

Fig. 12 Deformation parameters of cut block assembly versus contact area depth d_k



Because $W(\sigma)_h$ increases monotonously with increasing a, it attains a maximum in the limit case of free contact. If we consider a monolithic block b_v-b_{v-1} of a beam (Fig 12) loaded by P with constant eccentricity m_z=m and make in the symmetry section a cut of depth a, which deminish the uncracked depth to $d_k = d-a = \kappa d$, the released energy is

$$W(\sigma_h) = P^2 d\delta_h(a)/2EA$$

where δ_h is an increasing function of a, everywhere where we have a tensile stress-singularity until we rearch the limit state depth $a_o = d-d_o$. If we then furher increase the depth a and the cut is of finite thickness we will get a compressive stress singularity at the tip of the cut and therefore the energy $W(\sigma_h)$ will start again to increase. We have in this case an inflection point at a_o. If again the cut is infinitely thin it will not affect the state of stress and strain for values $a > a_o$ because the section is a symmetry plane. In this case the function $\delta_h(a)$ will remain constant (Fig 12). We have thus at a_o only in the latter case where the inpenetrability condition $\gamma_x \geq 0$ is not violated a true maximum of $\delta_h(a)$

$\delta_h = \delta_h(a_o) = \max\delta_h(a)$; $(\partial\delta_h/\partial a)_{ao} = 0$; $(\partial^2\delta_h/\partial a^2)_{ao} = 0$ 39')

Along the section a_v we have principal stresses σ_x, σ_z with $\sigma_x = 0$ at a_o and $\sigma_x < 0$ for $a > a_o$. The question arises where the longidutinal crack will be initiated along the opened joint (Fig 13). The strain energy in the uncracked state is

$$W_o = P^2 d(\lambda\dot{\delta}_e(m) + \delta_h(m,\lambda))/2E'A \qquad 40)$$

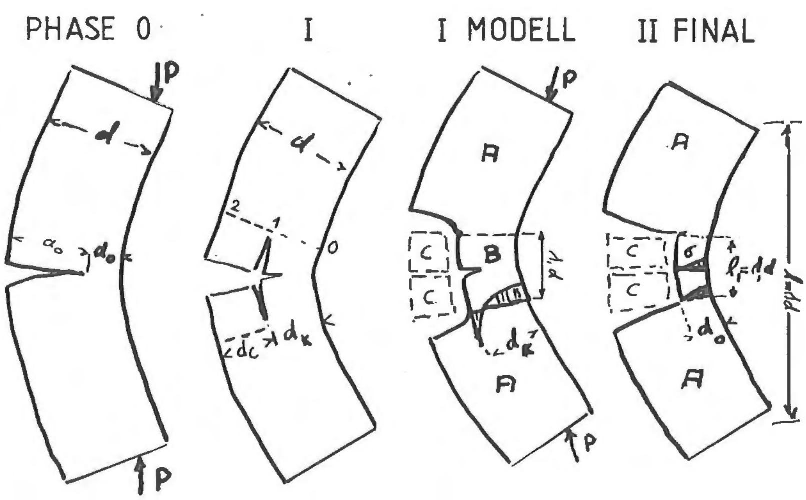
If the crack developes at height $z \leq z_o$ (z_o tip of gap in dry joint) with lenght l_1 we can distinguish three regions (Fig 13):

A with depth d and total lenght $l - l_1$.

B with depth d_k and total lenght l_1

C two unloaded cantilevers with depth $d_c = d - d_k$ and total lenght l_1 clamped to parts A along line 1-2.

Fig. 13
Lower bound solution of longitudinal cracking.



An approximate solution is sought by making fictitious cuts along 1-2 separating the accordingly unstressed parts C from the structure. Denoting $d_k = \kappa d$; $\lambda_1 = l_1/d$; $\lambda_k = l_1/d_k$; $A/A_k = q$; $m = e/k$; $m_k = e_k/k_k$ we get after cracking strain energies W_A, W_B and $\Delta W = W_A + W_B - W_o$

$$W_A = (P^2 d/2EA)((\lambda-\lambda_1)\dot{\delta}_e(m) + \delta_h(m,\lambda-\lambda_1,\kappa)) \qquad 40a)$$

$$W_B = (P^2 d_k/2EA_k)((\lambda_k\dot{\delta}_e(m_k) + \delta_h(m_k,\lambda_k)) \qquad 40b)$$

$$\Delta W = (P^2d/2EA)(\lambda_1(q\dot{\delta}_e(m_k)-\dot{\delta}_e(m)) +$$

$$+ \delta_h(m,\lambda-\lambda_1,\kappa)+q\kappa\ \delta_h(m_k,\lambda_k)-\delta_h(m,\lambda)) \qquad 40c)$$

The model satisfies the kinematical- and stress-conditions on the joint a_v of part B. On the lines 0-1 the kinematical conditions are satisfied but the equilibrium conditions are satisfied only for the generalized forces N_B, M_B. In the fictitious cut 1-2 the kinematical conditions are not satisfied but the stresses are permissible, which justifies the separation of parts C.

ΔW attains its maximum when $m_k = 1$ with corresponding $\kappa \therefore \kappa_o$; $A_k = A_o$; $\gamma_k = 0$. Indeed in this case the distribution σ_x in B is linear, P acting at a core point . Hence $\delta_h(1,\lambda_k)=0$;$\delta_h(m,\lambda-\lambda_1)=\max\delta_h(m,\lambda-\lambda_1,\kappa)$ and $q\delta_e(1)-\delta_e(m)=$ $=\delta_h(m)$ because for body B acc. to 20a): $u_p=Pl_1\dot{\delta}_e(1)/EA_o$ $=Pl_1\dot{\delta}_o(m_k)/EA_k$;$\forall A_k$ with $A_o \le A_k < A$ since A_o = Min A_B, from which $(A/A_o)\dot{\delta}_e(1) = \dot{\delta}_o(m)$, and max $(q\dot{\delta}_e(m_k)-\dot{\delta}_e(m)) = \dot{\delta}_h(m)$. Since we have uniaxial compression in B the rising of the crack above z_o inside the region d_o does not change the stresses in B and A. Therefore $(\partial\Delta W/\partial\kappa)_{\kappa o} = 0$ and

$$\Delta W_{max} \therefore (P^2d/2EA)(\lambda_1\dot{\delta}_h(m)+\delta_h(m,\lambda-\lambda_1)-\delta_h(m,\lambda)) \qquad 41)$$

The above result is close to a lower bound solution.

An upper bound solution is obtained assuming parts A being in the monolithic state $\{\sigma_e,\varepsilon_e\}$ parts B deform as in the lower bound solution and the cantilevers C remain undeformed and rigidly clamped to parts A along lines 1-2. W_o doesn't change. For parts A and B we get after cracking

$$W_A' = (P^2d/2EA)(\lambda-\lambda_1)\dot{\delta}_e(m) \qquad 40a')$$

$$W_B' = (Pd_k/2EA_k)(\lambda_1\dot{\delta}_e(m_k)+\delta_h(m_k,\lambda_k)) \qquad 40b')$$

Hence

$$\Delta W' = (P^2d/2EA)(\lambda_1(q\ \dot{\delta}_e(m_k)-\dot{\delta}_e(m))+q\kappa\delta_h(m_k,\lambda_k)-\delta_h(m,\lambda)) \qquad 40c')$$

By the same reasoning as in the lower bound solution the maximum is attained when κ and A_k corresponds to $m_k = 1$ which gives

$$\Delta W'_{max} = (P^2 d/2EA)(\lambda_1 \dot{\delta}_h(m) - \delta_h(m,\lambda)) \qquad 41')$$

This corresponds to a considerably smaller drop of potential energy π and stiffness D than that of the lower bound solution. Nevertheless both cases confirm the result of the stress calculation that according to lemma III the maximum of the tensile stresses σ_{zmax} are attained a little bit below the tip of the gap corresponding closely to the neutral line at height a_o according to the classical theory. Tests confirm also this result. The stress intensity factor K_I of the longitudinal crack are acc. to the lower bound solution

$$K_I = ((E'/b_o)\partial \Delta W/\partial a)^{1/2} = P(\dot{\delta}_h - \partial \dot{\delta}_h(m,\lambda-\lambda_1)/\partial\lambda)/b_o A)^{1/2}$$

and acc. to the upper bound solution

$$K_I = ((E'/b_o)\partial \Delta W'/\partial l_1)^{1/2} = P(\dot{\delta}_h/b_o A)^{1/2}$$

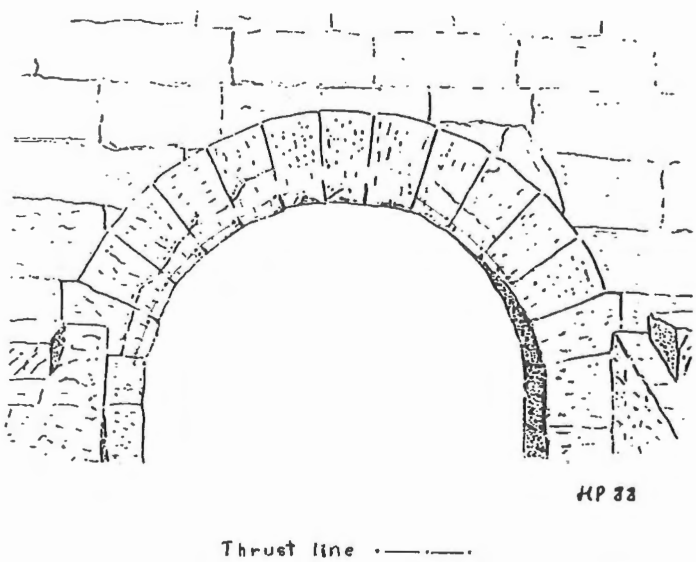


Fig. 14 Longitudinal cracking in stone arch (Arles,arena).

Longitudinal cracking occur also in sufficient slender ($\lambda > 1$) voussoirs of stone arches. From the cracks an estimate can be made about the position of the thrust-line of the arches because the resultant must pass on a distant $d_k/3$ from the compressed edge, where d_k is the depth of the uncracked region (Fig 14).

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