PARAMETRICALLY EXCITED VIBRATIONS - NUMERICAL AND ANALYTICAL INVESTIGATION OF SIMPLE MECHANICAL SYSTEMS

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ABSTRACT: In several mechanical systems some of the parameters like the stiffness may vary periodically. In those cases there is a risk for so called parametrically excited vibrations. By studying simple systems, one may get some general feeling for the problem and a rough estimate whether the phenomena may occur or not.

INTRODUCTION

The majority of machine vibrations consists of so called forced vibrations. However, there is another major class of machine vibrations, i.e. self-excited, which are rarer than forced vibrations, but they are difficult to anticipate before the fact and diagnose after the fact because of their subtlety /1/.

Parametric vibrations form a special subclass of self-excited vibrations. They can be explained by periodical variation of some parameters in the equations of motion. During the last years investigations have been performed by several authors, e.g. Todl /2/, Eicher /3, 4/ and Schmidt /5/.

Eicher /4, p. 8/ has given several technical applications were parametric excitation could occur, e.g.:

- centrifuges with anisotropic mass properties,
- wind power plants with two wings,
- gears with varying number of teeth in action,
- pendulum with varying length.

In order to understand the phenomena let us study the simplest possible equation written in nondimensional form:

$$\ddot{x} + (1 + \gamma\cos 2t)x = 0. \tag{1}$$

By moving the term with periodically varying coefficient over to the right hand side an equation

$$\ddot{x} + x = -\gamma\cos 2t \cdot x \tag{2}$$

is obtained which can be solved by means of iteration. The first order approximation is obtained from

$$\ddot{x}_o + x_o = 0 \tag{3}$$

which has the general solution

$$x_o = A \cos t + B \sin t. \tag{4}$$

The next approximation is obtained from

$$\ddot{x}_1 + x_1 = - \gamma \cos 2t \cdot x_o. \tag{5}$$

By using the trigonometrical formulae

$$\left.\begin{aligned} 2 \cos t \cos 2t &= \cos 3t + \cos t, \\ 2 \sin t \cos 2t &= \sin 3t - \sin t \end{aligned}\right\} \tag{6}$$

one obtains

$$\ddot{x}_1 + x_1 = - \frac{\gamma}{2} (A\cos 3t + A\cos t + B\sin 3t - B\sin t). \tag{7}$$

The particular solution of x_1 will now contain oscillating terms with linearly growing amplitude. In a similar way terms proportional to time squared are obtained in the next iteration.

In fact one can show that the general solution of (1) is /4, p. 10/

$$x(t) = C_1 \exp(\mu_1 t)\ P_1(t) + C_2 \exp(\mu_2 t)\ P_2(t). \tag{8}$$

C_1 and C_2 are real coefficients, $P_1(t)$ and $P_2(t)$ are periodical functions and μ_1, μ_2 are characteristic exponents.

In our analysis we have assumed that the frequency of parametric variation is equal to twice the eigenfrequency of the system. This is the case for the so called main parametric resonance. However, there is an infinite number of finite intervals for ω where unstable solutions are obtained for equation:

$$\ddot{x} + (1 + \gamma \cos \omega t)\ x = 0. \tag{9}$$

The intervals are located around the angular frequencies

$$\omega = \frac{2}{n} \tag{10}$$

where n is the order of the parametric resonance. However, the parametric resonances of higher order are generally so easily damped out, that they will hardly occur in practice.

INFLUENCE OF DAMPING

We shall now introduce a viscous damping term into our equation of motion:

$$\ddot{x} + 2D\dot{x} + f(t)x = 0. \tag{11}$$

D is a common nondimensional measure for damping, first introduced by Lehr /6/ in 1930, defined as the ratio between viscous damping and critical damping.

By making the transformation

$$x = y\,\exp(-Dt) \tag{12}$$

one obtains

$$\ddot{y} + (f(t)-D^2)y = 0. \tag{13}$$

In most practical cases $D \ll 1$ (otherwise a more careful investigation of parametrically excited vibrations is hardly required). It is readily seen that the amount of damping necessary to damp out the parametrically excited vibration is equal to the positive characteristic exponent.

The amount of damping necessary to damp out the vibrations of the first order parametric resonance is given by

$$D > \frac{\gamma}{4}. \tag{14}$$

The corresponding expression for the parametric resonance of second order is

$$D > \frac{\gamma^2}{12}. \tag{15}$$

In fact the amount of damping necessary to damp out the vibrations at the parametric resonance of n:th order is proportional to the n:th power of γ /5, p. 9/. In general $\gamma \ll 1$ which indicates that parametric resonances of higher order are of minor interest.

If the dynamic system is damped, but not enough damped, the vibrations will still grow exponentially, but with a lower rate.

INFLUENCES OF NONLINEARITIES

In the last section we noticed that parametrically excited vibrations may grow without bound even if a certain amount of damping is present in the system. However, every physical system is to a smaller or greater extent nonlinear. We shall now introduce a nonlinear term into our equation:

$$\ddot{x} + 2D\,\dot{x} + (1 + \gamma\cos 2t + \varepsilon x^2)x = 0. \tag{16}$$

The equation can be rewritten as two first order equations:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 , \\ \dot{x}_2 &= -(1 + \gamma\cos 2t + \varepsilon x_1^2)x_1 - 2Dx_2 \end{aligned}\right\} \tag{17}$$

and integrated by means of e.g. the Runge-Kutta method.

For the values $D = 0$, $\varepsilon = 0.001$, $\gamma = 0.25$ (undamped, slightly nonlinear case) the amplitude first grew exponentially as one might expect from the linear theory. However, after reaching its peak value it started to decrease again and then, after a while grew again. A low-frequent swaying was superimposed on the high-frequency vibration (Fig. 1).

In a slightly damped case ($D = 0.01$) the amplitude reached a limit value after a few cycles (Fig. 2).

THE METHOD OF YANO

Yano /7/ used a method which enables a transformation to an autonomous set of equations (i.e. the time does not occur explicitly in the set of equations).

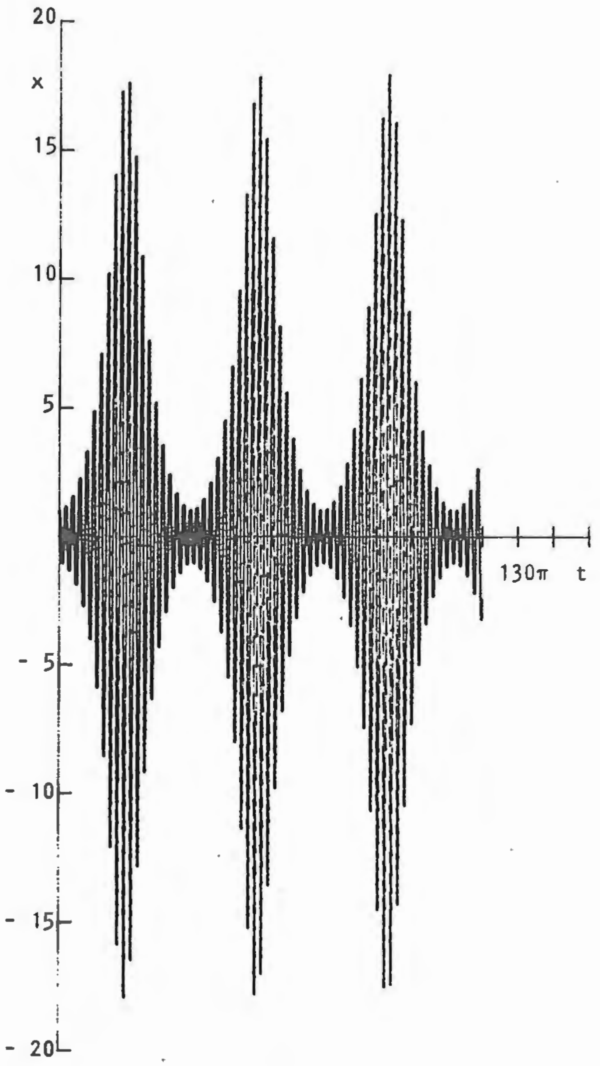

Figure 1. Main parametric resonance.
$\ddot{x} + (1 + \gamma\cos 2t + \varepsilon x^2)x = 0.$
$x(0) = 1,\ \dot{x}(0) = 0,\ \gamma = 0.25,\ \varepsilon = 0.001.$

The transformation method of Yano is used for our equation

$$\ddot{y} + 2D\dot{y} + (1 + \gamma\cos 2\eta t + \varepsilon y^2)y = 0 \tag{18}$$

which can be expressed as two first order differential equations

$$\left.\begin{aligned} \dot{y}_1 &= \eta y_2 \,, \\ \dot{y}_2 &= -2Dy_2 - \frac{1}{\eta}\,(1 + \gamma\cos 2\eta t + \varepsilon y_1^2)\,y_1 . \end{aligned}\right\} \tag{19}$$

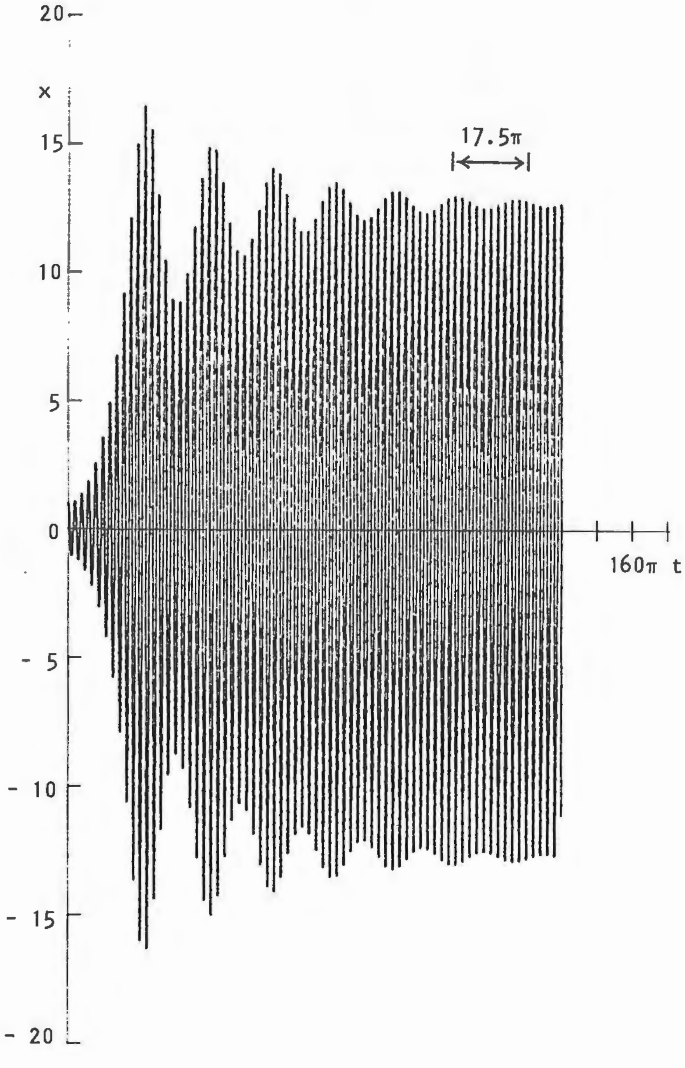

Figure 2. Main parametric resonance. Slightly damped case.
$\ddot{x} + 2D\dot{x} + (1 + \gamma\cos 2t + \varepsilon x^2)x = 0.$
$x(0) = 1,\ \dot{x}(0) = 0,\ D = 0.01,\ \gamma = 0.25.$

By introducing the following transformation

$$\left.\begin{aligned} y_1 &= \xi \cos\eta t + \zeta \sin\eta t, \\ y_2 &= \xi \sin\eta t + \zeta \cos\eta t \end{aligned}\right\} \tag{20}$$

and assuming that the parameters D, γ and ε are small enough, and that ξ, η are varying slowly a set of autonomous equations is obtained after an averaging procedure:

$$\left.\begin{aligned} \dot{A} &= -2DA - \frac{\gamma A}{4\eta} \sin 2\phi , \\ \dot{\phi} &= - \frac{1}{2\eta} (1 - \eta^2 + \frac{3\varepsilon}{4} A^2) - \frac{\gamma}{4\eta} \cos 2\phi \end{aligned}\right\} \tag{21}$$

where

$$\xi = A\cos\phi, \quad \zeta = A\sin\phi. \tag{22}$$

It is readily seen that

$$A^2 = y_1^2 + y_2^2 = y^2 + (\frac{\dot{y}}{\eta})^2. \tag{23}$$

By multiplying the first equation by A, our final set of equations is obtained:

$$\left.\begin{aligned} f_1 &= \frac{d}{dt} (A^2) = -2DA^2 - \frac{\gamma A^2}{2\eta} \sin 2\phi, \\ f_2 &= \frac{d\phi}{dt} = - \frac{1}{2\eta} (1 - \eta^2 + \frac{3\varepsilon}{4} A^2) - \frac{\gamma}{4\eta} \cos 2\phi. \end{aligned}\right\} \tag{24}$$

This set of equations can be studied analytically in the A^2- ϕ phase plane. Two important questions must be answered. First we must find the steady state solutions, i.e. the critical points in the phase plane. After that we must investigate their stability. The critical points are obtained from the conditions

$$f_1 = f_2 = 0. \tag{25}$$

For $\eta = 1$ one obtains the trivial solution

$$\left.\begin{aligned} A^2 &= 0, \\ \cos 2\phi &= 0 \end{aligned}\right\} \tag{26}$$

and the nontrivial solution

$$\left.\begin{aligned} A^2 &= -\frac{2\gamma}{3\varepsilon}\cos 2\phi, \\ \sin 2\phi &= -\frac{4D}{\gamma}\ . \end{aligned}\right\} \tag{27}$$

The stability conditions are obtained from the linearized eigenvalue equation

$$\begin{vmatrix} \frac{df_1}{dA^2} - \sigma & \frac{df_1}{d\phi} \\ \frac{df_2}{dA^2} & \frac{df_2}{d\phi} - \sigma \end{vmatrix} = 0. \tag{28}$$

For the trivial solution we get

$$\begin{vmatrix} -2D \pm \frac{\gamma}{2} - \sigma & 0 \\ -\frac{3\varepsilon}{8} & \pm\frac{\gamma}{2} - \sigma \end{vmatrix} = 0. \tag{29}$$

If $\gamma < D/4$ the time derivative of A^2 is always negative. Otherwise we get unstable eigensolutions. For the nontrivial solution we obtain

$$\begin{vmatrix} -\sigma\cdot i & \frac{2\gamma^2}{3\varepsilon} \\ -\frac{3\varepsilon}{8} & -2D - \sigma\cdot i \end{vmatrix} = 0 \tag{30}$$

and thus

$$i\cdot\sigma = -D \pm \sqrt{D^2 - \frac{\gamma^2}{4}}\ . \tag{31}$$

The solution is thus always stable.

In the damped case $\gamma = 0.25$ and $\varepsilon = 0.001$ one obtains the amplitude $A = 12.9$ and the swaying angular frequency $\sigma = 0.125$. In the damped slightly damped case good agreement with the results of numerical simulation were obtained. (See Fig. 2). In the undamped case the assumption of slowly varying amplitude is not valid any more.

TWO DIMENSIONAL CASE

If a rotor is mounted on an anisotropic shaft (Fig. 3), and we are considering

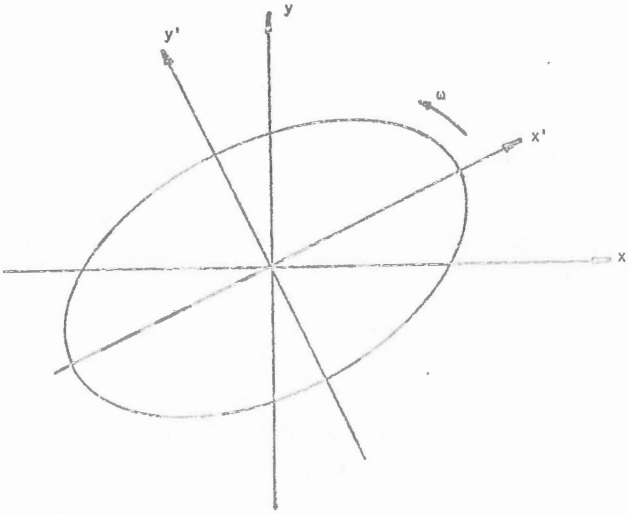

Figure 3. Cross-section of an anisotropic shaft.

the equations of motion in a nonrotating coordinate system, the terms in the stiffness matrix are varying periodically. This problem has been studied by several authors, e.g. Gasch /8/. If there is a gyroscopic coupling in the system, a velocity in the x-direction will cause a force in y-direction and vice versa. We shall include that effect into our analysis by means of a symbolic gyroscopic term:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + g\omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} -$$

$$\gamma \begin{bmatrix} \cos 2\omega t & \sin 2\omega t \\ \sin 2\omega t & -\cos 2\omega t \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} . \qquad (32)$$

ω is the angular speed of revolution. By introducing the complex notation

$$z = x + iy \qquad (33)$$

a single complex equation is obtained

$$\ddot{z} - i\omega g\dot{z} + z - \gamma z^*\exp(2i\omega t) = 0 \tag{34}$$

where z^* denotes the complex conjugate of z. Let us first consider the isotropic case, i.e. $\gamma = 0$. The eigenfrequencies are obtained by means of the common trial

$$z = z_o\exp(i\eta t). \tag{35}$$

Thus

$$\eta^2 - \omega g\eta - 1 = 0. \tag{36}$$

At each speed of revolution we obtain two eigenfrequencies, one with positive and the other with negative sign:

$$\eta = \frac{\omega g}{2} \pm \sqrt{1 + \left(\frac{\omega g}{2}\right)^2}. \tag{37}$$

The former is denoted as forward and the latter as backward precessional whirling frequency in common rotordynamic terminology.

When the frequency of the forward precessional whirl coincides with the spin speed (i.e. $\eta = \omega$) we have a forward precessional critical speed

$$\omega_+ = \sqrt{\frac{1}{1 - g}} \tag{38}$$

That critical speed is directly excited by an unbalance.

When the frequency of the backward precessional whirl coincides with the spin speed (i.e. $\eta = -\omega$) we have a backward precessional critical speed

$$\omega_- = \sqrt{\frac{1}{1 + g}} \tag{39}$$

That critical speed is normally not strongly excited by an unbalance. However, it can be noticed, e.g. if the bearings stiffnesses are anisotropic.

Equation (34) can be readily transformed to a rotating coordinate system by taking

$$z = z_o(t)\exp(i\omega t) \tag{40}$$

which gives

$$(\ddot{z}_o + 2i\omega\dot{z}_o - \omega^2 z_o - i\omega g\dot{z}_o + g\omega^2 z_o + z_o - \gamma z_o^*)\, e^{i\omega t} = 0. \tag{41}$$

We shall assume

$$\dot{z}_o = \lambda z_o, \quad \ddot{z}_o = \lambda^2 z_o \tag{42}$$

where λ is a real number, and introduce

$$z_o = z_R + iz_I \tag{43}$$

which gives, after identification of the real and imaginary parts, the following set of equations

$$\left.\begin{aligned} z_R(\lambda^2 - \omega^2 + \omega^2 g + 1 - \gamma) - z_I(2\omega g - \omega g\lambda) = 0, \\ z_R(2\omega\lambda - \omega g\lambda) + z_I(\lambda^2 - \omega^2 + \omega^2 g + 1 + \gamma) = 0. \end{aligned}\right\} \tag{44}$$

The condition for nontrivial solution gives

$$\lambda^4 + 2\lambda^2(1 + \omega^2 g - \omega^2) + \lambda^2\omega^2(2 - g)^2 = \gamma^2 - (1 + \omega^2 g - \omega^2)^2. \tag{45}$$

In this case we are thus able to calculate the growth rates of the characteristic equation analytically. If we assume $\lambda << 1$, we obtain

$$\lambda = \pm\sqrt{\frac{\gamma^2 - (1+\omega^2 g-\omega^2)^2}{2+g\omega^2}} . \tag{46}$$

The maximum value of λ is found in the vicinity of the forward precessional critical speed ω_+:

$$\lambda_{max} \approx \frac{\gamma}{\omega_+(2-g)} . \tag{47}$$

CONCLUSIONS

By studying simple mechanical systems, one can obtain some general ideas of the phenomena called parametrically excited vibrations. It is also possible to judge if the parameter variation is strong, and the damping weak enough, so that parametrical excitation might occur. The greatest risk for excitation is when the frequency of parameter variation is twice the eigenfrequency of the system (i.e. parametric resonance of first order). The condition for unstable solution is

$$\gamma > 4D. \tag{48}$$

γ is the relative variation of stiffness, and D is the ratio between viscous damping and critical viscous damping.

For a system with a hardening spring, the vibration amplitude does not grow without bound. In the undamped case a "burst" phenomenon can be observed. In the moderately damped system the limit amplitude was reached fairly quickly. By means of a transformation introduced by Yano, the equations can be transformed to an autonomous system and studied in the phase plane. For the damped case, a good approximation was obtained for the limit cycle and the "beat" frequency. In the undamped case the assumption of slowly varying energy is obviously not valid any more.

For the single-mass rotor with massless anisotropic shaft it is possible to study parametrically excited vibrations analytically.

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