Summary. The opening of vertical cracks and dry joints may by linearization be reduced to local dilatations and rotations of the centroid axis of the beam. The contact analysis for joints or cracks of beams with finite crack-spacing reveals that the positions of the rotation axis, the compressive force and the centroid of the crack volume comply with the known rules concerning beams with infinitesimal crack-spacing if the tendons are unbonded. This renders possible simple graphical methods to determine the positions mentioned, of beams with various cross-sections and provide information about the relation between the dilatational and rotational parameters. Also the stress distribution and the initiation of longitudinal cracks of the beam block is investigated. Estimates of the location of the longitudinal cracks are deduced. Some of the theoretical results are compared with corresponding tests with post-tensioned segmental beams with dry joints.

1. Introduction

Research concerning the elastic contact problems of cracked beams started in the first half of this century. The first investigation carried out by H. M. Westergaard (1933), was based on the use of stressfunctions in polar coordinates, which led to an approximately parabolic stressdistribution in the compressive zone of the cracked section [1]. Among other investigations worth mentioning is the work led by P. J. Vasiliev 1970-81 where the contact problem of the cracked section has been solved by using Greens functions [2]. For beams with rectangular cross-section and unbonded tendons graphs of the structural quantities were given.

At the chair of structural mechanics of Tampere University of Technology, these problems have been investigated by a different approach using the extremum principles of stiffness [3]. The total state of stress and strain has by superposition been separated into the state, corresponding to the uncracked beam, and the state induced by the edge
effect of the cracks. Furthermore this approach makes it possible to incorporate the contact theory into the framework of the classical theory of bending of reinforced concrete beams with zero tensile stresses.

2. The non-monolithic elastic beam

We consider an elastic cylindrical beam with straight tendons, occupying a volume $V = LA$ with span $L$, and vertical dry joints or cracks in cross-sections $A_v$, which divide the beam into a sequence of segments (Fig 1a). The load $p^*$ acting on external surface $\Gamma_e$ induces then an additional state of stress $(\sigma - \sigma^0)$ and strain $\varepsilon$, the resulting state being $\{p, \sigma, u, \varepsilon, \gamma\}$. Here $p(x,y,z) = \{\sigma_x, \sigma_y, \sigma_z\}^T$ stress-vector in cross-section $A(x)$, and $u(x,y,z) = \{u_x, u_y, u_z\}^T$ displacement vector. The discontinuity $[u] = u_v(x,y,z) - u_{v-1}(x,y,z)$ defines the gap vector $\gamma$ in $A_v$.

$$\gamma_{v,v-1} = [u(y,z)] = \gamma_x i + \gamma_y j + \gamma_z k$$  \hspace{1cm} (1)

$\sigma^0$ is the initial prestress of the beam, $i,j,k$ are unit vectors in principal directions $x,y,z$ of $A$. If the beam would remain uncracked, the same load would induce a state $\{p_e, \sigma^e, u^e, \varepsilon^e, 0\}$. The superposition of the states of the monolithic beam $\{p_e, \sigma^e, u^e, \varepsilon^e\}$ and the edge-effect of crack $\{p_h, \sigma_h, u_h, \varepsilon_h, \gamma\}$ gives

$$\{p, \sigma, u, \varepsilon, \gamma\} = \{p_e, \sigma^e, u^e, \varepsilon^e\} + \{p_h, \sigma_h, u_h, \varepsilon_h, \gamma\}$$  \hspace{1cm} (2)

If $\{u', \varepsilon', \gamma'\}$ is a possible kinematic state and $\{p'', \sigma''\}$ a state of equilibrium induced by the load $p^*$ on $\Gamma_e$, there holds because of $p''_{v-1,v}(Y,z) = -p''_{v,v-1}(Y,z)$; $[u'(y,z)] = \gamma'_{v,v-1}$ on $A_v$ and Greens theorem:

$$\int_{\Gamma_e} p^* u'd\Gamma = \int_Y \sigma''_{ij} \varepsilon_{ij} dV + \int_{\Delta} p''_{\gamma} dA;$$  \hspace{1cm} (2')

where $p''_{\gamma} = p''_{v-1,v} \gamma_{v,v-1}$. The generalized cross-sectional forces (normal- and shearforce $N, Q$, moments $M$) define a vector $R = \{N_x, M_y, M_z, Q_y, Q_z, N_x\}$ whose components are
The generalized displacements (translations $U$, rotations $\phi$) of cross-section $A(x)$ form a vector $U={U_x, \phi_y, \phi_z, U_y, U_z, \phi_x}$. Using the state of stress $\{p^i_e, \sigma^i_e\}$ of an infinite uncracked beam subjected in $A(x)$ to only one component $R_i = 1$, denoting $p^i_e = g^i$, $E_k/E_c = n_k$ we define component $U^i$ by:

$$U^i(x) = \int_A g^i u dA$$

with:

$g^1 = (n/A)i; g^2 = (nz/I_y)i; g^3 = (-ny/I_z)i$

$g^4 = r^4 yj + r^4 zk; g^5 = r^5 yj + r^5 zk; g^6 = r^6 yj + r^6 zk$

We subject the beam to a state of equilibrium $\sigma^i_e$ induced by $R_i = 1$ and the state $\{u, \varepsilon, \gamma\}$ induced by load $p^*$. Then we get by eq. 2') and using twice Betti's rule due to $p^0 = 0$, and $p^* = p^*e$ on $\Gamma_e$ the mutual generalized displacements of the end faces:
\[ \Delta U^i = U^i(L) - U^i(0) = \int_V \nu^i \cdot g^i \, dA + \int_V \sigma^0 \cdot \epsilon^i \cdot \epsilon^j \, dV + \Sigma \int_A \gamma^i \cdot \gamma^j \, dA \quad (5) \]

and

\[ \int_V \sigma^0 \cdot \epsilon^i \cdot \epsilon^j \, dV = \int_V \left( \sum \sigma_{ij} \cdot \epsilon_{ij} \right) \, dV = \int \sigma_{ij} \cdot \epsilon_{ij} \, dV \]

\[ = \int_A g^i \cdot u^i \, dA = \Delta U^i_e \]

Hence

\[ \Delta U^i = \Delta U^i_e + \Delta U^i_h; \quad \Delta U^i_h = \Sigma \int_A \gamma^i \cdot \gamma^j \, dA \quad (6) \]

Lemma I. The generalized mutual displacement of the end faces equals the sum of respective displacements of the monolithic beam and the corresponding work at the cracks and joints.

Complementarity [3] requires with \( \int g^i \Delta u^i \cdot dA = 0; \ i = 1, \ldots 6 \)

\[ u(x, y, z) = \Sigma (g^i \cdot U^i_e(x) + g^i \cdot U^i_h(x)) + \Delta u_o \quad (7) \]

The dilatation \( v = \Delta U^i \) and respective mutual rotations \( \omega_Y = \Delta U^2, \omega_Z = \Delta U^3 \) of the end faces of the beam are

\[ v = \int_A g^i \cdot u^i \, dA = v_e + v_h = v_e + \Sigma \epsilon_h / A; \quad (V_h = \int_A \gamma^i \cdot \gamma^j \, dA) \]

\[ \omega_Y = \int_A g^2 \cdot u^i \, dA = \omega_{ey} + \omega_{hy} = \omega_{ey} + \Sigma \epsilon_z v_h / I_y; \quad (z_h V_h = \int_A z \cdot \gamma^i \cdot \gamma^j \, dA) \quad (8) \]

\[ \omega_Z = \int_A g^3 \cdot u^i \, dA = \omega_{ez} + \omega_{hz} = \omega_{ez} - \Sigma \epsilon_y v_h / I_z; \quad (y_h V_h = \int_A y \cdot \gamma^i \cdot \gamma^j \, dA) \]

where \( \Sigma \epsilon_h \) is the gap volume and \( \Sigma \epsilon_z v_h, \Sigma \epsilon_y v_h \) its moments with respect to the principal axes. Especially in a beam, where \( N = 0 \) and \( v_e = 0 \) there holds, if the end faces remain plane

\[ v = v_h = \Sigma \epsilon_h / A; \quad \omega_{hy} = \Sigma \epsilon_z v_h / I_y; \quad \omega_{hz} = -\Sigma \epsilon_y v_h / I_z \quad (9) \]

Hence:

Corollary. The total elongation of the centroid axis of a bent beam is proportional to the crack volume \( \Sigma \epsilon_h \) and independent of the depth of the crack. The additional mutual rotations \( \omega_{hy}, \omega_{hz} \) of the end faces caused by the gaps are proportional to the moments of the gap volume \( \Sigma \epsilon_h v_h \) and \( -\Sigma \epsilon_y v_h \) respectively.

Fig 2a, b show the linear dependence of \( v_h \) on \( v_h \) and \( \omega_h \) on \( z_h v_h \) of loaded prestressed granite beams with unbonded tendons and one dry joint in the middle section by measuring
the elongation $v_h^d$ and $\omega_h^d$ on a length $l_d$ and the corresponding gap volume quantities $V_h$, $z_hV_h$ of the joint [4].

Considering a beam with constant crack spacing $l = \lambda d$ subjected to a constant compressive force $P$ with eccentricities $y_p = -e_y$, $z_p = -e_z$, the mid sections $b_{v-1}, b_v$ between two cracks remain plane (Fig 1b). Hence extension

$$v_x(y,z) = v_{ex}(y,z) + v_{hx}(y,z) = v + \omega_y z - \omega_z y$$

$$v_{ex}(yz) = v_e + \omega_y z - \omega_z y; \ v_{hx}(y,z) = v_h - \omega_y z - \omega_z y;$$

where acc. to the bending theory of beams and conditions 9)

$$v_e = -Pd^3/EA; \ \omega_y = Pd^3 e_z/EI_y; \ \omega_z = -Pd^3 e_z/EI_z;$$

$$v_h = V_h/A; \ \omega_y = z_h V_h/I_y; \ \omega_z = -v_h V_h/I_z$$

Fig 2. Test of prestressed granite beam with unbonded tendons, net area of section $A_h$ and unbonded tendons

a) Measured extension $V_h^d$ of centroid axis versus gap volume $V_h$

b) Measured mutual rotation $\omega_h^d$ versus gap moment $z_hV_h$. 
The deformation $v_x$ may therefore be separated into the monolithic deformation $v_{ex}$ and the effect of the joint $v_{hx}$, which actually implies that the two halves of the period are subjected to deformations $v_{ex}/2$ and they are separated by a linear discontinuity $v_x = v_{hx}$ at the joint.

The real gap width $\gamma_x(y,z)$ can therefore be substituted by a linear distribution (Fig 1b,c).

$$v_x(z,y) = \left(1 + \frac{z}{z_h} \right) \frac{1}{2} y + \frac{v_{hx}}{y} \frac{1}{z} v_{hx}/A$$

where $v_x(y',z') = 0$ determines the axis of the additional rotation $\omega_h$ caused by the crack. According to this scheme the displacement $u_h$ of the half blocks corresponds to rigid body motions with interpenetration above axis $\omega_h$ at $z'_h$ (Fig 1c). This corresponds to an analogous stress distribution in the monolithic beam induced by a compressive force $P$ at $y_p = -e_y, z_p = -e_z$

$$v_e(y,z) = - \left(1 + \frac{z}{z_p} \right) \frac{1}{2} y + \frac{v_{ex}}{y} \frac{1}{z} v_{ex}/E/1$$

3. Deformation parameters and stiffness of block

The extensions and rotations depend on $P$, $e_y$, $e_z$ and $\gamma$

$$\nu = P d\beta/E; \quad \omega_y = P d\alpha_y/EAk_y; \quad \omega_z = -P d\alpha_z/EAk_z$$

The three parameters $\beta, \alpha_y, \alpha_z$ depend on the relative kern-distances $m_y, m_z$ ($m_j = e_j/k_j$) and the ratio $\lambda = 1/d$. They are not independent because $\nu, \omega_y, \omega_z$ are derivatives of the strain energy $W$ which is a second degree homogeneous function of $N, M_y, M_z$.

$$W = P^2 d\delta(m_y,m_z,\lambda)/2EA = Pu_p/2; \quad u_p = -v_x(y_p,z_p) \geq 0$$

From relations $\partial W/\partial N = \nu; \partial W/\partial M_y = \omega_y; \partial W/\partial M_z = \omega_z$; there follows

$$\alpha_y = (1/2)d\delta/\partial m_z; \quad \alpha_z = (1/2)d\delta/\partial m_z; \quad \beta = m_y \alpha_y + m_z \alpha_z - \delta$$

Because the state $\{v_h, \sigma_h\}$ corresponds to no external load the strain energies can be separated

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where \[ W(\sigma_e) = \frac{P u_{ep}}{2} = \frac{p^2 \delta_e}{2EA} \]

\[ W(\sigma_h) = \frac{P u_{hp}}{2} = \frac{p^2 \delta_h}{2EA} \]

The relations 15) hold separately for \( \sigma_e, \beta_e, \delta_e \) and \( \alpha_h, \beta_h, \delta_h \) since \( \delta = \delta_e + \delta_h \). The \( \alpha_e, \beta_e, \delta_e \) are determined by eq. 10a) whereas expressions 10b) become

\[ v_h = \frac{P d \delta_h}{EA}; \quad \omega_{hy} = \frac{P d \alpha_{hy}}{E A k_z}; \quad \omega_{hz} = \frac{P d \alpha_{hz}}{E A k_y} \]

and the interpenetration (Fig 1c) of the rigid blocks at \( y_p, z_p \) are according to 10)

\[ u_{hp} = \frac{P d \delta_h}{EA} = -v_h (y_p, z_p) = (1 + z_p^2 + y_p^2 + y_p^2) v_h / EA > 0 \]

The parameters \( \beta_h, \alpha_h, \delta_h \) can be determined either by solving the contact problem for the resulting state \( \{p, \sigma, u, e, \gamma\} \) and then obtain \( \alpha_h, \beta_h, \delta_h \) from the differences \( \delta = \delta_e + \delta_h \), \( \beta = \beta_e + \beta_h \), or by confining the problem exclusively to the state \( \{p_h, \delta_h, u_h, e_h, \gamma\} \). In this case the block halves of the period are loaded at the joint, where \( \gamma > 0 \) by known forces \( P_{hx} = -\sigma_h \), the end faces \( b_{v-1}, b_v \) remaining plane. The work carried out is according to 10b)

\[ \frac{h f A h y P_h y A}{A h y A} = \frac{P u_{hp}}{2} = W(\sigma_h) \]

confined to section \( a_v \) since \( \int P_h u_h y A = 0 \) on \( A_{by} \), \( P_{hx} = -\sigma_{hx} \) being an equilibrium distribution.

A more transparent geometrical interpretation provides the concept of stiffness. The stiffness of the period can be defined as the ratio of the load \( P \) to the load-displacement \( u_p \). The total stiffness \( D \) and the interpenetration stiffness \( D_h \) are

\[ D = \frac{P}{u_p} = \frac{EA}{d \delta}; \quad D_h = \frac{P}{u_{hp}} = \frac{EA}{d \delta_h} \]

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These quantities are defined either by an admissible state of stress \(\{p',\delta'\}\), which include the non-tension condition \(\sigma_x < 0\) at dry joints and the stress energy \(W(\sigma')\) (eq. 14, 15)

\[
D(\sigma') = p^2/2W(\sigma'); \quad D(h') = p^2/2W(h')
\]

or by an addmissible state of strain \(\{u',\varepsilon',\gamma'\}\), which includes the impenetrability condition \(\gamma_x \geq 0\) at the joint:

\[
D(\varepsilon') = 2W(\varepsilon')/(u_p')^2; \quad D_h(\varepsilon') = 2W(h')/(u'_{hp})^2
\]

If in the structure the unloaded state is devoid of initial stress \(\delta\) and gaps then, according to the extremum principles of stiffness \(D(\sigma')\) provide lower bounds to the actual stiffness \(D\) and \(D(\varepsilon')\) upper bounds to the actual stiffness \(D\) [3] Hence:

\[
D(\sigma') \leq D \leq D(\varepsilon'); \quad D_h(\sigma') \leq D_h \leq D_h(\varepsilon')
\]

The lower bound solution based on \(\{p',\delta'\}\) provide an upper bound \(\delta''\) whereas the upper bound solution based on \(\{u',\varepsilon',\gamma'\}\) provide a lower bound \(\delta'_h\) to \(\delta_h\) because of eq. 18).

The functions \(\beta_e(m_y,m_z,\lambda)\) are all proportional to the slenderness \(\lambda\) acc. to eq. 10a

\[
\beta_e = \lambda \beta_e(m_y,m_z); \quad \alpha_e = \lambda \alpha_e(m_y,m_z); \quad \delta_e = \lambda \delta_e(m_y,m_z); \quad \delta_s = \delta_e + \delta_h
\]

Closed expressions for parameters \(\beta_h...\delta_h\) cannot generally be obtained. The dependence on \(\lambda\) is rather complicated with one exception where the compressive stress-distribution in the cross-section is partly linear. This materializes in a beam with very dense crack spacing \(\lambda \to 0\) in accordance with the classical theory of bending of reinforced concrete beams. In this exceptional case the contact problem may be solved by elementary means. We get if \(\lambda \to 0\) with \(\beta^o = \beta_e + \beta_h\)

\[
\vdots; \quad \delta'e = \delta'_e + \delta'_h
\]
\[ \beta^o = \lambda \delta_0(m_y, m_z); \quad \alpha^o_S = \lambda \delta_0(m_y, m_z); \quad \delta^o = \lambda \delta_0(m_y, m_z) \quad (20a) \]

\[ \beta^o_h = \lambda \delta_h(m_y, m_z); \quad \alpha^o_{HS} = \lambda \delta_h(m_y, m_z); \quad \delta^o_h = \lambda \delta_h(m_y, m_z) \quad (20b) \]

In the following we confine ourselves to segments with unbonded tendons and accordingly to the deformations of the net area of the joint. For a rectangular section the quantities \( \beta_h \ldots \delta_h \) are with \( m_y = 0; \ m_z = m \)

\[ \delta_h = (|m| - 1)^2 / (3 - |m|)^2; \]

\[ \alpha_{hz} = (4 - |m|)(|m| - 1)^2 / 3(3 - |m|)^2; \]

\[ \delta_h = (|m| - 1)^3 / 3(3 - |m|) \]

If \( \lambda \neq 0 \) approximate solutions are sought by variational methods [6]. For a block \( a_v - a_{v+1} \) with narrow rectangular cross section a lower bound solution is obtained by the stress-function expansion

\[ \phi_a = \chi_g s_a(\eta)(1 - (-1)^s \cos \pi \xi) + f(\eta); \text{ where } z < z_0, \ f', \ f_{zz} < 0 \]

\[ \phi_b = \chi_g s_b(\eta)(1 - (-1)^s \cos \pi \xi); \text{ where } z > z_0, \ f', \ f_{zz} = 0 \quad (21) \]

with \( \eta = 2z/d, \ \xi = 2x/l, \ s = 1 \ldots n, \) and boundary conditions satisfying the equilibrium conditions at \( z = \pm d/2 \) and \( z_0, \) the integral conditions and stress-conditions at \( x = 0, \pm 1/2 \)

\[ \int f_{zz} dz = -P; \quad \int f_{zz} dz = Pkm; \quad \tau_{xz} = 0 \quad (21') \]

The solution is obtained by minimizing acc. to Kantorovitsh's method the stress energy by variation of functions \( f(\eta), \ g_{sa}(\eta), \ g_{sb}(\eta). \)

\[ W(\sigma', \gamma') = (\lambda E) f + 1/2 -1/2 \int d^2/2 -d/2 (\partial^2 \phi / \partial x^2 + \partial^2 \phi / \partial y^2)^2 \ dx \ dy \]

An upper bound solution is obtained by a kinematically admissible deformation-field \( (u_h', \ v_h', \ \gamma') \) of a half-block \( a_v - b_v \) clamped without friction to plane \( x = 0 \) (Fig 3) and
\[ u'_{hx} = \Sigma (A_{\mu}(\gamma_{\mu} - \theta_{\mu} \cos(\mu\pi x/l)) + G_{\mu}(\gamma_{\mu} \pm \theta_{\mu} \sin(\mu\pi x/l)))/2 + \]

\[ \Sigma \Sigma (B_{ik}(2\pi x/d) \sin(2k\pi x/l) + D_{ik}(\sin(2\pi x/d) \cos(2k\pi x/l)) + G_{\mu}(\gamma_{\mu}(\mu\pi x/l - (\mu^2 - 8)/4\mu) \pm \theta_{\mu} \cos(\mu\pi x/l)))/2\mu\pi) \]

\[ + \Sigma \Sigma (C_{ik}(2\pi x/d) \cos(2k\pi x/l)) + E_{ik}(\sin(2\pi x/d) \cos(2k\pi x/l)) \]

This field satisfies the condition at \( x=0 \): \( u'_{hx} = \gamma/2 \geq 0 \) and at \( x = 1/2 \): \( u'_{hx} = \gamma/2 \). Here \( \gamma = \Sigma A_{\mu} \gamma_{\mu} \geq 0 \); \( \bar{\gamma} = \Sigma A_{\mu} \bar{\gamma}_{\mu} \); \( \theta_{\mu} = \bar{\gamma}_{\mu} - \gamma_{\mu} \) which represents an equilibrium distribution on \( b_{\nu} \). The solution is obtained by the minimum of the potential energy

\[ \pi = W(\epsilon') - \int_{A_{\nu}} f_{Av} \rho_{hx} u'_{hx} dA \]

with respect to the constants \( A_{\mu}, G_{\mu}, B_{ik}, C_{ik}, D_{ik}, E_{ik} \).

Another upper bound solution is obtained by the finite element method where the displacement field between neighbouring elements has no discontinuities.

a) Fig 3. Use of \( \{u_h, \epsilon_h, \gamma_h\} \) for an upper bound solution

From these solutions upper bounds \( \delta_h \) and lower bounds \( \delta_h' \) for parameters \( \delta_h \) are obtained, with corresponding values for \( a_h, \beta_h \) and for varying values \( m = -z_d/k_z \) along the symmetry axis of the cross-section. \( \delta_h \) and \( \beta_h \) assume only non-negative values whereas \( a_h \)'s sign depends on the direction of the bending moment (Fig 4). Of special interest are the parameters dependence on the slenderness \( \lambda \). For small values we have at given \( m_z \) a linear relationship according to formulae 20), 20b). Because the edge effect of the joint vanishes at a certain distance from the joint if \( \lambda \to \infty \) the parameters approach constant values (Fig 5). For a rectangular section we have approximate values
Fig 4. Deformation parameters of dry joint of T beam.

\[ \delta_h(m, r) = a(1 - |m| - 2)^2 r \delta_h; \]

\[ \beta_h(m, r) = (3 + r(2 - |m|)|m|) \delta_h(m, r)/(1 - (2 - |m|)^2) \]

\[ \alpha_h(4 - |m| + r(2 - |m|)) \delta_h(m, r)/(1 - (|m| - 2)^2); \]

with \( a \approx 0.74; \ r \approx 0.55 \)

With increasing eccentricity \( m \) the difference between upper bound and lower bound parameter-values increases, which in some cases leads to a decrease of the parameter-values of the upper bound solutions. An estimation of the limit value of \( \delta_h \) when \( \lambda \to \) is possible taking into consideration that \( \delta_h \) is proportional to \( W_h = \int w(\sigma_h) dV \), where \( w(\sigma_h) = \sigma_{hij} e_{hij}/2; \)

\( \partial w/\partial \sigma_{hij} = e_{hij}. \)

Therefore if the length \( l \) of the period \( b \)

\( v-l-b_v \) is increased by \( dl \) we get with \( \partial w/\partial l = e_{hij} \partial \sigma_{hij}/\partial l; \)

\[ dW_h = dl \int V \partial w_h/\partial l dV + dl \int (f_A w dA)_{1/2} \delta l = dl \int V \partial \sigma_{hij}/\partial l e_{hij} dV + (f_A w dA)_v \]
but \[ \int_V \left( \frac{\partial \sigma_{ij}}{\partial y} \right) e_{ij} \, dV - \int_A (y \partial \phi_h/\partial y) u_{hx} \, dA - \int_A \phi_h \, y \, dA = 0 \]

since on the endface \( b_v \) \( u_{hx} \) is linear and \( \partial \phi_h/\partial y \) is an equilibrium distribution. On \( A_v \) \( \partial \phi_h/\partial y = 0 \) only where \( y = 0 \) because of \( \gamma \)'s continuity. Hence \( dW_h = dl \, w \, dA \geq 0 \), \( \partial \phi_h/\partial y > 0 \).

Thus the parameter \( \delta_h \) increases and the stiffness \( D \) monotonously decreases with increasing \( \lambda \) at given eccentricity of the load \( P \).

In Fig. 5b the parameters of a T beam are calculated by the finite element method, which provide lower bounds to the parameters \( \delta_h \). With increasing length the parameter values are decreasing if \( \lambda > 1 \) which implies an increasing error. The optimal values are therefore the highest values attained approximately at \( \lambda = 1 \). The parameters \( a_h \) and \( b_h \) seem to follow the same pattern as \( \delta_h \) with \( \delta_h' = \delta_h > 0 \). A sufficient condition for this is \( \beta'' - \beta' h > 0 \). Then from relation \( \delta_h' = \delta_h > 0 \) and eq. 15) there follows:

\[ \beta'' - \beta' h > 0; \quad |a''_h| - |a'_h| > 0. \]

4. Smoothed stress- and gap-distributions at dry joints

Next we investigate the effect of the slenderness \( \lambda \) of the period on the distributions \( \sigma_x, \gamma_x \) at the joint. The centroid \( H(y_h,z_h) \) of the gap volume \( V_h \) is determined by the parameters \( a_h, b_h \) if the centroid \( e_y, e_z \) of \( P \) is known (eq. 9,10).

\[ z_h = c z_{hy}/b_h \quad \quad y_h = c y_{hz}/b_h \tag{23} \]

The zero-line of \( \gamma(z,y) \): \( l(H) = \gamma(y_h', z_h') = 0 \) intersects the principal axes at

\[ y'a = -i^2 z/y_h; \quad z'a = -i^2 y/z_h \tag{23'} \]

In the limiting case \( \lambda \to 0 \) corresponding to the classical theory of reinforced concrete the continuous compressive stress distribution \( \sigma(y,z) \leq 0 \) is linear on a part \( A_+ \) of \( A \). Only the part above the neutral-line \( l(\sigma) = 0 \) is strained by \( \varepsilon = \sigma/E \). If we consider a symmetric period \( b_{v-1}, b_v \) the mutual displacement of the endfaces is \( v_x = e_{x} = 1 \varepsilon_2 / E \). Starting from \( l(\sigma) \) we have a gap-distribution \( \gamma'(y,z) > 0 \) linear on \( A_+ = A-A_- \) (Fig 6).
Fig. 5 Dilatation parameters $\beta_h$, rotation parameters $\alpha_h$ and interpenetration parameters $\delta_h$ versus slenderness ratio $\lambda$. 
The symmetry planes \( b_{v-1}, b_v \) remain plane. The parts below \( l(\gamma') \) remain unstrained. Hence \( v_{x+} = \gamma' \) below \( l(\sigma_-) \) is an extension of the plane \( v_{x-} \). The mutual displacement of planes \( b_{v-1}, b_v \) is therefore a linear distribution over \( A \)

\[
v_{x}(y,z) = l\sigma_{-}/E + \gamma'
\]

24) to which corresponds a stress distribution

\[
\sigma_{x}(y,z) = \sigma_{-} + \sigma_{+}'
\]

25) where \( \sigma_{-} = E\gamma' / l \) is the fictitious stress induced by the gap. \( \sigma_{+}' \) represents thus two stressblocks: the real compressive block \( \sigma_{-} \) with resultant \( P = -P_{1} \) (\( P \geq 0 \)) at point \( P = \{ y_p, z_p \} \), and the tensile block \( \sigma_{+}' \) with resultant \( H = H_{1} \) (\( H \geq 0 \)), at point \( H = \{ y_h, z_h \} \). The fictitious force-void couple \((P,H)\) induces linear distributions \( \sigma_{e} \) and \( \sigma_{y}' = E\gamma_{x}/l \).

\[
\sigma_{e} = -(1 + z_p z / i_y y_p y / i_z) P / A; \quad \sigma_{y}' = (1 + z_h z / i_y y_h y / i_z)^2 H / A
\]

26) where \( H = E\gamma_{h}/d\lambda = \dot{P}_{h} \); \( y_h = c_{y} \dot{h} \); \( z_h = c_{z} \dot{h} \); Superposition gives
$$\sigma^* = \sigma^* - \sigma^* = \sigma_e + \sigma_\gamma \quad \sigma^* - \sigma_e = -(\sigma^* - \sigma_\gamma)$$

which is induced by a force $Q = (\beta_h - 1) \pi$ acting at $Q(\gamma_q \gamma_q)$ on line $PH$. The zero lines $l(\sigma^*), l(\sigma_e), l(\sigma_\gamma)$ define mappings, point to line, $Q \to l(Q); P \to l(P); H \to l(H)$. $l(Q)$ determines the axis of local rotation $\omega$

$L(P) = 1 + z_p z / i_y^2 + y_p y / i_z^2 = 0; L(H) = 1 + z_h z / i_y^2 + y_h y / i_z^2 = 0$

$L(Q) = 1 + z_q z / i_y^2 + y_q y / i_z^2 = 0$  \(27\)

Because of the similarity of the expressions for $\sigma_e(y, z)$ and $\gamma(y, z)$ we have the following dual relations known from the theory of eccentrically loaded elastic beams.

i) If $P$ moves along line $a_0$ through the origin $O$ then the neutral line $l(P)$ experiences a translation in the same direction.

ii) If $P$ moves along a line $a_p$ outside the origin $O$ the neutral line $l(P)$ turns round a fixed point $P'(a_p)$ which is the point of action of $P$ whose $\sigma_e$-distribution has neutral line $l(P') = a_p$.

i') If centroid $H$ of $V_h$ moves along line $a_h$ through $O$ then axis $\omega_h = l(H)$ experiences a translation in the same direction.

ii') If $H$ moves along a line $a_h$ outside $O$, the axis $\omega_h = l(H)$ turns round a fixed point $H'(a_h)$ which is the centroid of gap volume $V_h'$ whose $\gamma(y, z)$ distribution has zero line $l(H') = a_h$.

If we consider corresponding collinear points $Q, P, H$ the neutral lines $l(P)$ and $l(H)$ have common point $S$ in the plane $\Pi$. Hence $\sigma_e(S) = 0; \sigma_\gamma(S) = 0$. But because of (27) we have also $\sigma^*(S) = \sigma_e(S) + \sigma_\gamma(S) = 0$. Therefore if we denote the points of intersection of the axis $PH$ with $l(P), l(H)$ and $l(Q), P', H'$ and $Q'$ respectively:

iii) The neutral lines $l(P), l(H)$ and $l(Q)$ have common point $S$ which is the point of action of $Q$ or $P$ or $H$ whose neutral line $L(S)$ is $PH$ and contrarily, if $Q, P$ or $H$ are moving along $PH$, the neutral lines $l(Q), l(P)$ and $l(H)$ respectively are turning round $S$. Furthermore $l(P') = SP; l(H') = SH; l(Q') = SQ$ (Fig 6).
Because the point of action of \( P \) and \( H \) are independent of the magnitude of the force-void couple \((P,H)\), there is a one to one mapping \( H = T(P) \) of points \( P \in A \) to points \( H \in A \) in the projective plane. Compressive force \( P \) at point \( P \) induces tensile force \( H = -\beta_h(P)P \) at point \( H \). Since \( \sigma^\circ + \sigma^+ \) generates a plane \( \sigma^\circ(y,z) \) induced by force \( P \) at \( P \) and \( H \) at \( H \), then \((-\sigma^-) + (-\sigma^+) \) generates a plane \(-\sigma^+(z,y) \) induced by load \( P^\circ = -H = \beta_h(P)P \) at \( H \), and force \( H^\circ = -\beta_h(H)P^\circ = -P \) at \( P \). Hence

\[
T(H) = TT(P) = P; \quad TT = I; \quad \beta_h(P)\beta_h(H) = 1
\]

Since \( V_h = Pd\beta_h(P)/E \) we have

iv) If compressive force \( P \) with centroid at \( P \) induces gap volume \( V_h \) with centroid at \( H \) then force \( P \) at \( H \) induces gap volume \( V'_h \) at point \( P \) (Theorem of reciprocity)

From this follows

v) If \( P \) moves from \( P' \), a point on \( \partial K \), the contour of the kern, to a point \( P \) on the contour \( \partial A \) of the convex hull A of A, Q moves from \( Q_{k-} = P' \) through infinity to \( Q_{k+} \) on \( \partial K \), and \( l(Q) \) moves from the tangent of \( \partial A \) through \( H' \) to a tangent point \( Q_{k+} \) on \( \partial K \). \( l(Q) = \omega^\circ \) moves from \( l(Q_{k-}) = l(P') \) to a tangent \( l(Q_{k+}) \) of \( \partial A \) through \( P \).

v') If \( H \) moves from point \( H \) on \( \partial A \) to a point \( H_k = Q_{k+} \) on \( \partial K \), the axis \( \omega_h = l(H) \) moves from a tangent of \( \partial K \) through \( P' \) to a tangent through \( P \) on \( \partial A \) (Fig. 1d).

A numerical comparison shows that the stressblock \( \sigma_\circ \) for different \( \lambda \)-values doesn't differ much from the linear one on \( A_\circ \) of the classical theory with \( \lambda \rightarrow 0 \). Also the relations between the deformation parameters \( a_h, \beta_h, \delta_h \) closely approach the corresponding values for \( \lambda \rightarrow 0 \) with same \( m_y, m_z \). Hence

\[
\begin{align*}
\frac{a_h(\lambda)}{\beta_h(\lambda)} &= \frac{a_h}{\beta_h}; \quad \frac{\delta_h(\lambda)}{\beta_h(\lambda)} &= \frac{\delta_h}{\beta_h} \\
\end{align*}
\]

This means that the results concerning \( \sigma_\circ \) obtained for \( \lambda \rightarrow 0 \) can by induction be extended to values \( \lambda \neq 0 \). Relations 29) indicate that for given eccentricity of load \( P \) the centroid of \( V_h \) and the lines \( l(P), l(Q) \) and \( l(H) \) are approximately independent of \( \lambda \) and can be considered as invariants together with their points of intersections \( P', Q' \).
Fig. 7  Compressive stress distribution $\sigma_x$ and geometric determination of relative gap width $\beta_G(z)$ of dry joint.
and \( H' \) with \( PH \). In order to determine an approximate distribution \( \gamma_c \) affine with \( \sigma^+ \) of the real one \( \gamma \) we can start from distribution 11).

\[
\gamma(y,z) = \frac{\beta_h(y,z)}{EA}
\]

where \( \beta_h(y,z) \) is a linear function of \( y,z \)

\[
\beta_h(y,z) = \beta_h(1 + \frac{z}{i_y} y^2 + \frac{y}{i_z} z^2)
\]

Especially \( \beta_h(0,0) = \beta_h; \beta_h(y_p,z_p) = -\delta_h \). The smoothed gap width \( \gamma_c \) at the joint is therefore a linear distribution on \( \Lambda_+ \)

\[
\gamma_c(y,z) = \frac{Pd\beta_c(y,z)}{EA}
\]

which is determined by the \( \beta_c \)-plane spanned by the lines \( l(Q) \) and \( \beta_h(l(P)) \) [4].

On \( HP \) \( \gamma_c \) is determined by the \( \beta_c \) line through zeropoint \( Q' \) and the point \( \beta_h(P') \). \( \gamma(PH) \) is determined by the line through zeropoint \( H' \) and point \( \beta_h(O') \) through \( 0' = PH \land SO \), (Fig. 6) or by the straight line through \( \beta_h(P') = -\delta_h \) at point \( P \) and \( \beta_h(O') \) at \( 0' \). If the line passes through the origin \( 0 \) then \( \beta_h(0) = \beta_h \). For some symmetric profiles the \( \gamma \) distribution along the vertical symmetry line \( y = 0 \) has been calculated. In Fig 7a the distribution \( \beta_c \) is determined for different \( \lambda \)-values by using for \( \beta_h \) the line through \( \beta_h(P) \) and \( \beta_h(O') \). \( \gamma \) is zeropoint is very close to the theoretical point \( H' \). In Fig 7b the theoretical point \( H' \) is used. The diagrams for \( \sigma_x \) and \( \gamma \) calculated, show good agreement with the linear values \( \sigma^- \) and \( \gamma_c \) although the difference between calculated \( \gamma \) and smoothed \( \gamma_c \) increases with growing \( \lambda \), but the difference at the maximum gap width does not exceed 7 % when \( \lambda = 2 \) which actually corresponds to infinite \( \lambda \).

Also the reciprocity relation between the position vectors of \( H \) and \( P \) seems well established as is seen from Fig 9. If the cross-section has double symmetry the distance \( \Delta z \) between \( H \) and \( P \) remains approximately constant.
Fig. 9  Reciprocity relation of ratios $\rho_h = z_h / k$, $\rho_p = z_p / k$ for various slenderness ratios $\lambda$. 
Fig. 8 Measured gaps $\gamma$ of dry joint of prestressed concrete beam with unbonded tendons. Obs. linearity of $\gamma$ in $A$.

The numerical comparison carried out, confined to symmetric profiles, covers only in part the general theory developed. Nevertheless they show conclusively that in the most important case of symmetrical profiles with force-void line in the axis of symmetry the asymptotic approach to the classical theory with $\lambda \to 0$ provides a satisfactory base for the evaluation of stress and gap width at the joint.

5. The longitudinal cracking at dry joints

The stress-distribution of a block with rectangular profile between two dry joints loaded by a compressive force $P$ is shown in fig 10a. The $\sigma_x$ distribution start at the end faces as compressive stress close to the linear distribution $\sigma_x^0 < 0$ approaching with increasing $\lambda$ at the middle part of the block the distribution $\sigma_x$ of the monolithic beam. At the end faces appear tensile stresses $\sigma_z$ with a maximum below the limit line $\sigma_x = 0$. This maximum increases with $\lambda$ from 0 at $\lambda = 0$ to a limit value $\sigma_{z_{\text{max}}}$ when
Fig. 10 a) Stress distribution in segmental block with rectangular cross-section. b) Dependence of relative-
\[ \sigma_{x \text{min}}, \sigma_{x \text{max}} \] 
\[ \sigma_{z \text{max}} \] on eccentricity \( m \).

\( \lambda \geq 2 \). \( \sigma_{z \text{max}} \) increases with increasing eccentricity \( m \) and is considerably greater than the maximum tensile stress \( \sigma_{z \text{max}} \) at the lower edge of the block. The maximum values of the compressive stress \( \sigma_{x \text{max}} \) coincide with those corresponding to \( \lambda = 0 \) independently of the solutions used. A considerable difference occurs between the \( \sigma_{z} \) values of the upper bound solution and two lower bound solutions (Fig 10b). The \( \sigma_{z} \) stresses give rise to longitudinal cracks which start from the vertical joint. In some cases they are initiated earlier than the vertical cracks caused by the bending of beams (Fig 11). The predisposition to longitudinal cracking is very pronounced in prestressed beams with unbonded tendons. In segmental beams with dry joints an important question is the location and extension of possible cracks. This location can be determined by the lemma:

Lemma III: A possible crack of mode I will be initiated at that location of the body where a minute cut induces the greatest drop in the potential energy \( \Delta W = -W(\sigma_{n}) \) and therefore also the greatest drop in the stiffness \( D \) of the structure.
Proof. We consider a non-monolithic uncracked structure with dry joints \( \Gamma_k \) and uniformly distributed microcracks, loaded by external loads \( p^* \), in which we make a minute cut \( \Gamma_h(x) = a \cdot \Delta s \) normal to the principal tensile stress at \( x \). The state \( \{ p, \sigma, u, \varepsilon, \gamma \} \) is separated into the states \( \{ p', \sigma', u', \varepsilon', \gamma' \} \) before cracking and \( \{ p_h, \sigma_h, u_h, \varepsilon_h, \gamma_h \} \) induced by the cut: \( \{ p, \sigma, u, \varepsilon, \gamma \} = \{ p', \sigma', u', \varepsilon', \gamma' \} + \{ p_h, \sigma_h, u_h, \varepsilon_h, \gamma_h \} \)

\[ \pi_0 = W(\sigma'_e) - \int p^* \cdot u'_e \, d\Gamma = - \frac{1}{2} \int p^* \cdot u'_e \, d\Gamma \]  \hspace{1cm} (33) 

since \( W(\sigma_e) = \frac{1}{2} \int p^* \cdot u'_e \, d\Gamma \). The potential energy after cutting is

\[ \pi_1 = W(\sigma) - \int p^* \cdot u \, d\Gamma = - \frac{1}{2} \int p^* \cdot u \, d\Gamma \]

provided the friction law excludes irreversible effects and the work on \( \Gamma_k \) and \( \Gamma_h \): \( p'_e \gamma'_e, p \gamma = 0 \). Hence
\[ \Delta \pi = \pi_1 - \pi_0 = -\frac{1}{2} \int \mathbf{p}^* \cdot \mathbf{u}_h \, d\Gamma = -\Delta W \] 33\)

where \( \Delta W = W(\sigma'_e + \sigma_h) - W(\sigma'_e) = W(\sigma_h) - \int \Gamma_h \mathbf{p}_h \mathbf{y}_e \, d\Gamma. \) Since \( \sigma_h, \mathbf{e}_h \) are permissible states we get using eq. 3)

\[ \Delta W = \frac{1}{2} \left( \int \Gamma_h (\mathbf{p}_h \mathbf{y}_e - \mathbf{p'}_e \mathbf{y}_h) d\Gamma + \int \Gamma_h \mathbf{p}'_h d\Gamma \right) \] 34\)

On \( \Gamma_h(\mathbf{r}) \) the stress \( \mathbf{p}_{h\mathbf{x}}(\mathbf{x}, \mathbf{r}) = \mathbf{p}'_e(\mathbf{x}, \mathbf{r}) = -\sigma'_e(\mathbf{x}, \mathbf{r}) \) induces a singularity at the depth \( \mathbf{x} = \mathbf{a} \). Therefore acc. to linear fracture mechanics [5]

\[ \gamma_h(\mathbf{x}, \mathbf{r}) = 8K_I(\mathbf{a}, \mathbf{r})((\mathbf{a} - \mathbf{x})/2\pi)^{\frac{1}{2}} / E' \] 34\)

Because of the smallness of the cut, \( \mathbf{p}_h \) and \( \gamma_h \) differ on \( \Gamma_k \) from zero only where \( \sigma'_e, \mathbf{e}_e \) are very small. Therefore the first integral can be neglected compared with the second. Hence putting \( \sigma'_e(\mathbf{x}, \mathbf{r}) = \sigma'_e(0, \mathbf{r}) g(\mathbf{x}) \)

\[ (-\Delta \pi) = W(\sigma_h) = -\frac{1}{2} \int \mathbf{p}_h \gamma_h d\Gamma = \frac{1}{2} \sigma'_e(0, \mathbf{r}) \mathbf{ \Delta s} g(\mathbf{x}, \mathbf{r}) \gamma_h(\mathbf{x}) \, d\mathbf{x} \] 34\)

The integral depends also on \( W(\sigma_h) \) since \( \gamma(\mathbf{x}, \mathbf{r}) \) depends on the stress intensity factor

\[ K_I(a, \mathbf{r}) = ((E/\Delta s) \frac{\partial \sigma_h}{\partial \mathbf{a}})^{\frac{1}{2}} \] 35\)

Eq.34') therefore leads to a differential equation of \( W(\sigma_h) \). Its solution gives

\[ W(\sigma_h) = \sigma(0, \mathbf{r})^2 C(g(\mathbf{a})); \quad K_I(a, \mathbf{r}) = \sigma(0, \mathbf{r}) (E(\partial C/\partial \mathbf{a})/\Delta s)^{\frac{1}{2}} \] 36\)

Varying \( \mathbf{r} \), keeping \( \Delta s, \mathbf{a} \) constant, \( W(\sigma_h) \) simultaneously with \( K_I(a, \mathbf{r}) \) and \( \sigma(0, \mathbf{r}) \) attains maximum, where the crack is initiated, from which the lemma follows. =

The lemma describes the situation at the beginning of cracking. The cracking continues then leading to unstable growth or it stops when the total energy \( U = \pi + \pi_s \), where \( \pi_s \) is the surface energy, attains a minimum with \( \Delta \pi = -W(\sigma_h) \)

\[ \frac{\partial U}{\partial \mathbf{a}} = \frac{\partial (\pi_s - W(\sigma_h))/\partial \mathbf{a} = 0} \] 37\)

In mode I cracks this may happen when in compressed parts a state is attained where all singularities along the crack tip disappear. This case defines the limit state of free contact. Since along the crack \( \Delta s K_I(a, \mathbf{s}) = 0 \), there follows

\[ \frac{\partial W(\sigma_h)}{\partial \mathbf{a} = 0} \] 37\')
Because $W(\sigma)_h$ increases monotonously with increasing $a$, it attains a maximum in the limit case of free contact. If we consider a monolithic block $b_v-b_{v-1}$ of a beam (Fig 12) loaded by $P$ with constant eccentricity $m_z=m$ and make in the symmetry section a cut of depth $a$, which diminish the uncracked depth to $d_k = d-a = \kappa d$, the released energy is

$$W(\sigma)_h = P^2 d \delta_h(a)/2EA$$

where $\delta_h$ is an increasing function of $a$, everywhere where we have a tensile stress-singularity until we reach the limit state depth $a_o = d-d_0$. If we then further increase the depth $a$ and the cut is of finite thickness we will get a compressive stress singularity at the tip of the cut and therefore the energy $W(\sigma)_h$ will start again to increase. We have in this case an inflection point at $a_o$. If again the cut is infinitely thin it will not affect the state of stress and strain for values $a > a_o$ because the section is a symmetry plane. In this case the function $\delta_h(a)$ will remain constant (Fig 12). We have thus at $a_o$ only in the latter case where the impenetrability condition $\gamma_x \geq 0$ is not violated a true maximum of $\delta_h(a)$.
Along the section \( a_y \) we have principal stresses \( \sigma_x', \sigma_z \) with \( \sigma_x = 0 \) at \( a_0 \) and \( \sigma_x < 0 \) for \( a > a_0 \). The question arises where the longitudinal crack will be initiated along the opened joint (Fig 13). The strain energy in the uncracked state is

\[
W_0 = P^2d(\lambda \delta_e(m) + \delta_h(m, \lambda)) / 2E'A
\]

If the crack develops at height \( z \leq z_o \) (\( z_o \) tip of gap in dry joint) with length \( l_1 \) we can distinguish three regions (Fig 13):
- A with depth \( d \) and total length \( l - l_1 \).
- B with depth \( d_k \) and total length \( l_1 \).
- C two unloaded cantilevers with depth \( d_c = d - d_k \) and total length \( l_1 \) clamped to parts A along line 1-2.

An approximate solution is sought by making fictitious cuts along 1-2 separating the accordingly unstressed parts C from the structure. Denoting \( d_k = \lambda d; \lambda_1 = l_1/d; \lambda_k = l_1/d_k; A/A_k = q; m=e/k; m_k=e_k/k_k \) we get after cracking strain energies \( W_A, W_B \) and \( \Delta W = W_A + W_B - W_0 \)

\[
W_A = (P^2d/2EA)((\lambda-\lambda_1)\delta_e(m) + \delta_h(m, \lambda-\lambda_1, \lambda))
\]

\[
W_B = (P^2d_k/2EA_k)((\lambda_k\delta_e(m_k) + \delta_h(m_k, \lambda_k))
\]
\[ \Delta W = (P^2d/2EA)(\lambda_1(q\delta_e(m_k) - \delta_e(m)) + \\
+ \delta_h(m,\lambda - \lambda_1, \kappa) + q_k \delta_h(m_k, \lambda_k) - \delta_h(m, \lambda)) \]

The model satisfies the kinematical- and stress-conditions on the joint \( a_y \) of part B. On the lines 0-1 the kinematical conditions are satisfied but the equilibrium conditions are satisfied only for the generalized forces \( N_B, M_B \). In the fictitious cut 1-2 the kinematical conditions are not satisfied but the stresses are permissible, which justifies the separation of parts C.

\( \Delta W \) attains its maximum when \( m_k = 1 \) with corresponding \( \kappa = \kappa_0; A_k = A_0 \); \( \gamma_k = 0 \). Indeed in this case the distribution \( \sigma_x \) in B is linear, P acting at a core point. Hence

\[ \delta_h(1, \lambda_k) = 0; \delta_h(m, \lambda - \lambda_1) = \max \delta_h(m, \lambda - \lambda_1, \kappa) \] and \( q_{\delta_e}(1) - \delta_e(m) = \delta_h(m) \) because for body B acc. to 20a: \( u_p = P_l \delta_e(1)/EA_0 = P_l \delta_e(m_k)/EA_k \) \( \forall A_k \) with \( A_0 \leq A_k < A \) since \( A_0 = \text{Min} A_B \), from which \( (A/A_0) \delta_e(1) = \delta_0(m) \), and max \( (q_{\delta_e}(m_k) - \delta_e(m)) = \delta_h(m) \).

Since we have uniaxial compression in B the rising of the crack above \( z_0 \) inside the region \( d_0 \) does not change the stresses in B and A. Therefore \( (\partial \Delta W/\partial \kappa)_{\kappa_0} = 0 \) and

\[ \Delta W_{\text{max}} = (P^2d/2EA)(\lambda_1 \delta_h(m) + \delta_h(m, \lambda - \lambda_1) - \delta_h(m, \lambda)) \]

The above result is close to a lower bound solution.

An upper bound solution is obtained assuming parts A being in the monolithic state \( \{\sigma_e, t_e\} \) parts B deform as in the lower bound solution and the cantilevers C remain undeformed and rigidly clamped to parts A along lines 1-2. \( W_0 \) doesn't change. For parts A and B we get after cracking

\[ W_A' = (P^2d/2EA)(\lambda - \lambda_1) \delta_e(m) \]

\[ W_B' = (pd_k/2EA_k)(\lambda_1 \delta_e(m_k) + \delta_h(m_k, \lambda_k)) \]

Hence

\[ \Delta W' = (P^2d/2EA)(\lambda_1(q \delta_e(m_k) - \delta_e(m)) + q_k \delta_h(m_k, \lambda_k) - \delta_h(m, \lambda)) \]
By the same reasoning as in the lower bound solution the
maximum is attained when $k$ and $A_k$ corresponds to $m_k = 1$
which gives

$$
\Delta W'_{\text{max}} = (p^2d/2EA)(\lambda_1 \delta_h(m) - \delta_h(m, \lambda))
$$

This corresponds to a considerably smaller drop of
potential energy $\pi$ and stiffness $D$ than that of the lower
bound solution. Nevertheless both cases confirm the result
of the stress calculation that according to lemma III the
maximum of the tensile stresses $\sigma_{z_{\text{max}}}$ are attained a little
bit below the tip of the gap corresponding closely to the
neutral line at height $a_0$ according to the classical theory.
Tests confirm also this result. The stress intensity factor
$K_I$ of the longitudinal crack are acc. to the lower bound
solution

$$
K_I = \left(\frac{E'}{b_o} \frac{\partial \Delta W}{\partial a}\right)^{1/2} = \frac{P}{b_o} \left(\frac{\delta_h(m, \lambda - \lambda_1)}{\partial \lambda}\right)^{1/2}
$$

and acc. to the upper bound solution

$$
K_I = \left(\frac{E'}{b_o} \frac{\partial \Delta W'}{\partial a_1}\right)^{1/2} = \frac{P}{b_o} \left(\frac{\delta_h}{b_o A}\right)^{1/2}
$$

Fig. 14 Longitudinal cracking in stone arch (Arles, arena).
Longitudinal cracking occur also in sufficient slender $(\lambda > 1)$ voussoirs of stone arches. From the cracks an estimate can be made about the position of the thrust-line of the arches because the resultant must pass on a distant $d_k/3$ from the compressed edge, where $d_k$ is the depth of the uncracked region (Fig 14).

References


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