# STIFFNESS OF ELASTIC STRUCTURES WITH UNILATERAL SUPPORTS

Markku Heinisuo

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ABSTRACT: A classroom-type note concerned with a single degree of freedom system is given. The structure (the upper string) is supposed to behave linearly and elastically. The support (the lower string) can behave in a non-linear way. Static and dynamic cases are considered. Some general conclusions are found with this simple model. The initial stiffness of the structure is the same than the final one in the critical state of the support. The period (or the frequency) of the structure with the non-linear unilateral support in the free vibration is strongly dependent on the initial amplitude. The amplitude is not a unique function of the period.

## INTRODUCTION

Structures are in many cases supported with unilateral supports, i.e. supports which cannot assert tensile forces. In piping problems the supports are unilateral but they are located on both sides of the structure. In the following the supports are located on one side of the structure only. This is the case in many civil engineering problems. The supports are also often pointwise and thus the contact problem is in discrete form. If there are many potential contact points (points where contact is possible) the solution of the problem is obtained usually numerically. The most used methods are:

1. Direct solution of the complementary problem.
2. Minimization of the corresponding energy functional.
3. Solution of the corresponding saddle point problem.

In a forthcoming paper the cases described above are considered but the present paper deals with the simplest case: the system with one degree of freedom. Figure 1 presents two examples. Several phenomena can be described with this simple model and the exact solution is easy to derive. Many features of the problem still remain open especially in the dynamic loading case (effect of damping, forced vibration, the mass of the support etc.) and when there are frictional forces on the supports. Recently Gawecki has found some similar results as those presented in this paper, but the structure was much more complicated (Gawecki, 1986) than it is here.

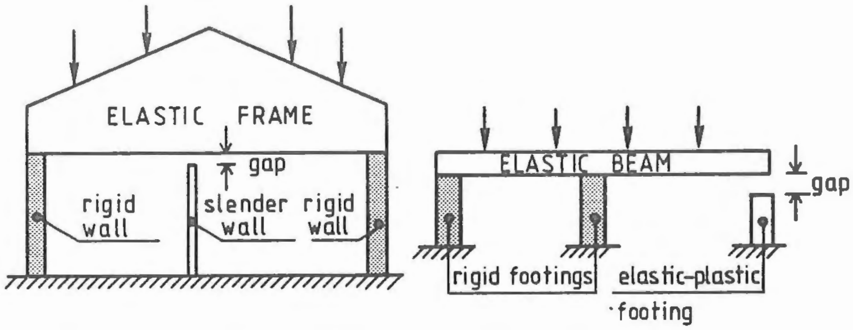


Fig. 1. Elastic structures with unilateral supports.

In this note the structure is considered to be linear and elastic with small deformations. The support is frictionless and unilateral. The support may behave in a non-linear way. Static and dynamic cases are handled.

STATIC CASE

Figure 2a presents a simple elastic and static model with elastic constants k and K and with an initial gap h. The lower string (the support) is supposed to be elastic

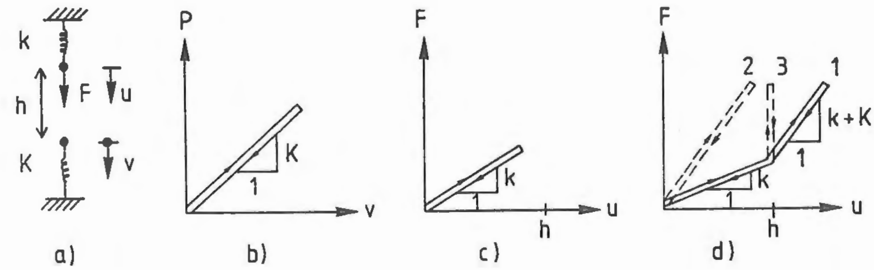


Figure 2. Linear support.

according to Fig. 2b (the force P is positive if compressive). If the initial gap h is large the (F, u)-curve is linear in loading and unloading (Fig. 2c). In the following the upper string (the structure) is considered to be elastic and linear in every case. If the initial gap h is small or the force F is large the (F, u)-curve is of the type presented (curve 1) in Figure 2d. It is seen that the (F, u)-curve is not linear because of the geometrical non-linearity in the boundary conditions. The (F, u)-curve is a linear one (curve 2, Fig.

2d) if the initial gap h=0. Figure 2d presents also the (F, u)-curve (curve 3) when the lower string is rigid ($K = \infty$).

Consider next the case where the lower string buckles elastically with the critical force $P_{cr}$ (Fig. 3a). Figure 3b presents the (F, u)-curve for this case. The critical deformation $v_{cr}$ is usually negligible small in the buckling case. Figure 3b presents also the (F, u)-curve in this case (the dashed line). Some remarks can be made on this case.

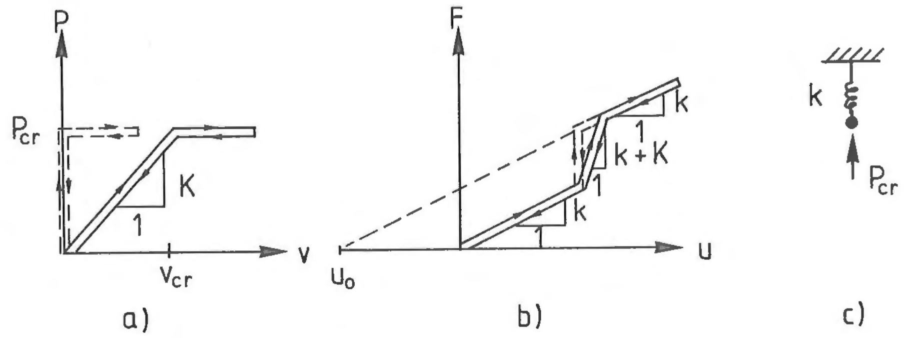


Figure 3. Buckling support.

Firstly, the (F, u)-curve would not be linear although the initial gap would be zero. This is due to the non-linear behaviour of the lower string. Secondly, the final stiffness (with large loading F) dF/du is the same than the initial stiffness (with small loading F) independently on the behaviour of the lower string provided that the lower string reaches its critical load i.e. $dP/dv = 0$. Thirdly, the final (F, u)-curve can be calculated directly. The point of intersection between the final (F, u)-curve and the line $F = 0$ gives the value $u_o$ which can be calculated accordingly to Fig. 3c. In many practical cases the initial and the final (F, u)-curves are those which are needed in the design procedure. As seen these can be evaluated if the initial stiffness and the displacement $u_o$ are known.

Figure 4a presents the behaviour of an elastic-plastic and ideal plastic string in loading and unloading. The corresponding (F, u)-curves are presented in Figure 4b. It is seen that now the (F, u)-curve is no longer unique. The final (F, u)-curve is dependent on whether there is loading or unloading. Hence the forementioned method to calculate the final (F, u)-curve still holds provided that the displacement $u_o$ is calculated according to Fig. 3c in loading.

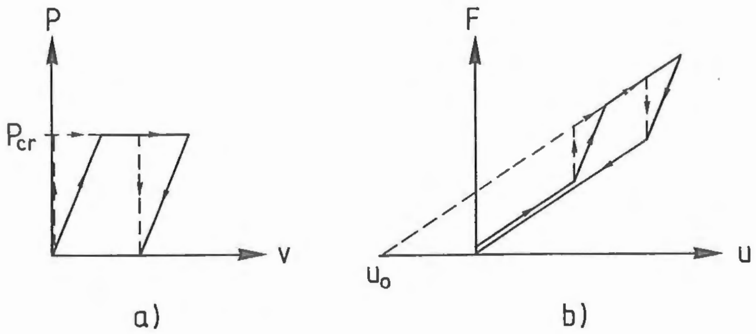


Figure 4. Plastic support.

In the elastic-plastic case it is also important to know the number of load cycles. Figure 5 presents the curves for these cycles and it is seen that the initial and the final (F, u)-curves are the same as in the previous case. Yet it can be seen that the shaded area in

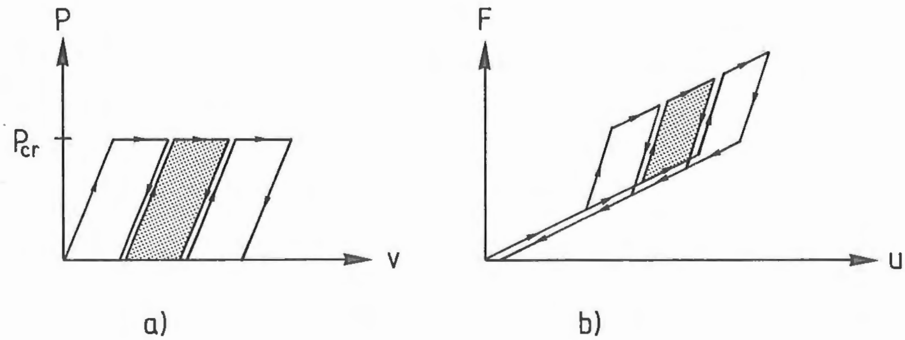


Figure 5. Three cycles.

Fig. 5b is a certain transform from the shaded area presented in the Fig. 5a. The similar result holds for all the previous cases.

By the generalization of the forementioned cases it is seen that if the supporting structure behaves in some non-linear way so that there exists the critical point where the equation $dP/dv = 0$ is valid (Fig. 6a) then the final (F-u)-curve never exeeds the linear (F-u)-curve determined by the method presented in the previous cases (Fig. 6b).

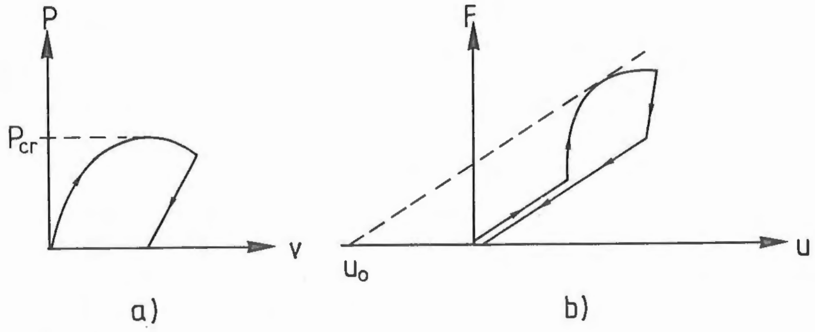


Figure 6. Non-linear support.

DYNAMIC CASE

Consider the case presented in the Fig. 2a but now the mass m of the upper string (the structure) taken into account. It is also supposed that the lower string (the support) can buckle elastically (Fig. 3a). It is seen from the Fig. 3b that the structure becomes first stiffer when the displacement is increasing and then the structure reaches its final stiffness which is the same than the initial stiffness of the structure. If the initial amplitude $\delta$ = u(t=0) is smaller than the initial gap then the response of the structure in free vibrations is as shown in Fig. 7 ( damping is neglected).

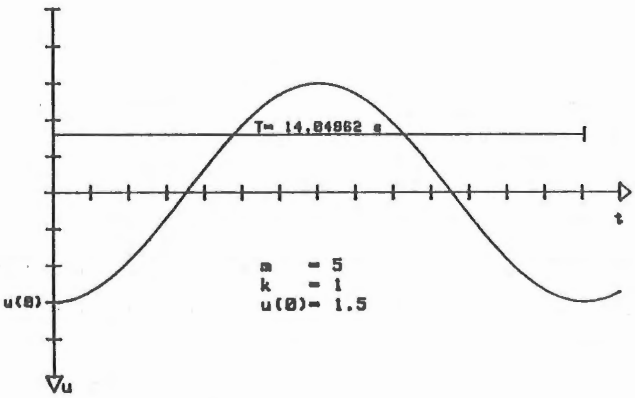


Figure 7. Small amplitude.

The period of the vibration is

$$T = 2\pi\sqrt{m/k}. \tag{1}$$

If the initial amplitude is in the range $h < \delta \leq h + v_{cr}$ then the response of the structure is of the type presented in Figure 8.

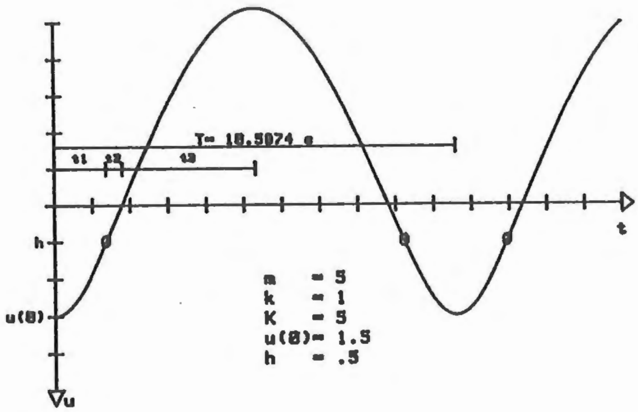


Figure 8. Medium amplitude.

It is seen that part of the time the structure vibrates with the period $T_1 = 2\pi\sqrt{m/(k+K)}$ and the rest of the time with the period $T_2 = 2\pi\sqrt{m/k}$. After some simple calculations the period T of the vibration (the calculations are left for the readers as an exercise) is found to be

$$T = 2(t_1 + t_2 + t_3),$$

$$t_1 = \sqrt{\frac{m}{k+k}} \arccos \frac{kh}{K(\delta - h) + k\delta},$$

$$t_2 = \sqrt{\frac{m}{k}} \operatorname{arc} \cot \left( \frac{K(\delta - h) + k\delta}{(K+k)h} \sqrt{\frac{K+k}{k}} \sin \sqrt{\frac{K+k}{m}}\, t_1 \right),$$

$$t_3 = \frac{\pi}{2}\sqrt{\frac{m}{k}}. \qquad (2)$$

It can be seen that the period is now smaller than in the previous case. It is also the same if the initial amplitude is positive or negative. When the support is rigid ($K = \infty$) this case is impossible if $\delta > h$.

If the initial amplitude is large $\delta > h + v_{cr}$ the response of the structure is as shown in Fig. 9. It is seen that now three separate times can be distinguished when the structure is vibrating freely. After some calculations the period T of the vibration is found to be

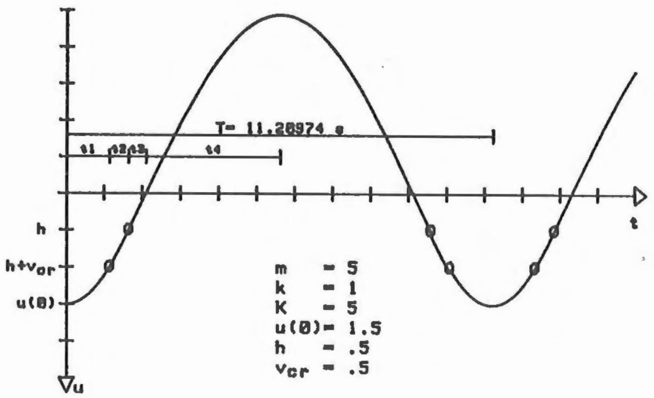


Figure 9. Big amplitude.

$$T = 2(t_1 + t_2 + t_3 + t_4),$$

$$t_1 = \sqrt{\frac{m}{k}}\,\overline{\text{arc}}\cos\frac{P_{cr}+k(h+v_{cr})}{P_{cr}+k\delta},$$

$$t_2 = \sqrt{\frac{m}{K+k}}\,(\overline{\text{arc}}\cos\frac{kh}{r(K+k)} + \overline{\text{arc}}\tan A/B),$$

$$t_3 = \sqrt{\frac{m}{k}}\,\overline{\text{arc}}\cot\left(\sqrt{\frac{K+k}{k}}\left(\frac{B}{h}\sin\sqrt{\frac{K+k}{m}}\,t_2 - \frac{A}{h}\cos\sqrt{\frac{K+k}{m}}\,t_2\right)\right),$$

$$t_4 = \frac{\pi}{2}\sqrt{\frac{m}{k}},$$

$$A = -\sqrt{\frac{k}{K+k}}\,(\delta + \frac{P_{cr}}{k})\sin\sqrt{\frac{k}{m}}\,t_1,$$

$$B = \frac{Kv_{cr}+k(h+v_{cr})}{K+k},$$

$$r = \sqrt{A^2 + B^2}. \qquad (3)$$

The period is now longer than in the previous case but shorter than in the first case.

Some limiting cases can be easily derived for the problems studied but this is not done in this study. The most interesting thing found is the fact that the period of the free vibration of the structure depends on the initial amplitude according to egs (1)-(3). On the other hand the amplitude is not a unique function of the period (or the frequency 1/T) as seen in Fig. 10. Figure 10 presents the "backbone"-curve of the structure.

This non-uniqueness makes the calculation procedure difficult when solving the problem of forced vibration as stated by Gawecki, 1986. Gawecki presented also one backbone-curve, but it was firstly ($h < \delta < h + v_{cr}$) on the other side than that presented in the Fig. 10 after the linear part of the curve ( $\delta \leq h$). Gawecki studied the case where the support (the lower string) is rigid and the structure (the upper string) is

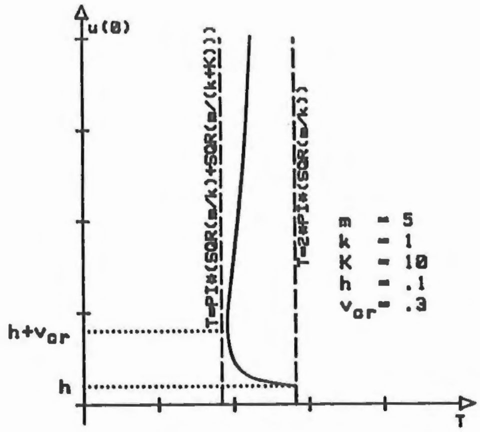


Figure 10. Backbone-curve.

elastic-plastic. This may cause the structure to become less stiff before the gaps are closed and the structure then stiffens afterwards. Also the limiting case, where the initial amplitude $\delta$ is very large, is quite different from the case studied here compared to that studied by Gawecki. Because the final stiffness of the structure is the same than the initial stiffness then in the limiting case $\delta \to \infty$ the period of the vibration is equal to the period in the case $\delta \leq h$ i.e. the case without the support.

CONCLUSIONS

The final stiffness of an elastic structure with a unilateral support is equal to the initial stiffness of the structure if the final stiffness is defined as the point where the support reaches its critical state $dP/dv = 0$. The stiffness of the structure has a significiant effect to the free vibration of the structure. The period of the structure is strongly dependent on the initial amplitude. Also the amplitude is not a unique function of the period. Many of these conclusions seem to be quite obvious but the author has not found them in the literature. Perhaps some of them can be found from elementary exercises in structural mechanics.

REFERENCE

Gawecki, A., Elastic-Plastic Beams and Frames with Unilateral Boundary Conditions, J. Struct. Mech., 14(1), 55-76 (1986).

Markku Heinisuo, Lic. tech., M.Sc. (Struct. eng.) Tampere University of Technology, Department of Civil Engineering, Structural Mechanics