VARIATIONAL METHODS FOR A NON LINEAR BEHAVIOR OF MULTILAYERED SOLIDS ; STRESS CALCULATION .

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Rakenteiden mekaniikka, Vol. 22
No 4, 1989, ss. 29-70.

SUMMARY : In the first part, the main tools of the numerical methods used in the analysis of elastoplastic multilayered solids are presented . First the local equations of the incremental elastoplastic problem are given. The variational theorem using incremental displacements is presented as well as the dual variational complementary energy. A mixed elastoplastic REISSNER formulation of the problem is developed . Two explicit schemes are given , using a gradient method with an auxiliary operator (MERCIER method). The convergence of the algorithm with suitable energy norms (bounded energy spaces) is shown . In the second part , a modified mixed variational method using the PIAN ideas is employed . An explicit incremental scheme based on MERCIER method is given . The example of a beam is chosen ; different results (stresses , deflections) calculated with the initial stress method and the presented hybrid method are compared . Convergence of the hybrid scheme can be faster than the initial stress method , depending on the value of the auxiliary parameter chosen in the numerical procedure .

PART 1 : MIXED MODEL

INTRODUCTION

We study in this part , the case of L solids $\Omega^{(l)}$ with adherent interfaces ($1 \leqslant l \leqslant L$) . We suppose that the L materials have each an elastoplastic standard behavior ; the behavior laws are given in an incremental form . We want to study the question of the calculation of the stresses between two layers, or the question of calculating the stress vector.

In this case, mixed methods are used especially under two forms: PIAN and REISSNER principles. The first one is indicated when we want to know an approximate solution satisfying the equilibrium equations; the second one is used especially when we want to evaluate the level of energy. The common base of both is the idea of minimizing the complementary energy on a special convex, where continuity conditions are not required: so, we need to use Lagrangian multipliers to relax these continuity conditions.

We proceed now in four steps: first, we give the local equations of the problem, then, their incremental form, the variational principles, and finally, the results that we can get.

GOVERNING EQUATIONS

Local equations of the problem

Consider L solids $\Omega^{(l)}$, $1 \leqslant l \leqslant L$, having different material properties and adherent interfaces (Fig.1). This implies the continuity of the increments of the stress and displacement vectors $\Delta\vec{T} = \Delta T_i$ and $\Delta\vec{u} = \Delta u_i$. The l^{th} interface between $\Omega^{(l)}$ and $\Omega^{(l+1)}$ is noted $\Gamma^{(l)}$. We suppose that the different solids have an elastoplastic standard behavior / 1, 2 /; the external forces $\Delta\vec{T}^{(d)} = \Delta T_i^{(d)}$ are applied slowly enough so that the hypothesis of quasi static deformations is valid. This means, for any plastic potential f that $\partial f / \partial t = d f / d t = \Delta f / \Delta t$.

In the layer $\Omega^{(l)}$, we can write, with (e) for elastic and (p) for plastic,

$$\Delta\sigma_{ij,j} = 0 \quad \text{(body forces are neglected)}, \tag{1}$$

$$\Delta\varepsilon_{ij} = \Delta\varepsilon_{ij}^{(e)} + \Delta\varepsilon_{ij}^{(p)}, \tag{2}$$

$$\Delta\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial(\Delta u_i)}{\partial x_j} + \frac{\partial(\Delta u_j)}{\partial x_i} \right) , \tag{3}$$

$$\Delta\varepsilon_{ij} = S_{ijkl} \, \Delta\sigma_{kl} + \Delta\lambda \frac{\partial f}{\partial \sigma_{ij}} , \tag{4}$$

$$\Delta\varepsilon_{ij} = - \Delta\lambda \frac{\partial f}{\partial A_{ij}} , \tag{5}$$

$$\begin{aligned} &\Delta\lambda \geqslant 0 \quad \text{if} \quad f = \Delta f = 0 , \\ &\Delta\lambda = 0 \quad \text{if} \quad f = 0 \text{ and } \Delta f < 0 , \text{ or if } f < 0 \text{ and } \Delta f < 0 . \end{aligned} \tag{6}$$

Plasticity occurs only in the first case , and the second case is governed by elastic laws . S_{ijkl} is the inverse of the elasticity tensor D_{ijkl} ; it has the following properties

$$S_{ijkl} = S_{jikl} = S_{klij} . \tag{7}$$

We leave , as an exercise , to show that the behavior law can be inverted . The material becomes plastic when the stress tensor satisfies a criterion which is convex in σ_{ij} and A_{ij} . (For sim-simplicity , tensors will be written without indices when involved in scalar forms) .

$$f (\sigma_{ij} , A_{ij}) = f (\sigma , A) = 0 . \tag{8}$$

$A_{ij} (\alpha)$ is a set of generalized forces depending on hidden internal parameters α_{ij} characterizing the internal state of the behavior ; $f (\sigma , A (\alpha))$ is the plastic potential function which is smooth enough so that , in the stress set , the normal $\partial f / \partial \sigma_{ij}$ can be defined . The material is at the σ_{ij} level , and the question is : when the surface forces increase from $\vec{T}^{(d)}$ to $\vec{T}^{(d)} + \Delta\vec{T}^{(d)}$ (the stresses are σ_{ij} , and the new stresses are $\sigma_{ij} + \Delta\sigma_{ij}$) , how can we evaluate the increment ? From exercise 1 (appendix) , we get

$$\Delta\sigma_{ij} = D_{ijkl}\left(\Delta\varepsilon_{kl} - \frac{\partial f}{\partial\sigma_{kl}}\,\frac{\left\langle \frac{\partial f}{\partial\sigma_{pq}} D_{pqrs}\,\Delta\varepsilon_{rs}\right\rangle}{h + \frac{\partial f}{\partial\sigma_{ab}} D_{abcd}\frac{\partial f}{\partial\sigma_{cd}}}\right) \tag{9}$$

with

$$h = \frac{\partial f}{\partial A_{ij}}\,\frac{\partial f}{\partial A_{rs}}\,\frac{\partial A_{ij}}{\partial\alpha_{rs}} \,. \tag{10}$$

h is the hardening factor for the level of stress σ_{ij}. We know (exercises 2 - 3 , appendix) the fundamental inequalities for each $\Omega^{(l)}$, when , from the level of stress σ_{ij} , σ^*_{ij} becomes ($\sigma_{ij} + \Delta\sigma_{ij}$) or ($\sigma_{ij} + \Delta\sigma^*_{ij}$) :

$$(\Delta\sigma_{ij} - \Delta\sigma^*_{ij})(\Delta\varepsilon_{ij} - \Delta\varepsilon^*_{ij}) \geqslant S_{ijkl}(\Delta\sigma_{ij} - \Delta\sigma^*_{ij})(\Delta\sigma_{kl} - \Delta\sigma^*_{kl}) \tag{11}$$

$$\Delta\sigma_{ij}\,\Delta\varepsilon_{ij} \geqslant h\,\Delta\varepsilon_{ij}\,D_{ijkl}\,\Delta\varepsilon_{kl}\,\frac{1}{h + \frac{\partial f}{\partial\sigma_{ab}} D_{abcd}\frac{\partial f}{\partial\sigma_{cd}}} \,. \tag{12}$$

Here , $\Delta\sigma_{ij}$ and $\Delta\sigma^*_{ij}$ are two arbitrary stress increments . These inequalities are used in the variational principles to get minimum conditions . On the interfaces $\Gamma^{(l)}$ (Fig.2) , $\vec{n}^{(l)}$ being the outward normal to $\partial\Omega^{(l)}$ along $\Gamma^{(l)}$, we get the continuity of the increment of the stress vector

$$\Delta\sigma^{(l)}_{ij}\,n^{(l)}_j = -\Delta\sigma^{(l+1)}_{ij}\,n^{(l+1)}_j \tag{13}$$

and the continuity of the increment of the displacement vector

$$\Delta u^{(l)}_i = \Delta u^{(l+1)}_i \,. \tag{14}$$

On Γ_u which is the part of $\partial\Omega$ where the displacements are prescribed (Fig. 1) ,

$$\Delta \vec{u} = 0 \ . \tag{15}$$

On Γ_σ which is the part of $\partial\Omega$ where the surface forces are known (Fig. 1) ,

$$\Delta\sigma_{ij}\, n_j = \Delta T_i^{(d)} \ . \tag{16}$$

Latin indices take the values (1 , 2 , 3) . The inequalities (11) and (12) are quite essential because they are the main tools in formulating a global or variational form of the problem.

Variational formulation / 3 /

a) Definition of the involved spaces (V and Σ_{ad})

We define now

$$V = \Big\{ \Delta\vec{v} : \varepsilon_{ij}(\Delta\vec{v}) \in L^2(\Omega^{(l)}) \ , \ 1 \leqslant l \leqslant L \ ,$$

$$\Delta\vec{v}^{(l+1)} = \Delta\vec{v}^{(l)} \ , \ l = 1 \ , \ \dots \ , \ L-1 \ ; \ \Delta\vec{v} = 0 \text{ on } \Gamma_u \Big\} \tag{17}$$

with the norm , using (12) ,

$$\left\| \Delta\varepsilon(\vec{v}) \right\|^2 = \sum_{l=1}^{L} \int_{\Omega^{(l)}} \Delta\sigma_{ij}(\Delta\varepsilon(\vec{v}))\, \Delta\varepsilon_{ij}(\vec{v})\, d\Omega \ . \tag{18}$$

Consider the functional defined on V

$$\mathcal{B}(\Delta\varepsilon(\vec{v})) = \frac{1}{2} \sum_{l=1}^{L} \int_{\Omega^{(l)}} \Delta\sigma_{ij}(\Delta\varepsilon(\vec{v}))\, \Delta\varepsilon_{ij}(\vec{v})\, d\Omega - \int_{\Gamma_\sigma} \Delta T_i^{(d)}\, \Delta v_i\, d\Gamma \ . \tag{19}$$

with

$$\Delta\sigma_{ij}(\Delta\varepsilon(\vec{v}))\, \Delta\varepsilon_{ij}(\vec{v}) = D_{ijkl}\, \Delta\varepsilon_{ij}(\vec{v})\, \Delta\varepsilon_{kl}(\vec{v}) - D_{ijkl} \frac{\partial f}{\partial\sigma_{ij}} \frac{\left\langle \dfrac{\partial f}{\partial\sigma_{pq}} D_{pqrs}\, \Delta\varepsilon_{rs}(\vec{v}) \right\rangle}{h + \dfrac{\partial f}{\partial\sigma_{ab}} D_{abcd} \dfrac{\partial f}{\partial\sigma_{cd}}}\, \Delta\varepsilon_{kl}(\vec{v}) \ . \tag{20}$$

If $(\Delta\vec{u}, \Delta\sigma)$ is the solution of (1) to (16), then, we have

$$\mathcal{B}(\Delta\varepsilon(\vec{u})) = \operatorname*{Inf}_{\Delta\vec{v} \in V} \mathcal{B}(\Delta\varepsilon(\vec{v})) . \tag{21}$$

The proof is given in exercise 4 of the appendix. We define

$$\Sigma_{ad} = \Big\{ \Delta\tau_{ij} : \Delta\tau_{ij,j} = 0 \text{ on } \Omega^{(l)} , 1 \leqslant l \leqslant L ,$$

$$\Delta\tau^{(l)}_{ij} n^{(l)}_j = - \Delta\tau^{(l+1)}_{ij} n^{(l+1)}_j , 1 \leqslant l \leqslant L-1 \text{ on } \Gamma^{(l)},$$

$$\Delta\tau_{ij} n_j = \Delta T^{(d)}_i \text{ on } \Gamma_\sigma \Big\}$$

with the norm

$$\|\Delta\tau\|^2_{\Sigma_{ad}} = \int_\Omega S_{ijkl} \Delta\tau_{ij} \Delta\tau_{kl} \, d\Omega .$$

Consider

$$\mathcal{A}(\Delta\tau) = \frac{1}{2} \sum_{l=1}^{L} \Big(\int_{\Omega^{(l)}} \Delta\tau_{ij} S_{ijkl} \Delta\tau_{kl} \, d\Omega$$

$$+ \int_{\Omega^{(l)}} \frac{1}{h} \left\langle \frac{\partial f}{\partial \sigma_{pq}} \Delta\sigma_{pq} \right\rangle \Delta\tau_{ij} \frac{\partial f}{\partial \sigma_{ij}} \, d\Omega \Big)$$

$$= \frac{1}{2} \int_\Omega \varepsilon_{ij}(\Delta\tau) \Delta\tau_{ij} \, d\Omega .$$

If $(\Delta\vec{u}, \Delta\sigma)$ is the solution of (1) to (16), then, we have

$$\mathcal{A}(\Delta\sigma(\Delta\vec{u})) = \operatorname*{Inf}_{\Delta\tau \in \Sigma_{ad}} \mathcal{A}(\Delta\tau) . \tag{22}$$

The proof is given in exercise 5.

b) Mixed formulation (functional equations)

We choose $\Delta\vec{v} \in V$, and

$$\Sigma = \Big\{ \Delta\tau : \Delta\tau_{ij} = \Delta\tau_{ji} , \Delta\tau_{ij} \in L^2(\Omega) \Big\} ,$$

and we define

$$a(\Delta\sigma,\Delta\tau) = \int_\Omega \Delta\sigma_{ij}\, S_{ijkl}\, \Delta\tau_{kl}\, d\Omega + \frac{1}{h}\int_\Omega \left\langle \frac{\partial f}{\partial \sigma_{pq}} \Delta\sigma_{pq} \right\rangle \frac{\partial f}{\partial \sigma_{ij}} \Delta\tau_{ij}\, d\Omega ,$$

$$b(\Delta\tau, \Delta\vec{v}) = \int_\Omega \Delta\tau_{ij}\, \varepsilon_{ij}(\Delta\vec{v})\, d\Omega ,$$

$$\|\Delta\tau\|^2 = \int_\Omega S_{ijkl}\, \Delta\tau_{ij}\, \Delta\tau_{kl}\, d\Omega ,$$

$$L(\Delta\vec{v}) = \int_{\Gamma_\sigma} \Delta T_i^{(d)}\, \Delta v_i\, d\Gamma ,$$

$$\|\Delta\vec{v}\|_V^2 = \int_\Omega D_{ijkl}\, \varepsilon_{ij}(\Delta\vec{v})\, \varepsilon_{kl}(\Delta\vec{v})\, d\Omega .$$

c) Properties of the functional

We have the problem :

$$\exists (\Delta\vec{u}, \Delta\sigma_{ij}) \in V \times \Sigma, \ \forall\ \Delta\tau_{ij} \in \Sigma, \ \forall\ \Delta\vec{v} \in V,$$

$$\left.\begin{aligned} &a(\Delta\sigma,\Delta\tau) - b(\Delta\tau,\Delta\vec{u}) = 0 , \\ &b(\Delta\sigma,\Delta\vec{v}) = L(\Delta\vec{v}) . \end{aligned}\right\} \qquad (23)$$

This problem is a saddle point of the functional

$$\mathcal{L}(\Delta\tau,\Delta\vec{v}) = -\int_\Omega \Delta\tau_{ij}\, \varepsilon_{ij}(\Delta\vec{v})\, d\Omega + \int_{\Gamma_\sigma} \Delta T_i^{(d)}\, \Delta v_i\, d\Gamma$$

$$+ \frac{1}{2}\int_\Omega \Delta\tau_{ij}\, S_{ijkl}\, \Delta\tau_{kl}\, d\Omega + \frac{1}{2h}\int_\Omega \left\langle \frac{\partial f}{\partial \sigma_{pq}} \Delta\tau_{pq} \right\rangle^2 d\Omega . \qquad (24)$$

The BREZZI BABUSKA theorem shows us that (see exercise 6)

$$\operatorname*{Max}_{\Delta\vec{v}\in V} \mathcal{L}(\Delta\sigma,\Delta\vec{v}) \leq \mathcal{L}(\Delta\sigma,\Delta\vec{u}) \leq \operatorname*{Inf}_{\Delta\tau\in\Sigma} \mathcal{L}(\Delta\tau,\Delta\vec{u}) . \qquad (25)$$

ASPECTS OF THE NUMERICAL SOLUTION

We outline now some aspects about the numerical analysis of the problem , which is : we know $(\sigma_{ij}, \vec{u})$ under surface loads $\vec{T}^{(d)}$; we give an increment $\Delta\vec{T}^{(d)}$; calculate $(\Delta\sigma_{ij}, \Delta\vec{u})$.

Two different schemes are given here ; the proof of convergence can be found in exercises 7 and 8 .

Note that we write

$$\left[\Delta\sigma , \Delta\tau\right] = \int_\Omega \Delta\sigma_{ij} \; S_{ijkl} \; \Delta\tau_{kl} \, d\Omega ,$$

$$(\Delta\tau, \Delta\varepsilon(\vec{v})) = \int_\Omega \Delta\tau_{ij} \; \Delta\varepsilon_{ij}(\vec{v}) \, d\Omega ,$$

$$\left\{ \varepsilon(\Delta\vec{v}) , \varepsilon(\Delta\vec{v}) \right\} = \int_\Omega \varepsilon_{ij}(\Delta\vec{v}) \; D_{ijkl} \; \varepsilon_{kl}(\Delta\vec{v}) \, d\Omega .$$

First scheme

The first scheme consists of

$$\left.\begin{aligned} &\left[\Delta\sigma^{n+1}, \Delta\tau\right] - \frac{1}{\rho}(\varepsilon(\Delta\vec{u}^{n+1}), \Delta\tau) = \left[\Delta\sigma(\Delta\vec{u}^{n}), \Delta\tau\right] \\ &\qquad\qquad\qquad\qquad - \frac{1}{\rho}(\varepsilon(\Delta\vec{u}^{n}), \Delta\tau), \\ &(\Delta\sigma^{n+1}, \Delta\varepsilon(\vec{v})) = L(\Delta\vec{v}) . \end{aligned}\right\} \quad (26)$$

This algorithm converges for $0 \leq \rho \leq 2$ (see exercise 7).

Second scheme

In the second scheme , we give another gradient method (MERCIER / 4 /) to find the saddle point :

$$\left.\begin{aligned} &\left[\Delta\sigma^{n+1}, \Delta\tau\right] - \rho(\Delta\varepsilon(\vec{u}^{n+1}), \Delta\tau) = \left[\Delta\sigma^{n}, \Delta\tau\right] \\ &\qquad\qquad\qquad\qquad - \rho(\varepsilon(\Delta\sigma^{n}), \Delta\tau), \\ &(\Delta\sigma^{n+1}, \varepsilon(\Delta\vec{v})) = L(\Delta\vec{v}) . \end{aligned}\right\} \quad (27)$$

The scheme (27) converges (see exercise 8) if $0 \leq \rho \leq M$, with

$$\exists M, \left\{ \varepsilon(\Delta\sigma) - \varepsilon(\Delta\tau) \right\}^2 < M \left[\Delta\sigma - \Delta\tau\right]^2 .$$

This second method can be considered as an explicit scheme . We proceed by increments .

Other possibility

A more recent method using convex analysis and differential equations on BANACH spaces can also be stated on ; an overlook on this last one is given below. The plasticity criterion is convex and has the form

$$f(\sigma, B) = 0 . \tag{28}$$

The behavior law is taken into account by the inequality

$$\overset{\circ}{\varepsilon}{}^{(p)}_{ij}(\sigma_{ij} - \tau_{ij}) - \overset{\circ}{\alpha}_{ij}(A_{ij} - B_{ij}) > 0 ,$$

$$\forall (\tau, B) , \quad f(\tau, B) \leqslant 0 .$$

By integrating on $\Omega = \overset{L}{\underset{l=1}{U}} \Omega^{(l)}$, we get

$$\int_{\Omega} (\overset{\circ}{\varepsilon}{}^{(p)}_{ij}(\sigma_{ij} - \tau_{ij}) - \overset{\circ}{\alpha}_{ij}(A_{ij} - B_{ij}))\, d\Omega , \tag{29}$$

$$\forall (\tau, B) \in K ,$$

where

$$K = \left\{ \tau \in \Sigma , B \in L^2(\Omega) : f(\tau, B) < 0 \right\} .$$

We define

$$\hat{\tau} = (\tau, B) ,$$

and

$$[\hat{\tau}, \hat{\tau}] = \int_{\Omega} \tau_{ij}\, S_{ijkl}\, \tau_{kl}\, d\Omega + \int_{\Omega} B_{ij}\, Z_{ijkl}\, B_{kl}\, d\Omega . \tag{30}$$

We have

$$\exists\, \varphi(\alpha) , \psi(A) ,$$

$$A_{ij} = \frac{\partial \varphi}{\partial \alpha_{ij}} , \quad \alpha_{ij} = \frac{\partial \psi}{\partial A_{ij}} , \quad \varphi + \psi = A_{ij}\, \alpha_{ij} .$$

Noting that $\Delta\varepsilon^{(p)}_{ij} = \Delta\varepsilon_{ij}(\vec{u}) - S_{ijkl}\, \Delta\sigma_{kl}$, we get $\forall\, \hat{\tau} \in K$,

$$\int_{\Omega} (\overset{\circ}{\varepsilon}_{ij}(\vec{u}) - S_{ijkl}\, \overset{\circ}{\sigma}_{kl})(\sigma_{ij} - \tau_{ij})\, d\Omega$$

$$- \int_{\Omega} \overset{\circ}{\alpha}_{ij}(A_{ij} - B_{ij})\, d\Omega \geqslant 0 . \tag{31}$$

We define

$$\Sigma^{*} = \left\{ \tau : \tau_{ij} = \tau_{ji} , \tau_{ij} n_j = T^{(d)}_i \text{ on } \Gamma_{\sigma} ,\right.$$

$$\tau_{ij,j} = 0 \text{ on } \Omega^{(l)} , \quad 1 \leq l \leq L ,$$

$$\left. \tau^{(l)}_{ij} n^{(l)}_{j} = - \tau^{(l+1)}_{ij} n^{(l)}_{j} , \quad 1 \leq l \leq L-1 \quad \text{on } \Gamma^{(l)} \right\} , \tag{32}$$

$$V^{*} = \left\{ \vec{v} = (v_{i}) : v^{(l+1)}_{i} = v^{(l)}_{i} , \quad 1 \leq l \leq L-1 , \right.$$

$$\left. \varepsilon_{ij} (\vec{v}) \in L^{2}(\Omega^{(l)}) , \quad 1 \leq l \leq L , \quad \vec{v} = 0 \text{ on } \Gamma_{u} \right\} . \tag{33}$$

We define

$$E(t) = \left\{ (\tau , B) : \tau \in \Sigma^{*} \right\} ,$$

and

$$K(t) = \left\{ K \cap E(t) \right\} .$$

We can obtain $E(t)$ by

$$E(t) = \left\{ \tau \in \Sigma , \ \forall \vec{v} \in V^{*} , \ b(\tau , \vec{v}) = L(\vec{v}) \right\} , \tag{34}$$

where

$$\Sigma = \left\{ \tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} , \ \tau_{ij} \in L^{2}(\Omega) \right\} ,$$

and

$$b(\tau , \vec{v}) = \int_{\Omega} \tau_{ij} \, \varepsilon_{ij} (\vec{v}) \, d\Omega , \quad L(\vec{v}) = \int_{\Gamma_{\sigma}} T^{(d)}_{i} \, v_{i} \, d\Gamma .$$

We define τ^{el}_{ij} as the solution of : $\exists \, \vec{w}^{el} \in V^{*}$, $\forall \vec{v} \in V^{*}$,

$$\int_{\Omega} D_{ijkl} \, \varepsilon_{ij} (\vec{w}^{el}) \, \varepsilon_{kl} (\vec{v}) \, d\Omega = L(\vec{v}) ,$$

and

$$\tau^{el}_{ij} = D_{ijkl} \, \varepsilon_{kl} (\vec{w}^{el}) .$$

We get , for $(\tau , \sigma) \in E(t)$,

$$\int_{\Omega} \tau_{ij} \, S_{ijkl} \, \mathring{\tau}^{el}_{kl} (\vec{w}^{el}) \, d\Omega = L(\vec{w}^{el}) ,$$

$$\int_{\Omega} \sigma_{ij} \, S_{ijkl} \, \mathring{\tau}^{el}_{kl} (\vec{w}^{el}) \, d\Omega = L(\vec{w}^{el}) ,$$

or

$$\int_{\Omega} (\tau_{ij} - \sigma_{ij})\, S_{ijkl}\, \overset{\circ}{\tau}_{kl}(\vec{w}^{el})\, d\Omega = 0 . \tag{35}$$

We have also

$$\int_{\Omega} \overset{\circ}{\varepsilon}_{ij}(\vec{u})\, (\tau_{ij} - \sigma_{ij})\, d\Omega = 0 , \tag{36}$$

and , by (31) : $\forall\, \hat{\tau} \in K(t)$,

$$\int_{\Omega} \Big(- S_{ijkl}\, (\overset{\circ}{\sigma}_{ij} - \overset{\circ}{\tau}{}^{el}_{kl})(\sigma_{kl} - \tau_{kl}) - \overset{\circ}{\alpha}_{ij}\, (A_{ij} - B_{ij})\Big)\, d\Omega \geqslant 0. \tag{37}$$

This inequation can also be written as

$$\frac{d}{d\,t}\left[\, \hat{\sigma} - \hat{\tau}^{el} \,,\, \hat{\sigma} - \hat{\tau} \,\right] \geqslant 0 . \tag{38}$$

Inequality (38) has an unique solution ; if $\hat{\sigma}^{*}$ is another solution of (38) , then we have

$$\left[\, \overset{\circ}{\hat{\sigma}}{}^{*} , \hat{\sigma} - \hat{\sigma}^{*} \,\right] \geqslant \left[\, \overset{\circ}{\tau}{}^{el} , \hat{\sigma} - \hat{\sigma}^{*} \,\right] ,$$

$$\left[\, \overset{\circ}{\hat{\sigma}} , \hat{\sigma}^{*} - \hat{\sigma} \,\right] \geqslant \left[\, \overset{\circ}{\tau}{}^{el} , \hat{\sigma}^{*} - \hat{\sigma} \,\right] ,$$

or

$$\left[\, \overset{\circ}{\hat{\sigma}}{}^{*} - \overset{\circ}{\hat{\sigma}} , \hat{\sigma}^{*} - \hat{\sigma} \,\right] \leqslant 0 ,$$

$$\frac{d}{d\,t}\left[\, \hat{\sigma}^{*} - \hat{\sigma} \,\right]^{2} \leqslant 0 .$$

The distance from $\hat{\sigma}^{*}$ to $\hat{\sigma}$ is a decreasing function of the time. For $t = 0$, the distance is 0 , and remains equal to 0 for all $t > 0$, and $\hat{\sigma}^{*} = \hat{\sigma}$. Modern formulations using an explicit scheme for solving (38) could be established (see the book of T. MIYOSHI / 5 /).

PART 2 : PIAN HYBRID MODEL

INTRODUCTION

The problem already presented in the first part will now be analized with the PIAN hybrid principle . This principle is indicated when an approximate solution satisfying the equilibrium equations is wanted. Its basic idea is to minimize the complementary energy on a special convex , where continuity conditions are not required on the stress vector : so , the use of Lagrangian multipliers is needed to relax these continuity conditions .

Here , we proceed in three steps : first , we develop the PIAN hybrid variational formulation ; we give some numerical aspects of the solution , and we propose a beam application .

VARIATIONAL FORMULATION

We try now to find a variational formulation / 3 / of the problem stated in the part 1 with the equations (1) to (16) .

Definition of the spaces U and Σ :

We define the space U of admissible displacements

$$U = \Big\{ \Delta \vec{v} = (\Delta v_i) : \ \varepsilon_{ij}(\Delta \vec{v}) \in L^2(\Omega^{(l)}) , \ 1 \leqslant l \leqslant L ,$$

$$\Delta \vec{v}^{(l+1)} = \Delta \vec{v}^{(l)} , \ l = 1 , \ldots , L-1 , \ \Delta \vec{v} = 0 \, / \Gamma_u \Big\} \quad (39)$$

and the space of admissible stresses

$$\Sigma = \Big\{ \Delta \tau = (\Delta \tau_{ij}) : \ \Delta \tau_{ij} = \Delta \tau_{ji} , \ \Delta \tau_{ij} \in L^2(\Omega^{(l)}) ,$$

$$\Delta \tau_{ij,j} \in L^2(\Omega^{(l)}) , \ \Delta \tau_{ij,j} = 0 \, / \Omega^{(l)} , \ 1 \leqslant l \leqslant L \Big\} . (40)$$

Mixed formulation

We consider the mixed hybrid dual principle of PIAN / 6 / written over $\Omega = \bigcup_{l=1}^{L} \Omega^{(l)}$

$$\mathcal{L}(\Delta\tau , \Delta\vec{v}) = \sum_{l=1}^{L} \Big(\int_{\partial\Omega^{(l)}} \Delta\tau_{ij} \, n_j \, \Delta v_i \, d\Gamma$$

$$- \frac{1}{2} \int_{\Omega^{(l)}} \Delta\tau_{ij} \, S_{ijkl} \, \Delta\tau_{kl} \, d\Omega \quad \ldots$$

$$- \frac{1}{2h} \int_{\Omega^{(l)}} \left\langle \frac{\partial f}{\partial \sigma_{pq}} \Delta \sigma_{pq} \right\rangle^2 d\Omega - \int_{\partial \Omega^{(l)} \in \Gamma_\sigma} \Delta T_i \, \Delta v_i \, d\Gamma \;) \; , \quad (41)$$

and we define

$$a\,(\Delta\sigma, \Delta\tau) = \sum_{l=1}^{L} \Big(\int_{\Omega^{(l)}} \Delta\sigma_{ij} \, S_{ijkl} \, \Delta\tau_{kl} \, d\Omega$$

$$+ \frac{1}{h} \int_{\Omega^{(l)}} \left\langle \frac{\partial f}{\partial \sigma_{pq}} \Delta\sigma_{pq} \right\rangle \frac{\partial f}{\partial \sigma_{ij}} \Delta\tau_{ij} \, d\Omega \Big) \; , \quad (42)$$

$$b\,(\Delta\tau, \Delta\vec{v}\,) = \sum_{l=1}^{L} \int_{\partial\Omega^{(l)}} \Delta\tau_{ij} \, n_j \, \Delta v_i \, d\Gamma \; , \quad (43)$$

$$L(\Delta\vec{v}\,) = \sum_{l=1}^{L} \int_{\partial\Omega^{(l)} \in \Gamma_\sigma} \Delta T_i^{(d)} \, \Delta v_i \, d\Gamma \; . \quad (44)$$

Using (2) and (4) , the form $a\,(\Delta\sigma, \Delta\tau)$ can also be written as

$$a\,(\Delta\varepsilon(\Delta\sigma)\,, \Delta\tau) = \sum_{l=1}^{L} \int_{\Omega^{(l)}} \Delta\varepsilon_{ij}(\Delta\sigma)\, \Delta\tau_{ij} \, d\Omega \; . \quad (45)$$

Properties of the functional

We have now the problem

$$\exists\,(\Delta\sigma_{ij}\,, \Delta\vec{u}\,) \in \Sigma \times U \; , \; \forall\; \Delta\tau_{ij} \in \Sigma \; , \; \forall\; \Delta\vec{v} \in U \; ,$$

$$a\,(\Delta\varepsilon(\Delta\sigma)\,, \Delta\tau) - b\,(\Delta\vec{u}\,, \Delta\tau) = 0 \; , \quad (46)$$

$$b\,(\Delta\sigma, \Delta\vec{v}\,) = L\,(\Delta\vec{v}\,) \; . \quad (47)$$

This problem is a saddle point of the PIAN functional (41) ; the BREZZI BABUSKA theorem shows that

$$\underset{\Delta\tau \in \Sigma}{\text{Max}}\; \mathcal{L}(\Delta\tau, \Delta\vec{u}\,) \leq \mathcal{L}\,(\Delta\sigma, \Delta\vec{u}\,) \leq \underset{\Delta\vec{v} \in U}{\text{Inf}}\; \mathcal{L}(\Delta\sigma, \Delta\vec{v}\,) \; . \quad (48)$$

ASPECTS OF THE NUMERICAL SOLUTION

Iterative scheme

In the following , we shall write

$$(\Delta\sigma\,, \Delta\tau) = \int_{\Omega} \Delta\sigma_{ij} \, \Delta\tau_{ij} \, d\Omega \; ,$$

and we adopt the following scheme , which is a gradient method primary suggested by MERCIER / 4 /

$$(\Delta\sigma^{n+1}, \Delta\tau) - \rho b\,(\Delta\vec{u}^{\,n+1}, \Delta\tau) = (\Delta\sigma^{n}, \Delta\tau) - \rho a\,(\Delta\mathcal{E}(\Delta\sigma^{n}), \Delta\tau) , \qquad (49)$$

$$b\,(\Delta\sigma^{n+1}, \Delta\vec{v}) = L\,(\Delta\vec{v}) . \qquad (50)$$

Proof of convergence

The exercise 9 shows that , as always exists such as

$$(\Delta\mathcal{E}(\Delta\sigma) - \Delta\mathcal{E}(\Delta\sigma^{n}), \Delta\sigma - \Delta\sigma^{n}) \geqslant \gamma\,(\Delta\sigma - \Delta\sigma^{n})^{2} , \qquad (51)$$

because the hardening parameter h is strictly > 0 , $\exists\, M > 0$

such that

$$(\Delta\mathcal{E}(\Delta\sigma) - \Delta\mathcal{E}(\Delta\sigma)^{n})^{2} \leqslant M\,(\Delta\sigma - \Delta\sigma^{n})^{2} . \qquad (52)$$

Then

$$(\Delta\sigma^{n+1} - \Delta\sigma)^{2} \leqslant (\Delta\sigma^{n} - \Delta\sigma)^{2} + (\rho^{2} M - 2\rho\gamma)\,(\Delta\sigma^{n} - \Delta\sigma)^{2} . \qquad (53)$$

To prove the convergence of the scheme (49,50) , ρ has to satisfy

$$(\rho^{2} M - 2\rho\gamma) < 0 , \text{ that is } 0 < \rho < \frac{2\gamma}{M} .$$

Modification of the algorithm

The scheme (49,50) is modified in order to simplify the equations to be computed : the term

$$(\Delta\sigma, \Delta\tau) = \sum_{l=1}^{L} \int_{\Omega^{(l)}} \Delta\sigma_{ij}\, \Delta\tau_{ij}\, d\Omega$$

is replaced by

$$[\Delta\sigma, \Delta\tau] = \sum_{l=1}^{L} \int_{\Omega^{(l)}} \Delta\sigma_{ij}\, S_{ijkl}\, \Delta\tau_{kl}\, d\Omega .$$

This can be done without changing the conditions of convergence because the two terms have both equivalent norms . The system to be solved is now

$$\exists\,(\Delta\sigma_{ij}^{n+1}, \Delta\vec{u}^{\,n+1}) \in \Sigma \times U , \; \forall\, \Delta\tau_{ij} \in \Sigma , \; \forall\, \Delta\vec{v} \in U ,$$

$$[\Delta\sigma^{n+1}, \Delta\tau] - \rho b\,(\Delta\vec{u}^{\,n+1}, \Delta\tau) = [\Delta\sigma^{n}, \Delta\tau] - \rho a\,(\Delta\mathcal{E}(\Delta\sigma^{n}), \Delta\tau) , \qquad (54)$$

$$b\,(\Delta\sigma^{n+1}, \Delta\vec{v}) = L\,(\Delta\vec{v}) . \qquad (55)$$

a) Elementary level : the approximation fields for the displacements and the stresses are chosen as :

$$\sigma = P(x_i)\,\beta \;, \text{ or } \quad \Delta\sigma = P\,\Delta\beta \;, \text{ and } \quad \Delta\tau = P\,\widetilde{\Delta\beta} \tag{56}$$

$$u = N(x_i)\,U_{nod} \;, \text{ or } \quad \Delta u = N\,\Delta U \;, \text{ and } \quad \Delta v = N\,\widetilde{\Delta U} \tag{57}$$

Each part of the equations (54) and (55) can be evaluated in terms of P , N , $\Delta\beta$, and ΔU . For one hybrid dual element $\Omega^{(1)}$

$$[\Delta\sigma^{n+1}, \Delta\tau] = \int_{\Omega^{(1)}} \Delta\tau_{ij}\, S_{ijkl}\, \Delta\sigma^{n+1}_{kl}\, d\Omega$$

$$= \widetilde{\Delta\beta}^{t} \left(\int_{\Omega^{(1)}} P^{t}\, S\, P\, d\Omega \right) \Delta\beta \;;$$

taking $H = \int_{\Omega^{(1)}} P^{t}\, S\, P\, d\Omega$, we get

$$[\Delta\sigma^{n+1}, \Delta\tau] = \widetilde{\Delta\beta}^{t}\, H\, \Delta\beta^{n} \;. \tag{58}$$

Similarly , we get

$$[\Delta\sigma^{n}, \Delta\tau] = \widetilde{\Delta\beta}^{t}\, H\, \Delta\beta^{n} \;.$$

$$b(\Delta\vec{u}^{n+1}, \Delta\tau) = \int_{\partial\Omega^{(1)}} \Delta\tau_{ij}\, n_j\, \Delta u_i\, d\Gamma \;;$$

using (57) and introducing $\Delta\tau_{ij}\, n_j = R\,\Delta\beta$, we get

$$b(\Delta\vec{u}^{n+1}, \Delta\tau) = \int_{\partial\Omega^{(1)}} \widetilde{\Delta\beta}^{t}\, R^{t}\, N\, \Delta U^{n+1}\, d\Gamma$$

$$= \widetilde{\Delta\beta}^{t} \left(\int_{\partial\Omega^{(1)}} R^{t}\, N\, d\Gamma \right) \Delta U^{n+1} \;;$$

using $T = \int_{\partial\Omega^{(1)}} R^{t}\, L\, d\Gamma$, we finally get

$$b(\Delta\vec{u}^{n+1}, \Delta\tau) = \widetilde{\Delta\beta}^{t}\, T\, \Delta U^{n+1} \;. \tag{59}$$

$$a(\Delta\varepsilon(\Delta\sigma^{n}), \Delta\tau) = \int_{\Omega^{(1)}} \Delta\varepsilon_{ij}(\Delta\sigma^{n})\, \Delta\tau_{ij}\, d\Omega \;;$$

using (4), we separate this contribution in two parts

$$a(\Delta\varepsilon(\Delta\sigma^n), \Delta\tau) = \int_{\Omega^{(1)}} \Delta\tau_{ij}\, S_{ijkl}\, \Delta\sigma^n_{kl}\, d\Omega$$

$$+ \frac{1}{h}\int_{\Omega^{(1)}} \Delta\tau_{ij} < \frac{\partial f}{\partial\sigma_{pq}} \Delta\sigma >_{pq} \frac{\partial f}{\partial\sigma_{ij}}\, d\Omega$$

$$= \tilde{\Delta\beta}^t H \Delta\beta^n + \tilde{\Delta\beta}^t \int_{\Omega^{(1)}} \frac{1}{h} P^t < \frac{\partial f}{\partial\sigma_{pq}}^n \Delta\sigma^n >_{pq} \frac{\partial f}{\partial\sigma_{ij}}^n\, d\Omega\,;$$

introducing

$$G^n = \frac{1}{h}\int_{\Omega^{(1)}} P^t < \frac{\partial f}{\partial\sigma_{pq}}^n \Delta\sigma^n >_{pq} \frac{\partial f}{\partial\sigma_{ij}}^n\, d\Omega\,, \tag{60}$$

we obtain

$$a(\Delta\varepsilon(\Delta\sigma^n), \Delta\tau) = \tilde{\Delta\beta}^t (H\Delta\beta^n + G^n)\,. \tag{61}$$

$$L(\Delta\vec{v}) = \int_{\partial\Omega^{(1)} \in \Gamma_\sigma} \Delta T_i^{(d)}\, \Delta v_i\, d\Gamma = \tilde{\Delta U}^t \Delta F\,. \tag{62}$$

Using (56 to 62) in (54) and (55) results in

$$\tilde{\Delta\beta}^t (H\Delta\beta^{n+1} - \rho T \Delta U^{n+1} = H\Delta\beta^n - \rho(H\Delta\beta^n + G^n))\,,$$

$$\tilde{\Delta U}^t (T^t \Delta\beta^{n+1} = \Delta F)$$

or

$$H\Delta\beta^{n+1} - \rho T \Delta U^{n+1} = (1-\rho) H \Delta\beta^n - \rho G^n\,, \tag{63}$$

$$T^t \Delta\beta^{n+1} = \Delta F\,. \tag{64}$$

b) Calculation of the stress parameters $\Delta\beta^{n+1}$: in each hybrid element, these unknowns can be evaluated independantly in terms of the nodal displacements of the element ; using (63)

$$H\Delta\beta^{n+1} = \rho T \Delta U^{n+1} + H\Delta\beta^n - \rho H \Delta\beta^n - \rho G^n\,,$$

we get consequently

$$\Delta\beta^{n+1} = \rho H^{-1} T \Delta U^{n+1} + (1-\rho)\Delta\beta^n - \rho H^{-1} G^n\,. \tag{65}$$

c) Equivalent stiffness matrix K : using (65) in (64) results in

$$ {}^{t}T\,\Delta\beta^{n+1} = \Delta F = \rho\,{}^{t}T\,H^{-1}\,T\,\Delta U^{n+1} + (1-\rho)\,{}^{t}T\,\Delta\beta^{n} - \rho\,{}^{t}T\,H^{-1}\,G^{n} ; $$

as ${}^{t}T\,H^{-1}\,T$ is equivalent / 7 / to the initial elastic stiffness matrix K of the element , we write

$$ \Delta F = \rho\,K\,\Delta U^{n+1} + (1-\rho)\,{}^{t}T\,\Delta\beta^{n} - \rho\,{}^{t}T\,H^{-1}\,G^{n} . $$

d) Computations for one iteration : writing $(A) = \sum_{l=1}^{L} A$, after the n^{th} iteration inside the increment ΔF of the external loads , we suppose to know the values of

$$ (G^{n}) = \sum_{l=1}^{L} G^{n} \quad \text{by (60)} , $$

$$ (\Delta\beta^{n}) = \sum_{l=1}^{L} \Delta\beta^{n} \quad \text{by (65)} , $$

and (ΔU^{n}) .

We begin the $(n+1)^{th}$ iteration by solving the system

$$ (K)(\Delta U^{n+1}) = \frac{1}{\rho} (\Delta F) + \frac{\rho - 1}{\rho} ({}^{t}T)(\Delta\beta^{n}) + ({}^{t}T)(H^{-1})(G^{n}) . \tag{66} $$

The matrices (K) , (T) , and (H) contain the initial values of the elastic stiffnesses for (K) , of the boundary terms for (T) , and of the compliances for (H) ; only the terms of the plastic contribution (G^{n}) have to be evaluated at each step of the computation , because they depend on the evolution of the plasticity at each integration point . Once (ΔU^{n+1}) is known for the whole mesh , it is easy to deduce ΔU^{n+1} for each element and then to calculate the stress parameters $\Delta\beta^{n+1}$, according to (65) ; we get the complete stress state by using

$$ \sigma^{n+1} = \sigma^{n} + P\,\Delta\beta^{n+1} \tag{67} $$

at each integration point ; there , we evaluate the new values of G^{n+1} in the element , and finally in the whole mesh . At this stage we are able to begin a new iteration provided that the norm (68) of the stress parameters is not small enough compared with a given tolerance . So , the iterations inside the increment ΔF are stopped if , M being the total number of integration points

$$\frac{\sqrt{\sum_{m=1}^{M} (\Delta\beta^{n+1} - \Delta\beta^{n})^2}}{\sqrt{\sum_{m=1}^{M} (\beta^{n})^2}} \times 100 \leq \text{Tolerance in } \% \; . \qquad (68)$$

EXAMPLE

Geometry , load and mechanical characteristics

We propose to compare the numerical results issued from different finite element methods on the example of a clamped beam , loaded by a concentrated force at the end . The geometry and the mechanical characteristics are given below ; the mesh used in all the calculations is made of 4 X 10 finite elements (Fig. 3) .

YOUNG elastic modulus : E = 210000 MPa

POISSON's ratio : ν = 0 . 25

Uniaxial yield stress : σ_o = 240 MPa

Strain hardening parameter : h = 80000 MPa

The TRESCA plasticity criterion is used in all cases , and the tolerance admitted for the norm of the stress parameters β is 1 % .

The elements and algorithms used

In the first case , four noded displacement finite elements are used in the mesh ; in the second case , eight noded displacement elements are utilized ; in both cases , the initial stress method of ZIENKIEWICZ gives the algorithm of plasticity . In the third case , we choose hybrid four noded elements with linear interpolations for the displacements and the stresses , and the algorithm developed above for the plasticity in the hybrid method is taken into account . We have for exemple , for P in the equation (56)

$$\sigma_{xx} = \beta_1 + \beta_4 x$$

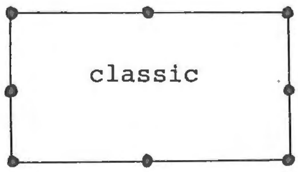
$$\sigma_{yy} = \beta_2 + \beta_5 y$$

$$\sigma_{xy} = \beta_3$$

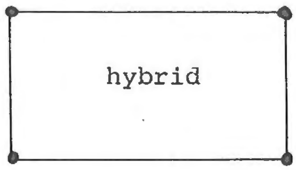
and $$P = \begin{bmatrix} 1 & 0 & 0 & x & 0 \\ 0 & 1 & 0 & 0 & y \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

case 1	case 2	case 3

four noded
displacement
element



eight noded
displacement
element



four noded
hybrid dual
element

Results in the elastic phase F_{init}

F_{init} = 410 N ; we give below the values of the deflection f_{max} and of the tensile stress at the GAUSS point indicated on the figure .

case 1	case 2	case 3
f_{max} = 0.05581 cm	f_{max} = 0.07849 cm	f_{max} = 0.07809 cm
σ_{xx} = 161.1 MPa	σ_{xx} = 235.2 MPa	σ_{xx} = 221.3 MPa
four noded displacement element	eight noded displacement element	four noded hybrid dual element

Plastic results for $F + \Delta F = 1.35\ F_{init}$

No plasticity occurs in the case 1 where the mesh is made of 4 noded classic elements with linear interpolations for the displacements ; of course , this element is too stiff for the analysis of bending : this numerical behavior is well known and need no further comment . In the two other cases , the results are quite similar with slight differences for the hybrid method , depending on the value fixed for the parameter ρ .

a) Results obtained with $\rho = 0.45$ (hybrid method)

case 1	case 2	case 3
$f_{max} = 0.07535$ cm	$f_{max} = 0.10945$ cm	$f_{max} = 0.1095$ cm
$\sigma_{xx} = 216.1$ MPa	$\sigma_{xx} = 289.1$ MPa	$\sigma_{xx} = 268.1$ MPa
1 iteration	7 iterations	4 iterations
four noded displacement element	eight noded displacement element	four noded hybrid dual element

b) Evolution of the results with ρ

$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$
$f_{max} = 0.11006$ cm	$f_{max} = 0.10983$ cm	$f_{max} = 0.1096$ cm
$\sigma_{xx} = 271$ MPa	$\sigma_{xx} = 269.2$ MPa	$\sigma_{xx} = 268.2$ MPa
11 iterations	7 iterations	6 iterations
$\rho = 0.4$	$\rho = 0.45$	$\rho = 0.5$
$f_{max} = 0.10952$ cm	$f_{max} = 0.1095$ cm	$f_{max} = 0.1095$ cm
$\sigma_{xx} = 268$ MPa	$\sigma_{xx} = 268.1$ MPa	$\sigma_{xx} = 267.9$ MPa
5 iterations	4 iterations	5 iterations

For $\rho = 0.6$, divergence appears after 2 iterations ; if we approximate the value of γ in the equation (51) by a mean value which would be the compliance 1 / E of the material , and if we choose the mean value of M in the equation (52) to be the equivalent elastic plastic compliance , the sufficient condition of convergence of the scheme proposed for the hybrid dual method is realized for $0 < \rho < \frac{2}{M} \simeq 0.55$. This condition is coherent with the numerical results obtained .

CONCLUSION

The optimization of the choice of the parameter ρ is a problem not completely solved yet . But it seems that a mean value of the auxiliary parameter ρ can be evaluated for given elastic modulus E and (here) constant hardening factor h . This hybrid method gives, with a hybrid four noded element and linear interpolations , better convergence than an initial stress method , with a classic eight noded element and quadratic kinematic interpolations .

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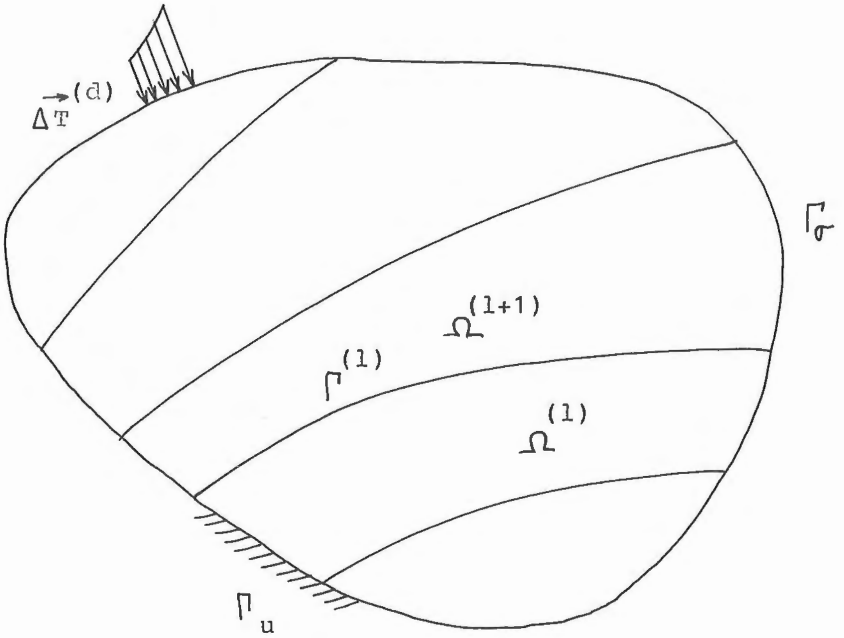

Figure 1 . Body composed of L solids

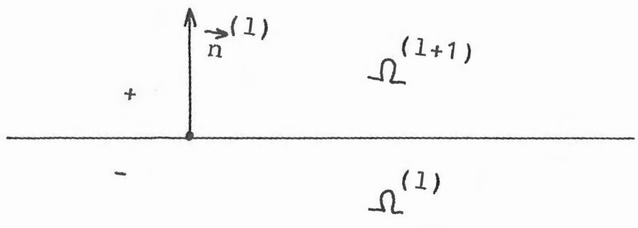

Figure 2 . Interface $\Gamma^{(l)}$

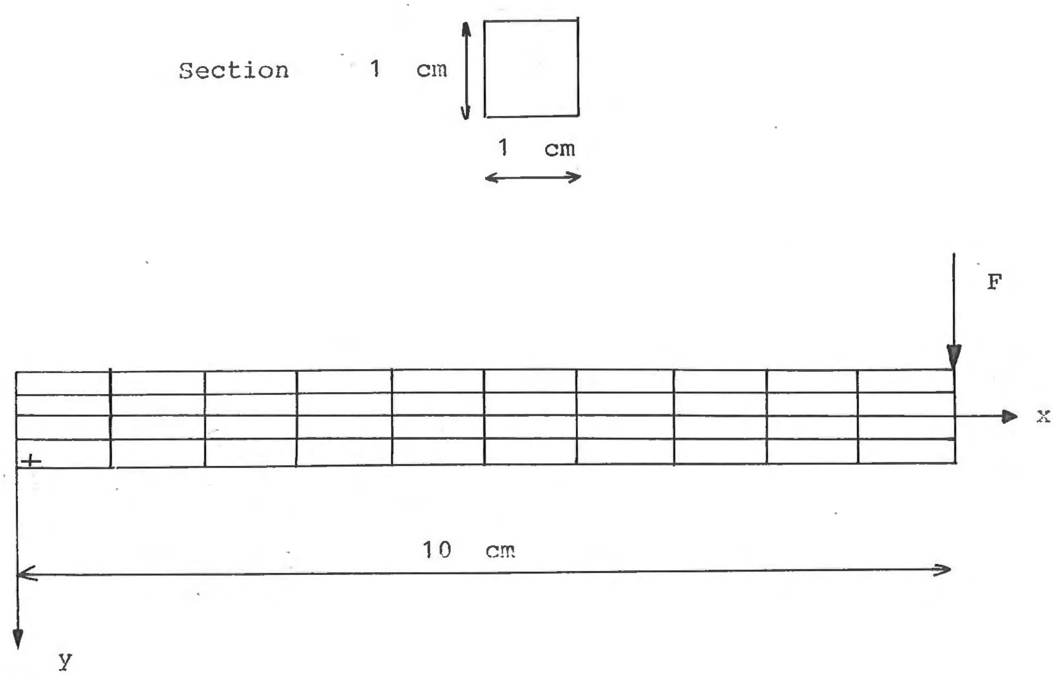

Figure 3 . Cantilever beam

APPENDIX

Exercise 1

Show that, with the notation $\begin{cases} \langle x \rangle = x & x \geqslant 0 , \\ \langle x \rangle = 0 & x < 0 , \end{cases}$

$$\Delta \varepsilon_{ij} = S_{ijkl} \, \Delta\sigma_{kl} + \Delta\lambda \frac{\partial f}{\partial \sigma_{kl}} \tag{1}$$

with

$$h = \frac{\partial f}{\partial A_{ij}} \frac{\partial f}{\partial A_{rs}} \frac{\partial A_{ij}}{\partial \alpha_{rs}} ,$$

is equivalent to

$$\Delta\sigma_{ij} = D_{ijkl} \left(\Delta\varepsilon_{kl} - \frac{\partial f}{\partial \sigma_{kl}} \frac{\left\langle \frac{\partial f}{\partial \sigma_{pq}} D_{pqkl} \, \Delta\varepsilon_{kl} \right\rangle}{h + \frac{\partial f}{\partial \sigma_{ab}} D_{abcd} \frac{\partial f}{\partial \sigma_{cd}}} \right) . \tag{2}$$

Proof

1°) The behavior law indicates that

$$\Delta f = \frac{\partial f}{\partial \sigma_{ij}} \Delta\sigma_{ij} + \frac{\partial f}{\partial A_{pq}} \Delta A_{pq} = 0 .$$

But

$$\Delta A_{pq} = \frac{\partial A_{pq}}{\partial \alpha_{ij}} \Delta\alpha_{ij} ;$$

and introducing

$$\Delta\alpha_{ij} = - \Delta\lambda \frac{\partial f}{\partial A_{ij}} ,$$

we get with the equation for h

$$\frac{\partial f}{\partial \sigma_{ij}} \Delta\sigma_{ij} = \Delta\lambda \frac{\partial A_{pq}}{\partial \alpha_{ij}} \frac{\partial f}{\partial A_{ij}} \frac{\partial f}{\partial A_{pq}} = h \, \Delta\lambda . \tag{3}$$

Equation (1) can be written in the form

$$S_{ijkl}\, \Delta\sigma_{kl} = \Delta\varepsilon_{ij} - \Delta\lambda \frac{\partial f}{\partial \sigma_{ij}} .$$

Multiplying by the stiffness D_{ijkl}, we obtain

$$\Delta\sigma_{ij} = D_{ijkl} \left(\Delta\varepsilon_{kl} - \Delta\lambda \frac{\partial f}{\partial \sigma_{kl}} \right) . \tag{4}$$

Substituting this equation in (3) results in

$$\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \left(\Delta\varepsilon_{kl} - \Delta\lambda \frac{\partial f}{\partial \sigma_{kl}} \right) - h\, \Delta\lambda = 0 .$$

2°) We get the increment $\Delta\lambda$ as a function of the strain increment

$$\Delta\lambda(\Delta\varepsilon) = \frac{\left\langle \dfrac{\partial f}{\partial \sigma_{ij}} D_{ijkl}\, \Delta\varepsilon_{kl} \right\rangle}{\dfrac{\partial f}{\partial \sigma_{pq}} D_{pqrs} \dfrac{\partial f}{\partial \sigma_{rs}} + h} .$$

Now rewritting $\Delta\sigma_{ij}$ in (4), we get the equation (2)

$$\Delta\sigma_{ij} = D_{ijkl} \left(\Delta\varepsilon_{kl} - \frac{\partial f}{\partial \sigma_{kl}} \frac{\left\langle \dfrac{\partial f}{\partial \sigma_{pq}} D_{pqkl}\, \Delta\varepsilon_{kl} \right\rangle}{h + \dfrac{\partial f}{\partial \sigma_{ab}} D_{abcd} \dfrac{\partial f}{\partial \sigma_{cd}}} \right) . \tag{2}$$

3°) Conversely, $\Delta\lambda$ could be written as a function of the increment $\Delta\sigma$. We have from (3)

$$\frac{\partial f}{\partial \sigma_{ij}} \Delta\sigma_{ij} = \Delta\lambda \frac{\partial f}{\partial A_{ij}} \frac{\partial f}{\partial A_{pq}} \frac{\partial A_{pq}}{\partial \alpha_{ij}} = \Delta\Lambda\, h ,$$

and

$$\Delta\lambda = \frac{1}{h} \left\langle \frac{\partial f}{\partial \sigma_{ij}} \Delta\sigma_{ij} \right\rangle .$$

Then (1) results in

$$\Delta\varepsilon_{ij} = S_{ijkl}\, \Delta\sigma_{kl} + \frac{1}{h} \left\langle \frac{\partial f}{\partial \sigma_{pq}} \Delta\sigma_{pq} \right\rangle \frac{\partial f}{\partial \sigma_{ij}} .$$

Exercise 2

We get two increments of stresses $\Delta\sigma_{ij}$ and $\Delta\sigma^{*}_{ij}$, with the associate increments of strains $\Delta\varepsilon_{ij}$ and $\Delta\varepsilon^{*}_{ij}$, corresponding to the standard elastoplastic behavior law . Show if $h > 0$, that the equations (1) and (2) are true

$$(\Delta\sigma_{ij} - \Delta\sigma^{*}_{ij})(\Delta\varepsilon^{(p)}_{ij} - \Delta\varepsilon^{(p)*}_{ij}) \geqslant 0 , \tag{1}$$

$$(\Delta\sigma_{ij} - \Delta\sigma^{*}_{ij})(\Delta\varepsilon_{ij} - \Delta\varepsilon^{*}_{ij}) \geqslant S_{ijkl}(\Delta\sigma_{ij} - \Delta\sigma^{*}_{ij})(\Delta\sigma_{kl} - \Delta\sigma^{*}_{kl}) . \tag{2}$$

$\Delta\varepsilon^{(p)}_{ij}$ and $\Delta\varepsilon^{(p)*}_{ij}$ are the plastic parts of the increments of strain $\Delta\varepsilon_{ij}$ and $\Delta\varepsilon^{*}_{ij}$ respectively .

Proof

1°) We introduce δ as

$$\delta = (\Delta\sigma_{ij} - \Delta\sigma^{*}_{ij})(\Delta\lambda - \Delta\lambda^{*})\frac{\partial f}{\partial\sigma_{ij}}$$

$$= (\Delta\sigma_{ij} - \Delta\sigma^{*}_{ij})(\Delta\varepsilon^{(p)}_{ij} - \Delta\varepsilon^{(p)*}_{ij}) .$$

With

$$\Delta\lambda = \frac{1}{h}\left\langle \frac{\partial f}{\partial\sigma_{ij}}\Delta\sigma_{ij}\right\rangle ,$$

and

$$\Delta\lambda^{*} = \frac{1}{h}\left\langle \frac{\partial f}{\partial\sigma_{ij}}\Delta\sigma^{*}_{ij}\right\rangle ,$$

we get

$$\delta = \frac{1}{h}(\Delta\sigma_{ij} - \Delta\sigma^{*}_{ij})\left(\left\langle\frac{\partial f}{\partial\sigma_{pq}}\Delta\sigma_{pq}\right\rangle - \left\langle\frac{\partial f}{\partial\sigma_{pq}}\Delta\sigma^{*}_{pq}\right\rangle\right)\frac{\partial f}{\partial\sigma_{ij}} .$$

δ is a scalar product written for the vectors $\dfrac{\partial f}{\partial\sigma_{ij}}$ and $(\Delta\sigma_{ij} - \Delta\sigma^{*}_{ij})$.

We can examine the different possibilities for the signs of the quantities $\frac{\partial f}{\partial \sigma_{ij}} \Delta\sigma$, and $\frac{\partial f}{\partial \sigma_{ik}} \Delta\sigma^*$; in that way, we can show that $\delta > 0$.

2°) We calculate the term Δ_1

$$\Delta_1 = \Delta\sigma_{ij}\,\Delta\varepsilon_{ij} + \Delta\sigma^*_{ij}\,\Delta\varepsilon^*_{ij} - 2\,\Delta\varepsilon_{ij}\,\Delta\sigma^*_{ij}\ ,$$

with

$$\Delta\varepsilon_{ij} = s_{ijkl}\,\Delta\sigma_{kl} + \Delta\lambda\left(\frac{\partial f}{\partial \sigma_{ij}}\right),$$

and

$$\Delta\varepsilon^*_{ij} = s_{ijkl}\,\Delta\sigma^*_{kl} + \Delta\lambda^*\left(\frac{\partial f}{\partial \sigma_{ij}}\right).$$

We get

$$\begin{aligned}\Delta_1 &= \Delta\sigma_{ij}\,s_{ijkl}\,\Delta\sigma_{kl} + \Delta\sigma^*_{ij}\,s_{ijkl}\,\Delta\sigma^*_{kl} \\ &\quad + \Delta\lambda\frac{\partial f}{\partial \sigma_{ij}}\,\Delta\sigma_{ij} + \Delta\lambda^*\frac{\partial f}{\partial \sigma_{ij}}\,\Delta\sigma^*_{ij} \\ &\quad - 2\,s_{ijkl}\,\Delta\sigma^*_{ij}\,\Delta\sigma_{kl} - 2\,\Delta\lambda\frac{\partial f}{\partial \sigma_{ij}}\,\Delta\sigma^*_{ij}\end{aligned}$$

$$\begin{aligned}&= (\Delta\sigma_{ij} - \Delta\sigma^*_{ij})\,s_{ijkl}\,(\Delta\sigma_{kl} - \Delta\sigma^*_{kl}) \\ &\quad + \frac{1}{h}\Big(\frac{\partial f}{\partial \sigma_{ij}}\Delta\sigma_{ij}\frac{\partial f}{\partial \sigma_{pq}}\Delta\sigma_{pq} + \frac{\partial f}{\partial \sigma_{ij}}\Delta\sigma^*_{ij}\frac{\partial f}{\partial \sigma_{rs}}\Delta\sigma^*_{rs} \\ &\quad - 2\frac{\partial f}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{rs}}\Delta\sigma_{ij}\,\Delta\sigma^*_{rs}\Big)\end{aligned}$$

$$\begin{aligned}&= (\Delta\sigma_{ij} - \Delta\sigma^*_{ij})\,s_{ijkl}\,(\Delta\sigma_{kl} - \Delta\sigma^*_{kl}) \\ &\quad + \frac{1}{h}\,(\Delta\sigma_{ij} - \Delta\sigma^*_{ij})\,\frac{\partial f}{\partial \sigma_{ij}}\,\frac{\partial f}{\partial \sigma_{pq}}\,(\Delta\sigma_{pq} - \Delta\sigma^*_{pq})\ .\end{aligned}$$

So, we get, for Δ_1

$$\Delta_1 \geqslant S_{ijkl} \, (\Delta\sigma_{ij} - \Delta\sigma^*_{ij})\,(\Delta\sigma_{kl} - \Delta\sigma^*_{kl}) \; .$$

Using the same method, we get also

$$\Delta_2 \geqslant S_{ijkl} \, (\Delta\sigma_{ij} - \Delta\sigma^*_{ij})\,(\Delta\sigma_{kl} - \Delta\sigma^*_{kl})$$

with

$$\Delta_2 = \Delta\sigma_{ij}\,\Delta\varepsilon_{ij} + \Delta\sigma^*_{ij}\,\Delta\varepsilon^*_{ij} - 2\,\Delta\sigma_{ij}\,\Delta\varepsilon^*_{ij} \; .$$

Adding Δ_1 and Δ_2, we get

$$\Delta_1 + \Delta_2 = 2\,(\Delta\sigma_{ij}\,\Delta\varepsilon_{ij} + \Delta\sigma^*_{ij}\,\Delta\varepsilon^*_{ij} - \Delta\sigma_{ij}\,\Delta\varepsilon^*_{ij} - \Delta\sigma^*_{ij}\,\Delta\varepsilon_{ij})$$

$$= 2\,(\Delta\sigma_{ij} - \Delta\sigma^*_{ij})\,(\Delta\varepsilon_{ij} - \Delta\varepsilon^*_{ij})$$

$$\geqslant 2\,S_{ijkl}\,(\Delta\sigma_{ij} - \Delta\sigma^*_{ij})\,(\Delta\sigma_{kl} - \Delta\sigma^*_{kl}) \; .$$

Exercise 3

Show that

$$\Delta\sigma_{ij}\,\Delta\varepsilon_{ij} \geqslant h\,D_{ijkl}\,\Delta\varepsilon_{ij}\,\Delta\varepsilon_{kl}\;\frac{1}{h + \dfrac{\partial f}{\partial\sigma_{ab}}\,D_{abcd}\,\dfrac{\partial f}{\partial\sigma_{cd}}} \; .$$

Proof

We know, from exercise 1, that

$$\Delta\sigma_{ij} = D_{ijkl}\,\left(\Delta\varepsilon_{kl} - \frac{\partial f}{\partial\sigma_{kl}}\;\frac{\left\langle \dfrac{\partial f}{\partial\sigma_{pq}}\,D_{pqrs}\,\Delta\varepsilon_{rs} \right\rangle}{h + \dfrac{\partial f}{\partial\sigma_{ab}}\,D_{abcd}\,\dfrac{\partial f}{\partial\sigma_{cd}}}\right) \; .$$

We note β the quantity $\left(h + \dfrac{\partial f}{\partial\sigma_{ab}}\,D_{abcd}\,\dfrac{\partial f}{\partial\sigma_{cd}}\right)$, and we calculate the product

$$\beta \, \Delta\varepsilon_{ij} \, \Delta\sigma_{ij} = \Delta\varepsilon_{ij} \, D_{ijkl} \, \Delta\varepsilon_{kl} \left(h + \frac{\partial f}{\partial \sigma_{ab}} D_{abcd} \frac{\partial f}{\partial \sigma_{cd}} \right)$$

$$- D_{ijkl} \, \Delta\varepsilon_{ij} \frac{\partial f}{\partial \sigma_{kl}} \left\langle \frac{\partial f}{\partial \sigma_{pq}} D_{pqrs} \, \Delta\varepsilon_{rs} \right\rangle .$$

$$\Delta\varepsilon_{ij} \, \Delta\sigma_{ij} = \frac{1}{\beta} \left(h \, \Delta\varepsilon_{ij} \, D_{ijkl} \, \Delta\varepsilon_{kl} \right)$$

$$+ \frac{1}{\beta} \Delta\varepsilon_{ij} \, D_{ijkl} \left(\Delta\varepsilon_{kl} \frac{f}{\partial \sigma_{ab}} D_{abcd} \frac{\partial f}{\partial \sigma_{cd}} - \frac{f}{\partial \sigma_{kl}} \left\langle \frac{\partial f}{\partial \sigma_{pq}} D_{pqrs} \, \Delta\varepsilon_{rs} \right\rangle \right).$$

Noting that

$$\Delta\varepsilon_{ij} \, D_{ijkl} \, \Delta\varepsilon_{kl} = \| \vec{X} \|^2 ,$$

and

$$\| \vec{Y} \|^2 = \frac{\partial f}{\partial \sigma_{ab}} D_{abcd} \frac{\partial f}{\partial \sigma_{cd}} ,$$

and using

$$\| \vec{X} \|^2 \, \| \vec{Y} \|^2 \geqslant | \vec{X} \cdot \vec{Y} |^2 ,$$

we get what we want .

Exercise 4

Show that $\mathcal{B}(\Delta\varepsilon(\vec{u})) = \inf\limits_{\Delta\vec{v} \in V} \mathcal{B}(\Delta\varepsilon(\vec{v}))$

Proof

We note G the operator which associates $\Delta\sigma_{ij}$ to $\Delta\vec{v} \in V$

$$G(\Delta\vec{v}) = \Delta\sigma_{ij}(\Delta\vec{v})$$

$$= D_{ijkl} \left(\varepsilon_{kl}(\Delta\vec{v}) - \frac{\partial f}{\partial \sigma_{kl}} \frac{\left\langle \frac{\partial f}{\partial \sigma_{pq}} D_{pqrs} \, \varepsilon_{rs}(\Delta\vec{v}) \right\rangle}{h + \frac{\partial f}{\partial \sigma_{ab}} D_{abcd} \frac{\partial f}{\partial \sigma_{cd}}} \right).$$

$G(\Delta\vec{v})$ satisfies (1), (2) and (3)

$$\lim_{\|\Delta\vec{v}\| \to +\infty} \frac{(G(\Delta\vec{v}), \mathcal{E}(\Delta\vec{v}))}{\|\mathcal{E}(\Delta\vec{v})\|_V} \longrightarrow +\infty \; ; \qquad (1)$$

Remember that

$$(G(\Delta\vec{v}), \mathcal{E}(\Delta\vec{v})) = \int_\Omega \Delta\sigma_{ij}(\Delta\vec{v})\, \mathcal{E}_{ij}(\Delta\vec{v})\, d\Omega \; ;$$

$$(G(\Delta\vec{v}) - G(\Delta\vec{u}), \mathcal{E}(\Delta\vec{v}) - \mathcal{E}(\Delta\vec{u})) \geqslant 0 \,, \qquad (2)$$

$\forall\, \Delta\vec{v} \in V$, and $\forall\, \Delta\vec{u} \in V$;

if $\mathcal{E}(\Delta\vec{v})$ is bounded on V, then $G(\Delta\vec{v})$ is bounded on V.(3)

The operator G is hemicontinuous : $\exists\, \Delta\vec{u}$,

$$\mathcal{B}(\mathcal{E}(\Delta\vec{u})) = \operatorname*{Inf}_{\Delta\vec{v} \in V} \mathcal{B}(\mathcal{E}(\Delta\vec{v})) \,.$$

Remember that

$$a(\Delta\vec{u}, \Delta\vec{v}) = \int_\Omega \mathcal{E}_{ij}(\Delta\vec{u})\, D_{ijkl}\, \mathcal{E}_{kl}(\Delta\vec{v})\, d\Omega \; ;$$

The functional $\mathcal{B}(\mathcal{E}(\Delta\vec{v}))$ is lower semicontinuous because it is the convex envelope of a continuous functional defined on V ; we get

$$\lim_{\|\Delta\vec{v}\|_V \to +\infty} \mathcal{B}(\mathcal{E}(\Delta\vec{v})) \longrightarrow +\infty \,,$$

and

$$\forall\, \Delta\vec{v} \in V, \int_\Omega \sigma_{ij}(\Delta\vec{u})\,(\mathcal{E}_{ij}(\Delta\vec{v}) - \mathcal{E}_{ij}(\Delta\vec{u}))\, d\Omega \geqslant 0 \,.$$

Exercise 5

Show that $\mathcal{R}(\Delta\sigma(\vec{u})) \leq \operatorname*{Inf}_{\Delta\tau \in \Sigma_{ad}} \mathcal{R}(\Delta\tau)$,

remembering that

$$\Sigma_{ad} = \Big\{ \Delta\tau_{ij}\ ,\ \Delta\tau_{ij,j} = 0 \text{ on } \Omega^{(l)}\ ,\ 1 \leqslant l \leqslant L\ ,\ \Delta\tau \in L^2(\Omega^{(l)})$$

$$\Delta\tau^{(l)}_{ij}\, n^{(l)}_j = -\,\Delta\tau^{(l+1)}_{ij}\, n^{(l+1)}_j\ ,\ 1 \leqslant l \leqslant L-1 \quad \text{on } \Gamma^{(l)}\ ,$$

$$\Delta\tau_{ij}\, n_j = \Delta T^{(d)}_i \quad \text{on } \Gamma_\sigma \Big\}\ .$$

Proof

It is possible to inverse the behavior law ; so the operator G^{-1} can be defined . The operator G^{-1} satisfies (1), (2) and (3)

$$\lim_{\|\Delta\tau\| \to +\infty} \frac{(G^{-1}(\Delta\tau)\ ,\Delta\tau)}{\|\Delta\tau\|_{\Sigma_{ad}}} \longrightarrow +\infty\ , \tag{1}$$

where

$$(G^{-1}(\Delta\tau)\ ,\Delta\tau) = \int_\Omega \Delta\varepsilon_{ij}(\Delta\tau)\,\Delta\tau_{ij}\, d\Omega\ ,$$

$$(G^{-1}(\Delta\tau)\ ,\Delta\tau) = \int_\Omega \Delta\tau_{ij}\, S_{ijkl}\, \Delta\tau_{kl}\, d\Omega$$

$$+ \frac{1}{h}\int_\Omega \Big\langle \frac{\partial f}{\partial \sigma_{ij}}\, \Delta\tau_{ij} \Big\rangle \frac{\partial f}{\partial \sigma_{pq}}\, \Delta\tau_{pq}\, d\Omega\ .$$

We have

$$(G^{-1}(\Delta\sigma) - G^{-1}(\Delta\tau)\ ,\Delta\sigma - \Delta\tau) \geqslant 0\ , \tag{2}$$

$$\forall\ (\Delta\sigma,\Delta\tau) \in \Sigma_{ad} \times \Sigma_{ad}$$

If $\Delta\tau$ is bounded on Σ_{ad} , then $G^{-1}(\Delta\tau)$ is bounded on V. (3)

$$\lim_{\|\Delta\tau\|_{\Sigma_{ad}} \to +\infty} \frac{\mathcal{B}(\Delta\tau)}{\|\Delta\tau\|} \longrightarrow +\infty\ ,$$

$$\exists\ \Delta\sigma \in \Sigma_{ad}\ ,\ \mathcal{B}(\Delta\sigma) \leqq \inf_{\Delta\tau \in \Sigma_{ad}} \mathcal{B}(\Delta\tau)\ ,$$

or

$$\forall\ \Delta\tau \in \Sigma_{ad}\ ,\ \int_\Omega \Delta\varepsilon_{ij}(\Delta\sigma)\,(\Delta\tau_{ij} - \Delta\sigma_{ij})\,d\Omega \geqslant 0 \quad .$$

Note : in the elastic case , we have

$$K\,(\Delta\tau) = 1/2 \int_\Omega \Delta\tau_{ij}\ s_{ijkl}\ \Delta\tau_{kl}\ d\Omega \quad .$$

We take

$$\Delta\varepsilon_{ij}(\vec{u}) = s_{ijkl}\,\Delta\sigma_{kl} \quad .$$

We get

$$K\,(\Delta\sigma) = \operatorname*{Inf}_{\Delta\tau \in \Sigma_{ad}} K\,(\Delta\tau) \quad .$$

We calculate

$$K\,(\Delta\sigma + \Delta\tau) - K\,(\Delta\sigma)$$

$$= \int_\Omega \Delta\varepsilon_{ij}(\vec{u})\,\Delta\tau_{ij}\,d\Omega + 1/2 \int_\Omega \Delta\tau_{ij}\ s_{ijkl}\ \Delta\tau_{kl}\ d\Omega \quad .$$

But

$$\int_\Omega \Delta\varepsilon_{ij}(\vec{u})\,\Delta\tau_{ij}\,d\Omega = \sum_{l=1}^{L} \left(\int_{\partial\Omega^{(l)}} \Delta\tau_{ij}^{(l)}\,n_j^{(l)}\,\Delta u_i\,d\Gamma - \int_{\Omega^{(l)}} \Delta\tau_{ij,j}\,\Delta u_i\,d\Omega \right).$$

Because $\Delta\tau_{ij} \in \Sigma_{ad}$, the last term is equal to 0 ; taking into account the continuity conditions on the interfaces , and the boundary conditions , we get

$$\int_\Omega \Delta\varepsilon_{ij}(\vec{u})\,\Delta\tau_{ij}\,d\Omega = \int_{\Gamma_\sigma} \Delta T_i^{(d)}\,\Delta u_i\,d\Gamma \quad .$$

Exercise 6

Show that

$$\operatorname*{Max}_{\Delta\vec{v} \in V} \mathcal{L}(\Delta\sigma, \Delta\vec{v}) \leq \mathcal{L}(\Delta\sigma, \Delta\vec{u}) \leq \operatorname*{Min}_{\Delta\tau \in \Sigma} \mathcal{L}(\Delta\tau, \Delta\vec{u}) \quad ,$$

remembering that

$$V = \left\{ \varepsilon_{ij} (\Delta \vec{v}) \in L^2(\Omega^{(l)}) ,\ 1 \leq l \leq L ,\ \Delta \vec{v}^{(l+1)} = \Delta \vec{v}^{(l)} ,\right.$$
$$\left. l = 1 , \ldots , L-1 ;\ \Delta \vec{v} = 0 \text{ on } \Gamma_u \right\}$$

and

$$\Sigma = \left\{ \Delta\tau : \Delta\tau_{ij} = \Delta\tau_{ji} ,\ \Delta\tau_{ij} \in L^2 (\Omega) \right\} .$$

Proof

The forms $a(\Delta\sigma,\Delta\tau)$ and $b(\Delta\tau,\Delta\vec{v})$ are bicontinuous on $\Sigma \times \Sigma$ and $\Sigma \times V$ respectively . We have the inequality

$$\Delta\tau_{ij}\ \Delta\varepsilon_{ij}(\Delta\tau) \geqslant S_{ijkl}\ \Delta\tau_{ij}\ \Delta\tau_{kl} .$$

We define

$$Z = \left\{ \Delta\tau \in \Sigma ,\ \forall \Delta\vec{v} \in V ,\ b(\Delta\tau,\Delta\vec{v}) = 0 \right\} .$$

Using the inequality above , we get that $b(\Delta\tau,\Delta\vec{v})$ is Z-elliptic . $L(\Delta\vec{v})$ is continuous on V ; by KORN inequality ,

$$\exists\ C ,\ \sup_{\Delta\tau \in \Sigma} \frac{b(\Delta\tau,\Delta\vec{v})}{\left\|\Delta\tau\right\|_{\Sigma}} \geqslant C \left\|\Delta\vec{v}\right\|_V$$

and we get what we want .

Exercise 7

Show the convergence of the scheme described in (26) .

Proof

The equations of (26) can be written as

$$\rho\left[\Delta\sigma^{n+1},\Delta\tau\right] = \rho\left[\Delta\sigma(\Delta\vec{u}^{n}),\Delta\tau\right] + \left(\varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u}^{n}),\Delta\tau\right), \quad (1)$$

$$(\Delta\sigma^{n+1},\varepsilon(\Delta\vec{v})) = L(\Delta\vec{v}) . \quad (2)$$

We choose

$$\Delta\tau_{ij} = D_{ijkl}\,\Delta\varepsilon_{kl}(\vec{v}) \text{ in } (1),$$

and we get

$$\rho(\Delta\sigma^{n+1}, \Delta\varepsilon(\vec{v})) = \rho(\Delta\sigma(\Delta\vec{u}^{n}), \Delta\varepsilon(\vec{v})) + \left\{ \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u}^{n}), \Delta\varepsilon(\vec{v}) \right\}. \quad (3)$$

But , at the end of the iterations , we shall have

$$(\Delta\sigma, \Delta\varepsilon(\vec{v})) = L(\Delta\vec{v}) \quad ; \quad (4)$$

so , by substracting (4) from (2) , and choosing

$$\Delta\vec{v} = \Delta\vec{u}^{n+1} - \Delta\vec{u},$$

we get

$$(\Delta\sigma^{n+1}, \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u})) = (\Delta\sigma, \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u})). \quad (5)$$

Choosing again $\Delta\vec{v} = \Delta\vec{u}^{n+1} - \Delta\vec{u}$ in (3) , we write

$$\left\{ \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u}^{n}), \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u}) \right\}$$

$$= \rho(\Delta\sigma^{n+1} - \Delta\sigma(\Delta\vec{u}^{n}), \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u}))$$

$$= \rho(\Delta\sigma - \Delta\sigma(\Delta\vec{u}^{n}), \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u})) \quad , \text{ using } (5) .$$

We evaluate now

$$\left\{ \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u}), \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u}) \right\}$$

$$= \left\{ \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u}^{n}), \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u}) \right\}$$
$$+ \left\{ \varepsilon(\Delta\vec{u}^{n}) - \varepsilon(\Delta\vec{u}), \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u}) \right\}$$

$$= \left\{ \varepsilon(\Delta\vec{u}^{n}) - \varepsilon(\Delta\vec{u}), \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u}) \right\}$$
$$+ \rho(\Delta\sigma - \Delta\sigma(\Delta\vec{u}^{n}), \varepsilon(\Delta\vec{u}^{n+1}) - \varepsilon(\Delta\vec{u}))$$

$$= \left\{ \varepsilon(\Delta \vec{u}^{\,n}) - \varepsilon(\Delta \vec{u}) + \rho S\,(\Delta\sigma - \Delta\sigma(\Delta \vec{u}^{\,n})),\ \varepsilon(\Delta \vec{u}^{\,n+1}) - \varepsilon(\Delta \vec{u}) \right\}.$$

Using the CAUCHY inequality , and S D = 1 ,we get

$$\left\{ \varepsilon(\Delta \vec{u}^{\,n+1}) - \varepsilon(\Delta \vec{u}) \right\}^2 \leqslant \left\{ \varepsilon(\Delta \vec{u}^{\,n}) - \varepsilon(\Delta \vec{u}) \right\}^2$$

$$+ \rho^2 \left[\Delta\sigma - \Delta\sigma(\Delta \vec{u}^{\,n}) \right]^2$$

$$- 2\rho\,(\Delta\sigma - \Delta\sigma(\Delta \vec{u}^{\,n}),\ \varepsilon(\Delta \vec{u}^{\,n}) - \varepsilon(\Delta \vec{u})).$$

But

$$(\Delta\sigma - \Delta\sigma(\Delta \vec{u}^{\,n}),\ \varepsilon(\Delta \vec{u}) - \varepsilon(\Delta \vec{u}^{\,n})) \geqslant \left[\Delta\sigma - \Delta\sigma(\Delta \vec{u}^{\,n}) \right]^2 ,$$

and

$$\left\{ \varepsilon(\Delta \vec{u}^{\,n+1}) - \varepsilon(\Delta \vec{u}) \right\}^2 \leqslant \left\{ \varepsilon(\Delta \vec{u}^{\,n}) - \varepsilon(\Delta \vec{u}) \right\}^2$$

$$+ \rho(\rho - 2) \left[\Delta\sigma - \Delta\sigma(\Delta \vec{u}^{\,n}) \right]^2 .$$

The sequence $\left\{ \varepsilon(\Delta \vec{u}^{\,n}) - \varepsilon(\Delta \bar{u}) \right\}^2 \longrightarrow 0$ for $0 < \rho < 2$.

Note : for $\rho = 1$: we get in (3)

$$(\Delta\sigma^{n+1}, \Delta\varepsilon(\vec{v})) = (\Delta\sigma(\Delta \vec{u}^{\,n}), \Delta\varepsilon(\vec{v}))$$

$$+ \left\{ \varepsilon(\Delta \vec{u}^{\,n+1}) - \varepsilon(\Delta \bar{u}^{\,n}),\ \varepsilon(\Delta \vec{v}) \right\} = L(\Delta \vec{v}) ,$$

or

$$\left\{ \varepsilon(\Delta \vec{u}^{\,n+1}),\ \varepsilon(\Delta \vec{v}) \right\} = L(\Delta \vec{v}) + \int_\Omega \Delta\varepsilon_{ij}(\vec{u}^{\,n})\, D_{ijkl}\, \Delta\varepsilon(\vec{v})\, d\Omega$$

$$- \int_\Omega \Delta\sigma_{ij}(\Delta \vec{u}^{\,n})\, \Delta\varepsilon_{ij}(\vec{v})\, d\Omega .$$

But (see exercise 1)

$$\Delta\varepsilon_{ij}(\vec{u}^{\,n})\, D_{ijkl} - \Delta\sigma_{kl}(\Delta \vec{u}^{\,n}) = D_{ijkl}\, \Delta\varepsilon^{(p)}_{ij}(\Delta \vec{u}^{\,n}) ,$$

where

$$\Delta\varepsilon^{(p)}_{ij}(\Delta \vec{u}^{\,n}) = \frac{\partial f}{\partial \sigma_{ij}}\, \Delta\lambda(\Delta \vec{u}^{\,n}) ,$$

$$\Delta\lambda(\Delta\vec{u}^{\,n}) = \frac{\left\langle \dfrac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \Delta\varepsilon_{kl}(\Delta\vec{u}^{\,n}) \right\rangle}{\dfrac{\partial f}{\partial \sigma_{pq}} D_{pqrs} \dfrac{\partial f}{\partial \sigma_{rs}} + h} .$$

We have now

$$\int_\Omega \Delta\varepsilon_{ij}(\vec{u}^{\,n+1}) D_{ijkl} \Delta\varepsilon_{kl}(\vec{v})\, d\Omega$$

$$= L(\Delta\vec{v}) - \int_\Omega \Delta\varepsilon_{ij}^{(p)}(\Delta\vec{u}^{\,n}) D_{ijkl} \Delta\varepsilon_{kl}(\vec{v})\, d\Omega .$$

This scheme is the initial stress method proposed by ZIENKIEWICZ .

Exercise 8

Show the convergence of the scheme described in (27) .

Proof

The equations of (27) can be written as

$$[\Delta\sigma^{n+1}, \Delta\tau] = [\Delta\sigma^{n}, \Delta\tau] + \rho(\varepsilon(\Delta\vec{u}^{\,n+1}) - \varepsilon(\Delta\sigma^{n}), \Delta\tau) , \quad (1)$$

$$(\Delta\sigma^{n+1}, \varepsilon(\Delta\vec{v})) = L(\Delta\vec{v}) . \quad (2)$$

We choose $\Delta\tau = \Delta\sigma^{n+1} - \Delta\sigma$ in (1) , and we get

$$[\Delta\sigma^{n+1} - \Delta\sigma^{n}, \Delta\sigma^{n+1} - \Delta\sigma]$$

$$= \rho(\varepsilon(\Delta\vec{u}^{\,n+1}) - \Delta\varepsilon(\Delta\sigma^{n}), \Delta\sigma^{n+1} - \Delta\sigma) . \quad (3)$$

At the end of the iterations , we shall have

$$(\Delta\sigma, \varepsilon(\Delta\vec{v})) = L(\Delta\vec{v}) , \quad (4)$$

and we get , by substracting (4) from (2) , and choosing

$$\Delta\vec{v} = \Delta\vec{u}^{\,n+1} - \Delta\vec{u} ,$$

$$(\Delta\sigma^{n+1} - \Delta\sigma, \varepsilon(\Delta\vec{u}^{\,n+1})) = (\Delta\sigma^{n+1} - \Delta\sigma, \varepsilon(\Delta\vec{u})) . \quad (5)$$

Using (5) in (3) results in

$$\left[\Delta\sigma^{n+1} - \Delta\sigma^{n}, \Delta\sigma^{n+1} - \Delta\sigma\right] = \rho\left(\varepsilon(\Delta\vec{u}) - \Delta\varepsilon(\Delta\sigma^{n}), \Delta\sigma^{n+1} - \Delta\sigma\right).$$

We evaluate now

$$\left[\Delta\sigma^{n+1} - \Delta\sigma\right]^2 = \left[\Delta\sigma^{n+1} - \Delta\sigma^{n}, \Delta\sigma^{n+1} - \Delta\sigma\right] + \left[\Delta\sigma^{n} - \Delta\sigma, \Delta\sigma^{n+1} - \Delta\sigma\right]$$

$$= \rho\left(\varepsilon(\Delta\vec{u}) - \Delta\varepsilon(\Delta\sigma^{n}), \Delta\sigma^{n+1} - \Delta\sigma\right)$$

$$+ \left[\Delta\sigma^{n} - \Delta\sigma, \Delta\sigma^{n+1} - \Delta\sigma\right]$$

$$= \left(\rho\left(\varepsilon(\Delta\vec{u}) - \Delta\varepsilon(\Delta\sigma^{n})\right) + S(\Delta\sigma^{n} - \Delta\sigma), \Delta\sigma^{n+1} - \Delta\sigma\right).$$

Using the CAUCHY inequality , we get

$$\left[\Delta\sigma^{n+1} - \Delta\sigma\right]^2 \leqslant \left[\Delta\sigma^{n} - \Delta\sigma\right]^2 + \rho^2 \left\{ \varepsilon(\Delta\vec{u}) - \varepsilon(\Delta\sigma^{n})\right]^2$$

$$- 2\rho\left(\Delta\sigma^{n} - \Delta\sigma, \Delta\varepsilon(\Delta\sigma^{n}) - \varepsilon(\Delta\vec{u})\right) \quad .$$

Using the fundamental inequality , we find

$$\left[\Delta\sigma^{n+1} - \Delta\sigma\right]^2 \leqslant \left[\Delta\sigma^{n} - \Delta\sigma\right]^2 + \rho^2 \left\{ \varepsilon(\Delta\vec{u}) - \Delta\varepsilon(\Delta\sigma^{n})\right\}^2$$

$$- 2\rho\left[\Delta\sigma^{n} - \Delta\sigma\right]^2 \quad .$$

$\exists\, M > 0$, so that

$$\left\{ \varepsilon(\Delta\vec{u}) - \Delta\varepsilon(\Delta\sigma^{n})\right\}^2 \leq M\left[\Delta\sigma^{n} - \Delta\sigma\right]^2 \quad .$$

If $0 < \rho < \dfrac{2}{M}$, the sequence converges .

Exercise 9

Show the convergence of the hybrid dual scheme (22-23) .

Proof

At the end of the iterations in one increment of load ΔP , the incremental solution will satisfy the system

$$a\,(\Delta\varepsilon(\Delta\sigma),\Delta\tau) - b\,(\Delta\vec{u},\Delta\tau) = 0 \quad , \tag{1}$$

$$b\,(\Delta\sigma,\Delta\vec{v}) = L\,(\Delta\vec{v}) \quad . \tag{2}$$

The hybrid dual scheme can be written as

$$\begin{aligned}(\Delta\sigma^{n+1},\Delta\tau) - \rho\, b\,(\Delta\vec{u}^{n+1},\Delta\tau) &= (\Delta\sigma^{n},\Delta\tau) \\ &\quad - \rho\, a\,(\Delta\varepsilon(\Delta\sigma^{n}),\Delta\tau) \quad ,\end{aligned} \tag{3}$$

$$b\,(\Delta\sigma^{n+1},\Delta\vec{v}) = L\,(\Delta\vec{v}) \quad . \tag{4}$$

We substract (2) from (4)

$$b\,(\Delta\sigma^{n+1} - \Delta\sigma,\Delta\vec{v}) = 0 \quad ,$$

and , if we choose $\Delta\vec{v} = \Delta\vec{u}^{n+1} - \Delta\vec{u}$, we get

$$b\,(\Delta\vec{u}^{n+1},\Delta\sigma^{n+1} - \Delta\sigma) = b\,(\Delta\vec{u},\Delta\sigma^{n+1} - \Delta\sigma) \quad .$$

Taking now $\Delta\tau = \Delta\sigma^{n+1} - \Delta\sigma$ in (1) , we finally obtain

$$\begin{aligned}b\,(\Delta\vec{u}^{n+1},\Delta\sigma^{n+1} - \Delta\sigma) &= b\,(\Delta\vec{u},\Delta\sigma^{n+1} - \Delta\sigma) \\ &= a\,(\Delta\varepsilon(\Delta\sigma),\Delta\sigma^{n+1} - \Delta\sigma) \quad .\end{aligned} \tag{5}$$

Choosing $\Delta\tau = \Delta\sigma^{n+1} - \Delta\sigma$ in (3) results in

$$\begin{aligned}(\Delta\sigma^{n+1} - \Delta\sigma^{n},\Delta\sigma^{n+1} - \Delta\sigma) &= \rho\big(\, b\,(\Delta\vec{u}^{n+1},\Delta\sigma^{n+1} - \Delta\sigma) \\ &\quad - a\,(\Delta\varepsilon(\Delta\sigma^{n}),\Delta\sigma^{n+1} - \Delta\sigma)\big) \quad .\end{aligned}$$

Using (5) and the notation (,) , we finally get

$$\begin{aligned}(\Delta\sigma^{n+1} - \Delta\sigma^{n},\Delta\sigma^{n+1} - \Delta\sigma) &= \rho\big(\, a\,(\Delta\varepsilon(\Delta\sigma),\Delta\sigma^{n+1} - \Delta\sigma) \\ &\quad - a\,(\Delta\varepsilon(\Delta\sigma^{n}),\Delta\sigma^{n+1} - \Delta\sigma)\big)\end{aligned}$$

$$= \rho\,(\Delta\varepsilon(\Delta\sigma) - \Delta\varepsilon(\Delta\sigma^{n}),\, \Delta\sigma^{n+1} - \Delta\sigma) .$$

We evaluate now the quantity

$$(\Delta\sigma^{n+1} - \Delta\sigma,\, \Delta\sigma^{n+1} - \Delta\sigma) \tag{7}$$

$$= (\Delta\sigma^{n+1} - \Delta\sigma^{n} + \Delta\sigma^{n} - \Delta\sigma,\, \Delta\sigma^{n+1} - \Delta\sigma)$$

$$= (\rho\,(\Delta\varepsilon(\Delta\sigma) - \Delta\varepsilon(\Delta\sigma^{n})) + \Delta\sigma^{n} - \Delta\sigma,\, \Delta\sigma^{n+1} - \Delta\sigma) .$$

Using the inequality

$$2\int_{\Omega} \Delta\tau\, \Delta\theta\, d\Omega \leq \int_{\Omega} (\Delta\tau\,\Delta\tau + \Delta\theta\,\Delta\theta)\, d\Omega ,$$

we can write in our case

$$(\rho\,(\Delta\varepsilon(\Delta\sigma) - \Delta\varepsilon(\Delta\sigma^{n})) + \Delta\sigma^{n} - \Delta\sigma,\, \Delta\sigma^{n+1} - \Delta\sigma)$$

$$\leq \Big(\frac{\rho^2}{2} (\Delta\varepsilon(\Delta\sigma) - \Delta\varepsilon(\Delta\sigma^{n}),\, \Delta\varepsilon(\Delta\sigma) - \Delta\varepsilon(\Delta\sigma^{n}))$$

$$+ \frac{1}{2} (\Delta\sigma^{n} - \Delta\sigma,\, \Delta\sigma^{n} - \Delta\sigma) + \rho\,(\Delta\varepsilon(\Delta\sigma) - \Delta\varepsilon(\Delta\sigma^{n})),\, \Delta\sigma^{n} - \Delta\sigma)$$

$$+ \frac{1}{2} (\Delta\sigma^{n+1} - \Delta\sigma,\, \Delta\sigma^{n+1} - \Delta\sigma) .$$

So , we get for (7) , introducing $(- , -) = (-)^2$

$$(\Delta\sigma^{n+1} - \Delta\sigma)^2 \leq \rho^2 (\Delta\varepsilon(\Delta\sigma) - \Delta\varepsilon(\Delta\sigma^{n}))^2$$

$$+ (\Delta\sigma^{n} - \Delta\sigma)^2 \tag{8}$$

$$- 2\rho\,(\Delta\varepsilon(\Delta\sigma) - \Delta\varepsilon(\Delta\sigma^{n}),\, \Delta\sigma - \Delta\sigma^{n}) .$$

Using one of the fundamental inequalities established in the exercise 2 of the appendix

$$(\Delta\varepsilon(\Delta\sigma) - \Delta\varepsilon(\Delta\sigma^{n}))(\Delta\sigma - \Delta\sigma^{n})$$

$$\leq s_{ijkl}(\Delta\sigma - \Delta\sigma^{n})(\Delta\sigma - \Delta\sigma^{n}) ,$$

we write for the whole volume

$$(\Delta\varepsilon(\Delta\sigma) - \Delta\varepsilon(\Delta\sigma^{n}), \Delta\sigma - \Delta\sigma^{n}) \geqslant \gamma(\Delta\sigma - \Delta\sigma^{n})^{2} . \quad (9)$$

$\exists\, M > 0$ such that

$$(\Delta\varepsilon(\Delta\sigma) - \Delta\varepsilon(\Delta\sigma^{n}))^{2} \leq M(\Delta\sigma - \Delta\sigma^{n})^{2} .$$

Then

$$(\Delta\sigma^{n+1} - \Delta\sigma)^{2} \leq (\Delta\sigma^{n} - \Delta\sigma)^{2}$$

$$+ (\rho^{2} M - 2\rho\gamma)(\Delta\sigma^{n} - \Delta\sigma)^{2} .$$

To prove the convergence of the scheme (22 , 23) , has to satisfy : $(\rho^{2} M - 2\rho\gamma) < 0$, that is $0 < \rho < \frac{2}{M}$.