# MECHANICAL INTERPRETATION OF A COMPUTATIONAL PROCEDURE FOR THE FORCE METHOD

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SUMMARY: The purpose of this paper is two-fold. First the relationship between the superposition-equation method from elementary structural analysis and the matrix force method and force and displacement methods are given. Secondly a mechanical interpretation of an algorithm producing sparse and banded self-stresses is given. It is explained how the primary structure is found by this method and in which manner the numbering affects the effectiveness of the algorithm. The load cluster method is used for explanatory purpose. The method is also connected with the choice of minimal substructures.

## INTRODUCTION

The superposition-equation method or Maxwell's method is well-known from elementary structural analysis /4/, /13/. This method is widely used for the analysis of truss and frame structures with a low degree of indeterminacy. In matrix structural analysis the equivalent of the superposition-equation method is the force method.

With the propagation of digital computers matrix methods have become more and more important. In structural analysis there are two competitors, the displacement - and the force method. The methods are dual and are based on three elementary equations. In comparison between the displacement - and the force method the latter one has three great disadvantages:

(1) the necessary choice of the primary structure;

(2) the great amount of operations necessary for the analysis of the primary structure under only the applied forces;

(3) the great amount of operations necessary for the analysis of the primary structure under only the redundant forces.

During the 1960s the first algorithms were invented for an automatic choice of the primary structure /1/, /2/. In connection with these algorithms there exists one additional disadvantage: the stresses within the primary structure due only to the redundant forces range over the whole structure and result in a completely dense overall flexibility matrix. This is a great drawback of the method, as the overall stiffness matrix of the displacement method is banded or has a skyline form. This problem was recognized pretty early /3/ but there was no solution for a couple of years.

In elementary structural analysis the load cluster method is used for the reduction of numerical manipulations in connection with the superposition-equation method /4/. The primary structure under only the redundants is manipulated by combining different redundant restraints into a new cluster redundant restraint. This method can be applied with a computer.

In the following a review of the elementary equations of the force method will be given and its relation to the displacement method will be shown.

Secondly an algorithm for the numerical choice of the primary system will be presented. The analysis of the primary system under the redundant forces will be explained by use of the Gauss LU-factorization and a turn-back LU-factorization scheme first presented by Topcu /11/. This will be followed by a mechanical interpretation of both algorithms. Examples will illustrate the improvement of the turn-back LU-factorization scheme.

## ELEMENTARY EQUATIONS OF THE LINEAR ELASTIC ANALYSIS

The basic equations of the linear elastic analysis with small displacements are: equilibrium, material law and compatibility.

The equilibrium conditions are

$$\underline{a}^T \underline{F} = \underline{R} . \qquad (1)$$

Here the internal linearly independent forces $\underline{F} \in \mathbb{R}^m$ are transformed by the equilibrium matrix $\underline{a}^T \in \mathbb{R}^{n \times m}$ into the nodal loads $\underline{R} \in \mathbb{R}^n$.

A structure is statically determinate, if n = m; a structure is indeterminate to the ρ-th degree, if $\rho \equiv m - n > 0$; a structure is a mechanism, if n < m. In the following we examine only redundant structures.

The equilibrium matrix of the structure is built up from element equilibrium matrices as presented e.g. in /5/ and /6/. It is presumed that $\underline{a}^T$ has full row rank, which is always guaranteed by an accurate idealization of a redundant or statically determinate structure.

The material law is

$$\underline{f}\ \underline{F} = \underline{v}. \qquad (2)$$

Here the internal linearly independent forces $\underline{F}$ are transformed by the block diagonal flexibility matrix $\underline{f} \in \mathbb{R}^{m \times m}$ into the element deformations $\underline{v} \in \mathbb{R}^m$

and

$$\underline{F}_x = \underline{B}_x \, \underline{X}.$$

The forces $\underline{F}_o$ are the internal forces in the primary structure due only to the applied loads $\underline{R}$ ($\underline{X} \equiv \underline{0}$). The forces $\underline{F}_x$ are the internal forces in the primary structure due only to the redundant forces ($\underline{R} \equiv \underline{0}$). The components of $\underline{F}_x$ corresponding to $\underline{X}$ are unit forces.

Mathematically $\underline{B}_o \in \mathbb{R}^{m \times n}$ is the Moore-Penrose generalized inverse of $\underline{a}^T$, which means that

$$\underline{B}_o^T \, \underline{a} \equiv \underline{a}^T \, \underline{B}_o = \underline{I} \tag{7}$$

with $\underline{I} \in \mathbb{R}^{n \times n}$ as the identity matrix.Furthermore $\underline{B}_x \in \mathbb{R}^{m \times p}$ is the kernel of $\underline{a}^T$, which means that

$$\underline{B}_x^T \, \underline{a} \equiv \underline{a}^T \, \underline{B}_x = \underline{0} \tag{8}$$

with $\underline{0} \in \mathbb{R}^{n \times p}$ as zero matrix. $\underline{X}$ are the redundant forces.

In elementary structural analysis the primary structure is selected by the analyst. Theoretically there exists an infinite number of different primary structures. Having only discretized structures in mind, there exists only a finite number of different linearly independent redundant force vectors. The matrices $\underline{B}_o$ and $\underline{B}_x$ are therefore not definite.

By using equation (2) and replacing $\underline{F}$ by equation (6) as solution of (1) and replacing $\underline{v}$ by equation (4) one obtains

$$\underline{f} \, \underline{B}_o \, \underline{R} + \underline{f} \, \underline{B}_x \, \underline{X} = \underline{a} \, \underline{r}. \tag{9}$$

Premultiplying this equation by $\underline{B}_x^T$ the basic equation is found by use of equation (8):

$$\underline{B}_x^T \, \underline{f} \, \underline{B}_o \, \underline{R} + \underline{B}_x \, \underline{X} = \underline{0} \tag{10}$$

or

$$\underline{\delta} \, \underline{X} = \underline{\delta}_o \tag{11}$$

with

$$\underline{\delta} = \underline{B}_x^T \, \underline{f} \, \underline{B}_x \, ,$$

$$\underline{\delta}_o = -\underline{B}_x^T \, \underline{f} \, \underline{B}_o \, \underline{R}.$$

The mechanical interpretation of this is the following:

- Firstly equation (10) means that the work done by the redundant forces is equal to zero (Bernoulli's principle of virtual work).

a numerical procedure for finding the primary structure will be given and secondly the analysis of the primary structure under the redundant forces will be explained.

## EVALUATION OF THE PRIMARY STRUCTURE

By equation (1) for each degree of freedom at each node equilibrium is fulfilled. For a statically determinate structure $\underline{a}^T$ is a square matrix. By factorization of this square matrix within a single step the linearly independent internal forces can be analysed. Because of the unsymmetric form of $\underline{a}^T$ pivoting is necessary during farctorization, ensuring that the element of the main diagonal of the matrix remains always non-zero. For a redundant structure such a factorization is as well possible. As there are more unknowns than equations, it is necessary to add further equations. In case no non-zero diagonal element is found by pivoting the corresponding internal force $F_i$ is chosen as redundant force. This is done by adding the equation

$$F_i = 0.$$

During the whole factorization of the matrix $\underline{a}^T$ additional ρ equations are found and one ends up with a new square and regular matrix $\underline{\tilde{a}}^T$.

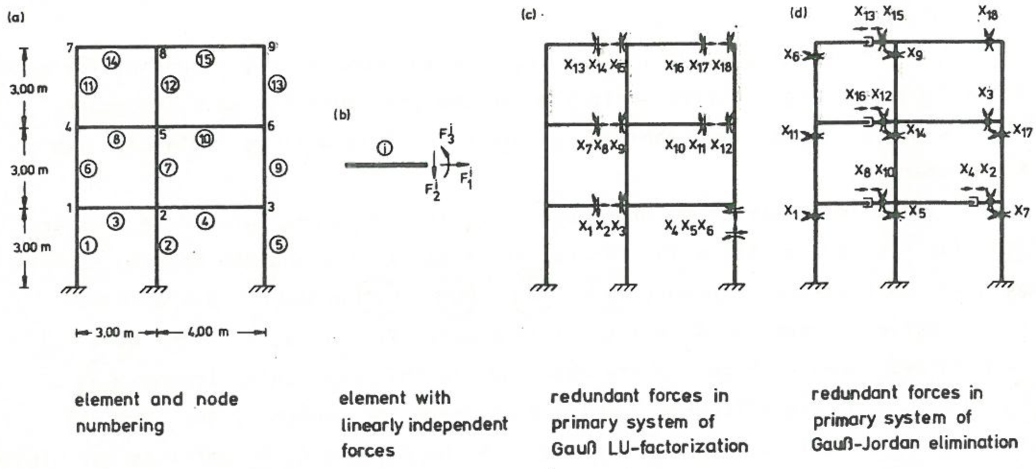


Figure 1. Two dimensional frame.

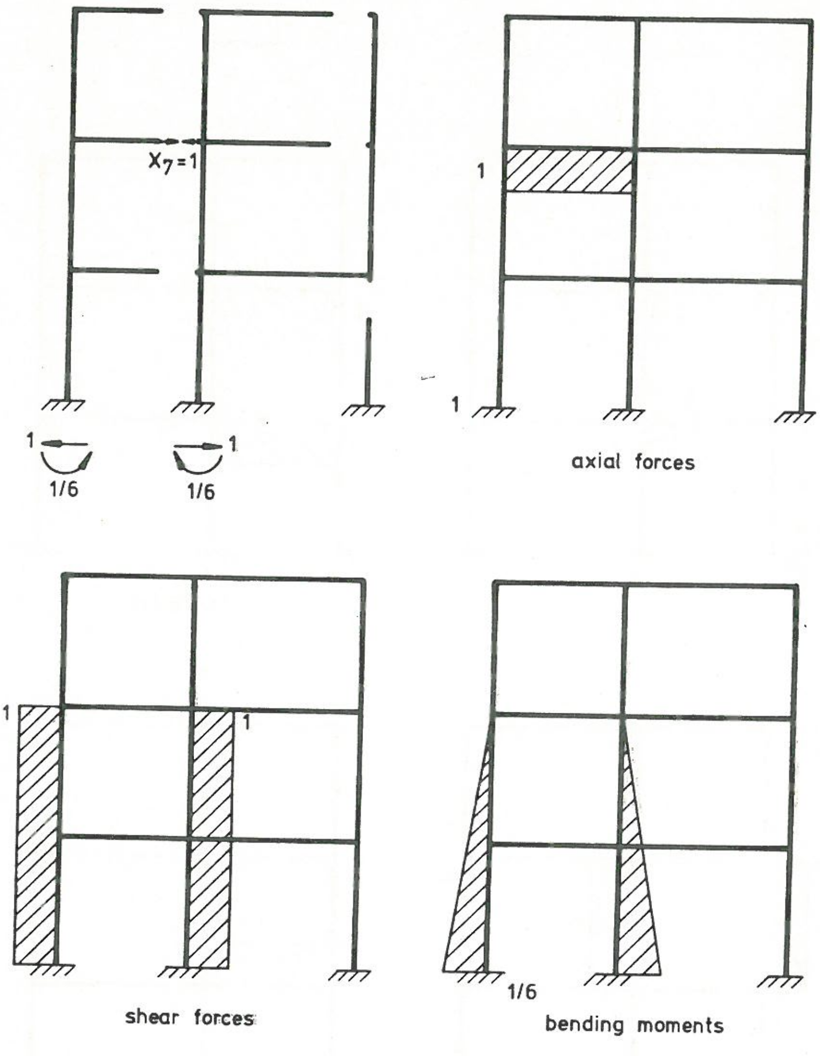


Figure 2. Internal forces due to $X_7=1$; $X_i=0$; $i \neq 7$.

The main problem of the load cluster method, the computation of the different coefficients can be stated as:
find a self equilibrated substructure using only previous redundant restraints and leaving the reactions of the primary structure as far as possible zero.

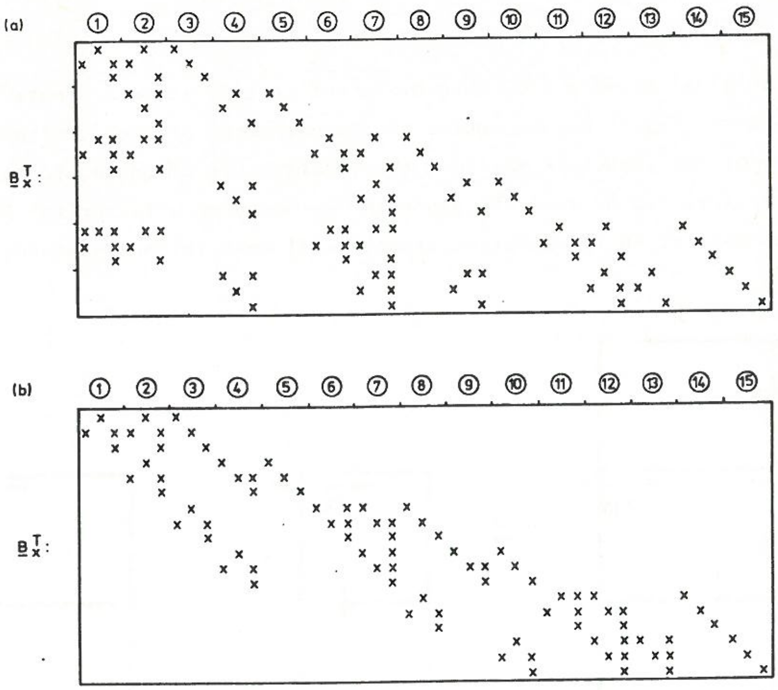


Figure 4. $\underline{B}_x$ matrices due to single loads (a) and load clusters (b) as redundant forces.

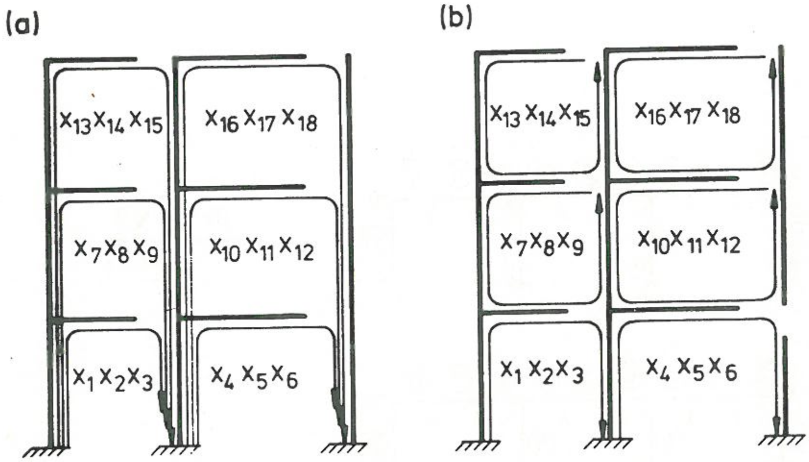


Figure 5. Physical extension within the primary structure due to single loads (a) and load clusters (b) as redundant forces.

The plane frame of Figure 7 has two different numberings. As redundant forces the forces $F_1^5$, $F_2^5$, $F_1^9$, $F_2^9$, $F_3^9$, $F_1^{13}$ and $F_2^{13}$ are chosen. (The superscribt identifies the element and the subscribt the force component.) The primary structures for $F_2^5$ and $F_2^9$ with both the numberings are examined. The results found by the turn-back LU-factorization are given in Figure 8. Not only different self-stresses but also different physical extensions of these self-stresses are found.

The algorithm always starts with the computation of the first self-stress. For higher self-stresses the algorithm tries to equilibrate the applied redundant force by choosing adequate values of the already examined redundant forces. With both numberings the redundant force $F_2^9$ is equilibrated by $F_2^5$ and the redundant

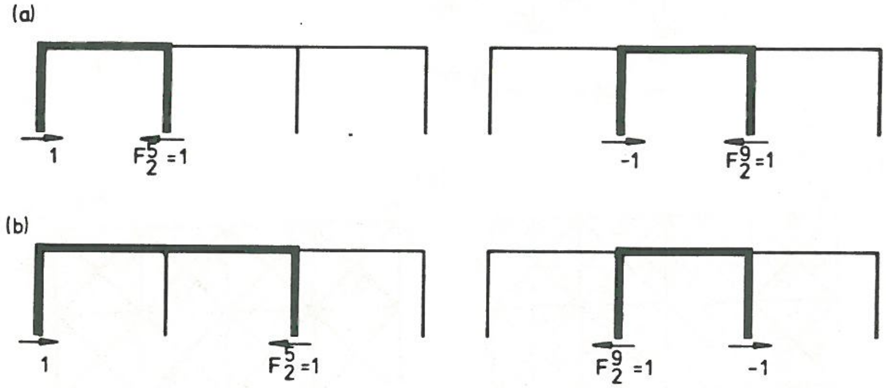


Figure 8. Influence of numbering on physical extension of self-stresses.

force $F_2^5$ is equilibrated by $F_2^1$. Although $F_2^5$ equilibrating by $F_2^9$ with the second numbering causes a smaller physical extension of the self-stress, this is not possible as $F_2^9$ is not known to the algorithm up to that point.

The example shows the necessity of a numbering with smallest bandwidth of the equilibrium matrix $\underline{a}^T$: As only previously computed redundant forces can be used within a cluster the minimal bandwidth is necessary to obtain self-stresses with minimal physical extensions and thus a banded matrix $\underline{B}_x$. This method is not only valid for a frame but also for general structures idealized by finite elements, as the following truss example will show. In Figure 9 the structure and its element and node numbering are given. Furthermore the redundant forces found by LU decom-

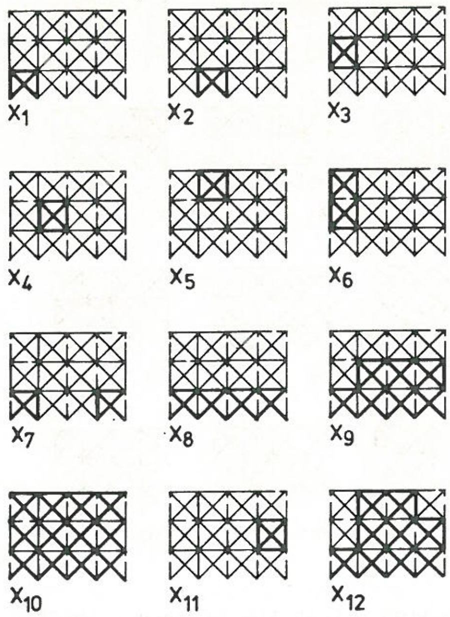


Figure 10. Physical extension within the primary structure due to single loads as redundant forces.

As an example we look at the self-stress belonging to $X_{12} = 1$. Whereas with the conventional procedure this self-stress spreads over the whole structure, with the load cluster only elements 11, 46, 47 and 48 are affected. The load cluster is formed by

$X_{12} = 1, \quad X_{10} = 1$ and $X_6 = 1$.

The turn back LU procedure can be applied as mentioned above to any finite element structure. Larger examples and examples of plane stress and plate bending structures are given in /10/. The mechanical interpretation is found by analogy but is not as obvious as with frame and truss structures.

## CONCLUSIONS

Using Gauss-Jordan elimination or Gauss LU-decomposition for the automatic analysis of the primary structure by the force method always leads to self-stresses that affect the whole structure in general. By the turn back LU-factorization only small parts of the structure are affected. This is an improvement of the matrix force method.

The method of load clusters is used for mechanical interpretation of the turn back LU-factorization. In /6/ an easy to understand test procedure for the turn back LU-factorization written in FORTRAN IV with explanations is given.

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