DYNAMIC STABILITY ANALYSIS OF ELASTIC SYSTEMS

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SUMMARY: Elastic systems under time-independent loading conditions are considered. A static analysis is not always sufficient to determine instability. Therefore, a dynamic analysis is utilized here, and the vibrations about the equilibrium state are analyzed. Attention is focussed on the variation of the vibration frequencies as the loading parameter is increased. The stability properties depend on the types of forces acting on the system, such as nonconservative or gyroscopic forces. These properties are illustrated by examples involving columns, panels, shafts, pipes, and arches. Various classifications of systems are then discussed in terms of a discretized set of equations of general form.

INTRODUCTION

The stability of elastic systems is often investigated by a static analysis, based on the equilibrium equations. For systems which only involve conservative, nongyroscopic, time-independent forces, such an analysis is justifiable. However, if there are nonconservative, gyroscopic, or time-dependent forces present, a dynamic analysis based on the equations of motion may be required. This can be illustrated by the following two examples.

Consider the cantilevered column shown in Figure 1 (a), with modulus of elasticity E, moment of inertia I of the cross-section, length L, mass per unit length m, and applied load P at the tip. Let $y(x)$ denote an equilibrium configuration. First, assume the load P acts vertically, as in Figure 1 (b). The linear equilibrium equation and boundary conditions are /1/

$$EIy''(x) + P[y(x) - y(L)] = 0, \quad y(0) = 0, \quad y'(0) = 0. \tag{1}$$

This has the trivial solution, $y(x) \equiv 0$, for any value of P, corresponding to the undeflected column. It also has nontrivial solutions (eigensolutions)

$$y(x) = A_n\{1 - \cos[(2n-1)\pi x/(2L)]\}, \quad n = 1,2,\dots, \tag{2}$$

corresponding to the loads (eigenvalues)

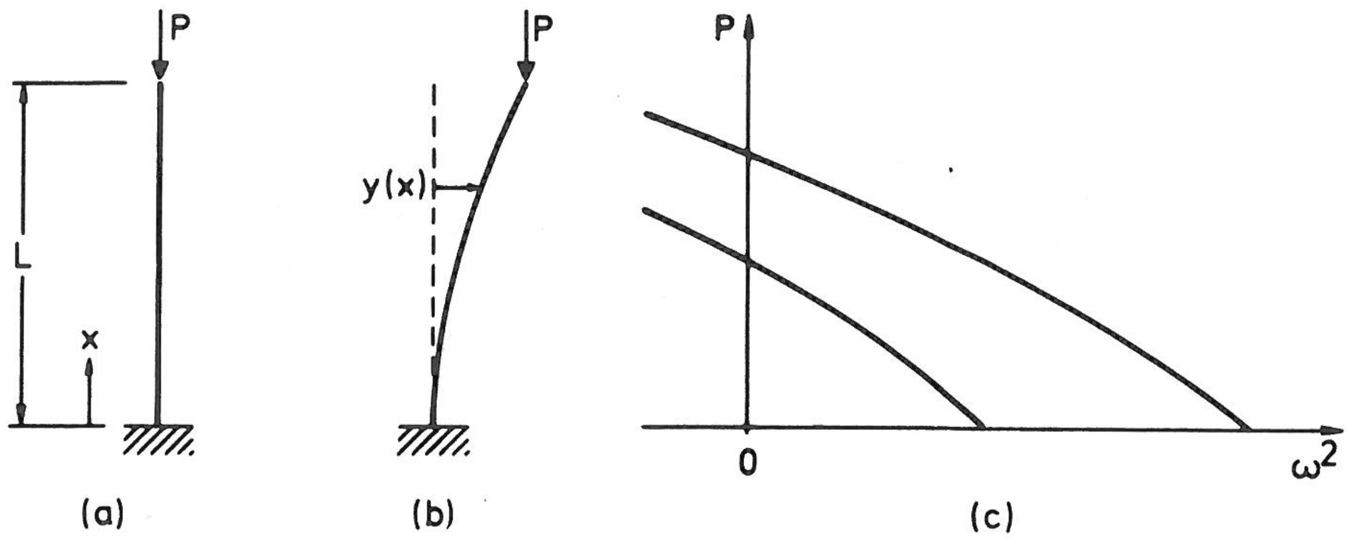

Fig. 1. Column with vertical tip load. (a) Undeflected configuration. (b) Deflected configuration. (c) Characteristic curves.

$$P = (2n-1)^2\pi^2 EI/(4L^2), \quad n = 1,2,\ldots \,. \tag{3}$$

According to the static stability criterion, the critical load is the lowest load at which the linear equilibrium equation possesses a nontrivial solution, i.e., $P = \pi^2 EI/(4L^2)$.

Now assume the load P is a follower load which acts tangentially to the tip of the column when it deflects, as depicted in Figure 2 (b). The linear equilibrium equation and boundary conditions are /2/

$$EIy''(x) + P[y(x) - y(L) + (L-x)y'(L)] = 0, \quad y(0) = 0, \quad y'(0) = 0\,. \tag{4}$$

In this case, the only solution is the trivial one, and the static analysis does not yield the critical load.

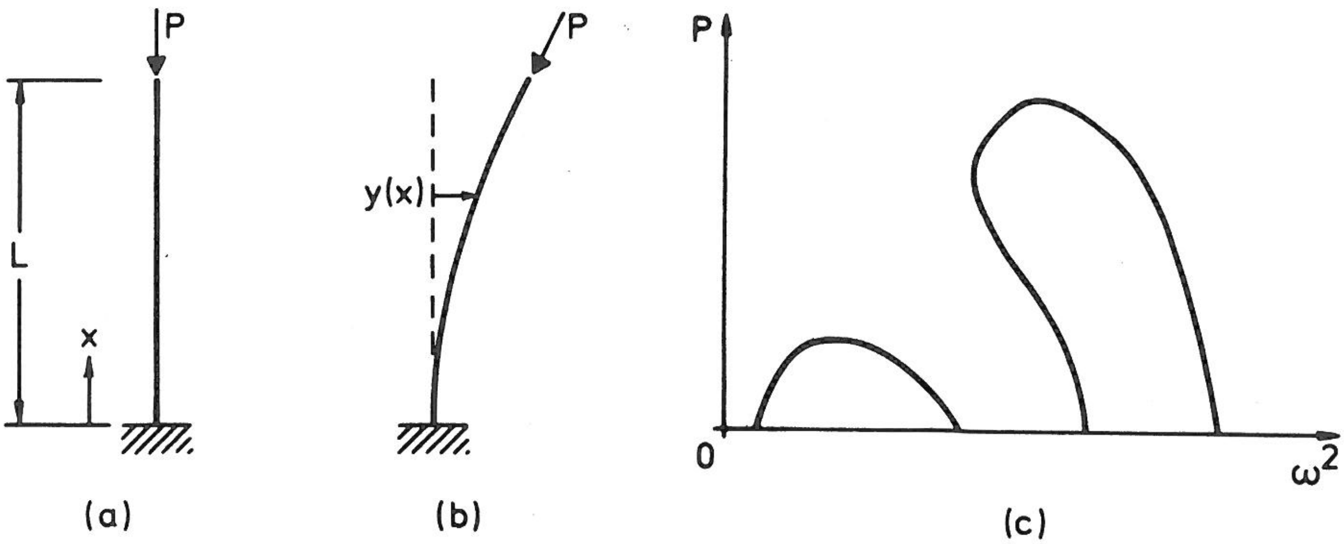

Fig. 2. Column with follower tip load. (a) Undeflected configuration. (b) Deflected configuration. (c) Characteristic curves.

A dynamic analysis will always furnish the critical load, even when a static analysis is sufficient. By investigating the vibration frequencies of the system, one can determine when unstable motions are possible. In some cases, as for the cantilever under a vertical load, the static stability criterion is satisfied when a stable motion changes to an unstable motion. In others, when instability is of the dynamic type, it is not.

Some one-dimensional structures are used here to demonstrate various forms of behavior. Different types of loads are applied. Vibration frequensies are plotted versus a loading parameter, and critical values are determined from these characteristic curves. A general set of discretized equations is then introduced, which is obtained when approximate methods of analysis are utilized, and the stability properties of certain classes of systems are discussed.

CONTINUOUS SYSTEMS

Column with vertical tip load

Consider the column in Figures 1 (a) and 1 (b). Let $y(x)e^{i\omega t}$ denote a motion with vibration mode $y(x)$ and frequency ω. The linear equation of motion is /2/

$$EIy'''' + Py'' - m\omega^2 y = 0, \tag{5}$$

with boundary conditions

$$y(0) = 0, \quad y'(0) = 0,$$
$$EIy''(L) = 0, \quad EIy'''(L) + Py'(L) = 0. \tag{6}$$

The general solution of (5) can be written as

$$y(x) = A \sin \sigma x + B \cos \sigma x + C \sinh \lambda x + D \cosh \lambda x \tag{7}$$

where

$$\sigma = \left[\frac{P + \sqrt{P^2 + 4EIm\omega^2}}{2EI}\right]^{1/2}, \qquad \lambda = \left[\frac{-P + \sqrt{P^2 + 4EIm\omega^2}}{2EI}\right]^{1/2}. \tag{8}$$

Application of (6) to the solution (7) yields four homogeneous equations in A, B, C, and D. One solution is $y(x) \equiv 0$. For a nontrivial solution, the determinant of the 4×4 coefficient matrix is set equal to zero, and one obtains the characteristic equation

$$2\lambda^2\sigma^2 + (\lambda^2 - \sigma^2)\lambda\sigma \sin \sigma L \sinh \lambda L + (\lambda^4 + \sigma^4)\cos \sigma L \cosh \lambda L = 0 \tag{9}$$

where λ and σ are defined by (8).

The transcendental equation (9) can be solved numerically for the vibration frequencies ω as a function of the load P, and the results can be plotted as characteristic curves in the plane of P versus ω^2. There are an infinite number of such curves, since the system is continuous. They are almost linear and have negative slope, as sketched in Figure 1 (c) for the two curves closest to the origin. When P is small, all roots ω^2 are real and positive, corresponding to harmonic motion for each mode. When a curve crosses into the left half plane, so that $\omega^2 < 0$, one of the solutions $y(x)e^{i\omega t}$ grows monotonically with time. This is called instability of the divergence type. The critical load in this example, $P = \pi^2 EI/(4L^2)$, is the lowest value of P at which $\omega^2 = 0$ is a root. This value was obtained in the previous section by a static analysis, which corresponds to the case of zero frequency in the dynamic analysis.

Column with follower tip load

Consider the column in Figures 2 (a) and 2 (b). The equation of motion is still (5), with general solution (7), but the last boundary condition in (6) is replaced by $EIy'''(L) = 0$. The first few characteristic curves have the form shown in Figure 2 (c). As in Figure 1 (c), all roots ω^2 are real and positive when P is small, and the column is then stable. However, the curves here do not intersect the P axis, which corresponds to the fact that there are no nontrivial static solutions, as was noted earlier.

As P is increased in Figure 2 (c), the lowest two roots ω^2 coalesce and then become a complex pair. Two of the resulting complex frequencies have negative imaginary part, and there exist unstable motions which oscillate with increasing amplitudes. This type of instability is called flutter. The critical load, $P = 2.031\ \pi^2 EI/(L^2)$, corresponds to the lowest coalescence point /2/.

In the previous problem, the vertical tip load was conservative, and instability was of the divergence type. Follower loads are nonconservative, since the work done depends on the path taken by the structure and not just on the initial and final states. Nonconservative loads do not always cause flutter instability, however, as demonstrated in the next example.

Column with distributed follower load

Consider the column shown in Figures 3 (a) and 3 (b). The distributed follower load has magnitude p per unit length. The equation of motion is now /3/

$$EIy'''' + (L-x)py'' - m\omega^2 y = 0 . \tag{10}$$

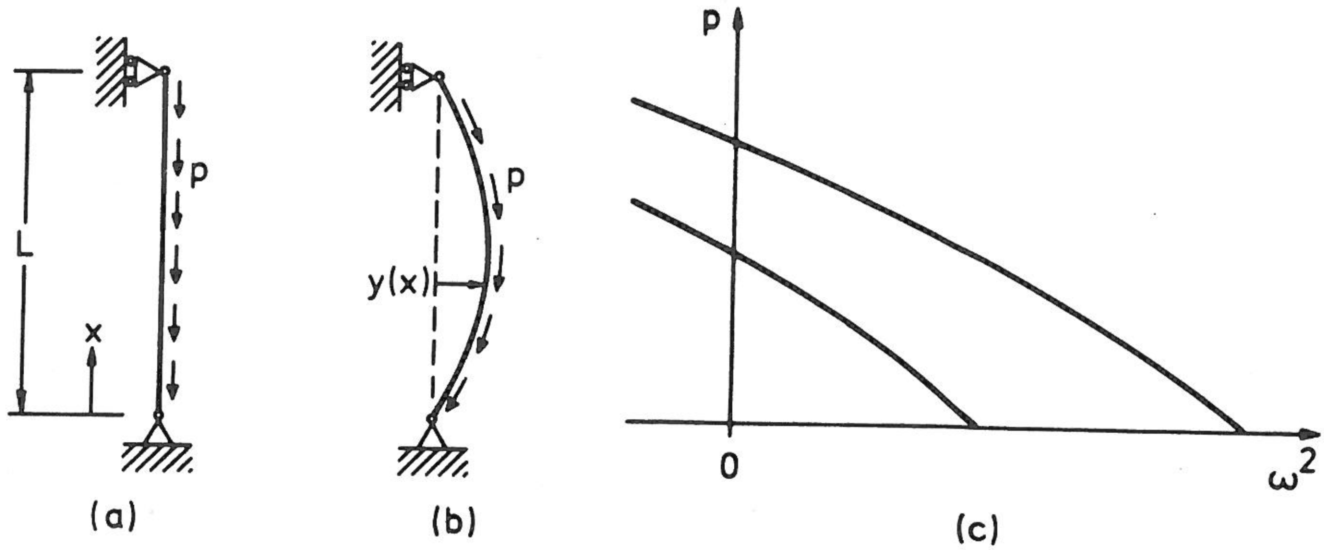

Fig. 3. Column with distributed follower load. (a) Undeflected configuration. (b) Deflected configuration. (c) Charasteristic curves.

At the simply supported ends, $y = 0$ and $EIy'' = 0$. Since (10) has a variable coefficient, it is solved by an approximate method (e.g., Galerkin's method). The resulting characteristic curves have the form depicted in Figure 3 (c), and instability is of the divergence type.

Panel in airflow

Aerodynamic loads are also nonconservative. Consider the cantilevered panel shown in Figures 4 (a) and 4 (b). Assume the panel is long in the direction perpendicular to the airflow, so that the mode shapes y are just functions of x. The panel has thickness h, density ρ, and Poisson's ratio ν. The flexural rigidity D is given by $D = Eh^3/[12(1 - \nu^2)]$. The density of the air is ρ_o, the speed of sound is c_o, and the airspeed is v. If the airflow is in the positive x direction and is supersonic, the linear equation for transverse vibrations is sometimes taken to be /2/

$$Dy'''' + \rho_o c_o v y' - \rho h \omega^2 y = 0 \qquad (11)$$

when damping is neglected. The boundary conditions are the same as for the column with a follower load at its tip. As the airspeed is increased, two frequencies coalesce and flutter instability occurs, as shown in Figure 4 (c). However, if the airflow is in the negative x direction, which corresponds to $v < 0$, divergence occurs, as seen in Figure 4 (c) when one root ω^2 becomes negative.

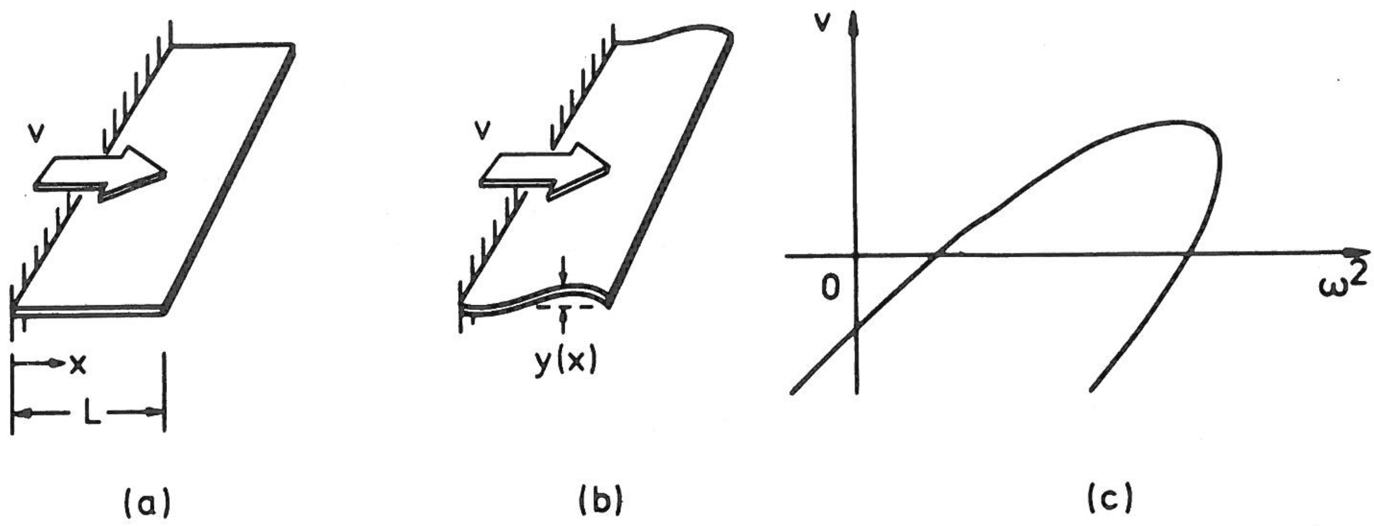

Fig. 4. Panel in airflow. (a) Undeflected configuration. (b) Deflected configuration. (c) Characteristic curves.

Rotating shaft

As an example of a conservative system with gyroscopic forces, consider the rotating shaft depicted in Figure 5 (a). Let modes $y_1(x)$ and $y_2(x)$ correspond to principal axes fixed in the cross-section, with principal moments of inertia I_1 and I_2. The angular velocity is denoted Ω. The governing linear vibration equations are /4/

$$\begin{aligned} EI_1 y_1'''' - 2i\omega m\Omega y_2 - (\Omega^2 + \omega^2) m y_1 &= 0, \\ EI_2 y_2'''' + 2i\omega m\Omega y_1 - (\Omega^2 + \omega^2) m y_2 &= 0, \end{aligned} \qquad (12)$$

and the modes satisfy simply supported boundary conditions at the ends. The equations are coupled due to the Coriolis forces, while the terms in (12) involving Ω^2 correspond to centrifugal forces.

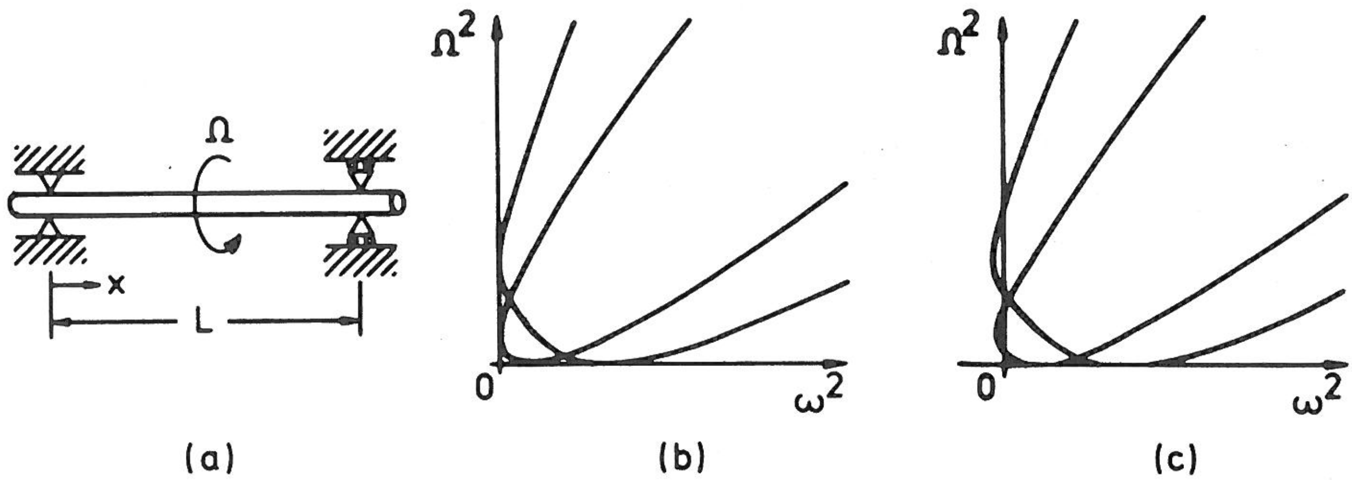

Fig. 5. Rotating shaft. (a) Undeflected configuration. (b) Characteristic curves when $I_1 = I_2$. (c) Characteristic curves when $I_1 \neq I_2$.

The characteristic equation is a function of Ω^2 and ω^2. In the plane of Ω^2 versus ω^2, the characteristic curves are parabolas which are symmetric with respect to the ray $\Omega^2 = \omega^2$. If $I_1 = I_2$, as for a circular cross-section, they are tangential to the axes, as sketched in Figure 5 (b). At the values of Ω^2 where $\omega^2 = 0$ is a root, there is a solution which grows with time and divergence occurs. Between these critical speeds, all roots ω^2 are real and positive, so that the system is stable. (This existence of stability past the initial critical value is called gyroscopic stabilization /4/.) If $I_1 \neq I_2$, the curves intersect the Ω^2 axis, as shown in Figure 5 (c), and there are intervals along the Ω^2 axis corresponding to divergence instability, where $\omega^2 \leq 0$ for at least one of the modes.

Pipe conveying fluid

Consider the simply supported pipe shown in Figures 6 (a) and 6 (b). A fluid with mass per unit length m_f flows through the pipe with speed u. The linear vibration equation is /5/

$$EIy'''' + m_f u^2 y'' + 2i\omega m_f u y' - (m_f + m)\omega^2 y = 0. \tag{13}$$

Coriolis and centrifugal forces are present, caused by the fluid flowing along a curved path when the pipe deflects. The characteristic equation is a function of u^2 and ω^2, and the characteristic curves have a form as shown in Figure 6 (c) /4/. The system is conservative, and initial instability is of the divergence type. Flutter instability may occur at higher values of the loading parameter, as shown.

If the pipe has a free end, however, the system is nonconservative. For the cantilevered pipe shown in Figures 7 (a) and 7 (b), the governing equation is (13) and the boundary conditions are the same as for the column with

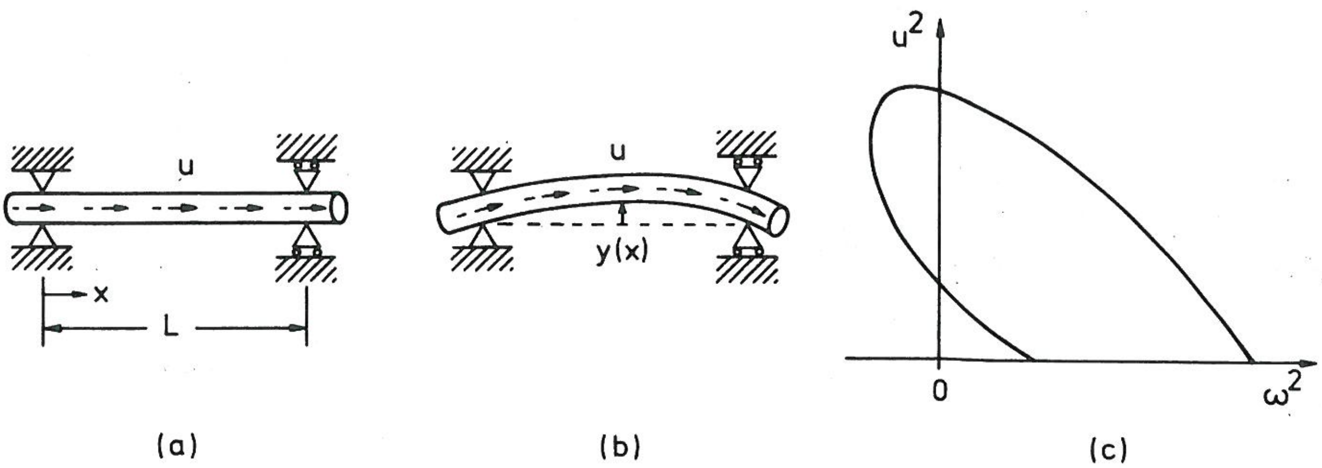

Fig. 6. Simply supported pipe conveying fluid. (a) Undeflected configuration. (b) Deflected configuration. (c) Characteristic curves.

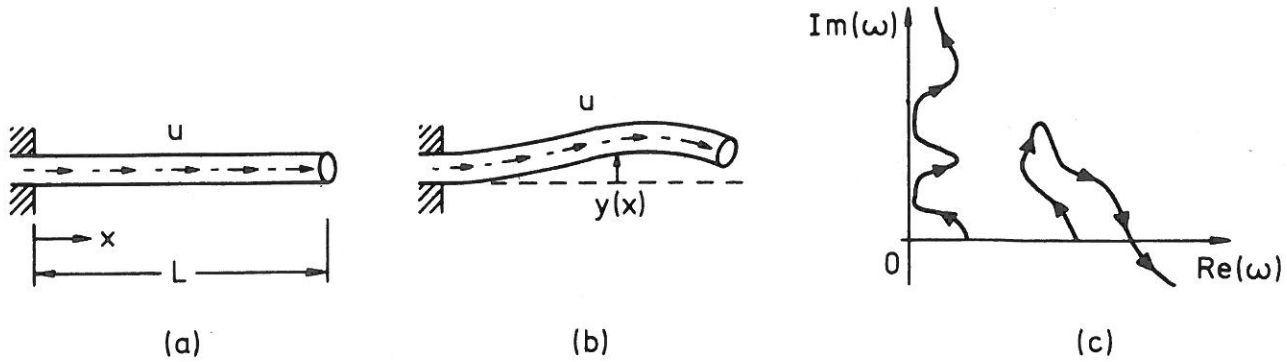

Fig. 7. Cantilevered pipe conveying fluid. (a) Undeflected configuration. (b) Deflected configuration. (c) Argand diagram.

a follower load at its tip. The critical speed would be different if the flow direction were reversed, and therefore the characteristic equation is a function of u and not just u^2. It is also a function of ω, rather than ω^2. For small values of u, the roots ω are generally complex. Dynamic instability occurs when the imaginary part of one of the roots becomes negative. Often the roots are plotted in an Argand diagram (real part of ω versus imaginary part of ω), as u is increased, until one of the modes becomes unstable /5/. This is sketched in Figure 7 (c) for two of the frequencies, where arrows indicate the direction of changing frequency as u increases. Instability here is of the flutter type and is sometimes called single-mode flutter, since it does not involve the coalescence of two frequencies and modes.

Shallow arch

In the previous examples, a trivial equilibrium state existed for any value of the loading parameter (P, p, v, Ω, or u). The stability of this state was determined by investigating the frequencies of small vibrations about that state. Some systems, however, do not possess such a trivial state and exhibit deformations before instability occurs. An example is provided by the shallow arch in Figure 8 (a).

Let $y_1(x)$ denote the equilibrium configuration of the arch under a uniformly distributed load q, and let $y_0(x)$ be the unloaded shape (i.e., when $q = 0$). The nonlinear equation for $y_1(x)$ is /6/

$$EI(y_1'''' - y_0'''') + \frac{EA}{2L}\, y_1'' \int_0^L [(y_0')^2 - (y_1')^2]dx = -q \qquad (14)$$

where A is the area of the cross-section. At the simply supported ends, $y_1 = 0$ and $EI(y_1'' - y_0'') = 0$. The system is conservative. For small vibrations about $y_1(x)$, with mode $y(x)$ and frequency ω, the governing

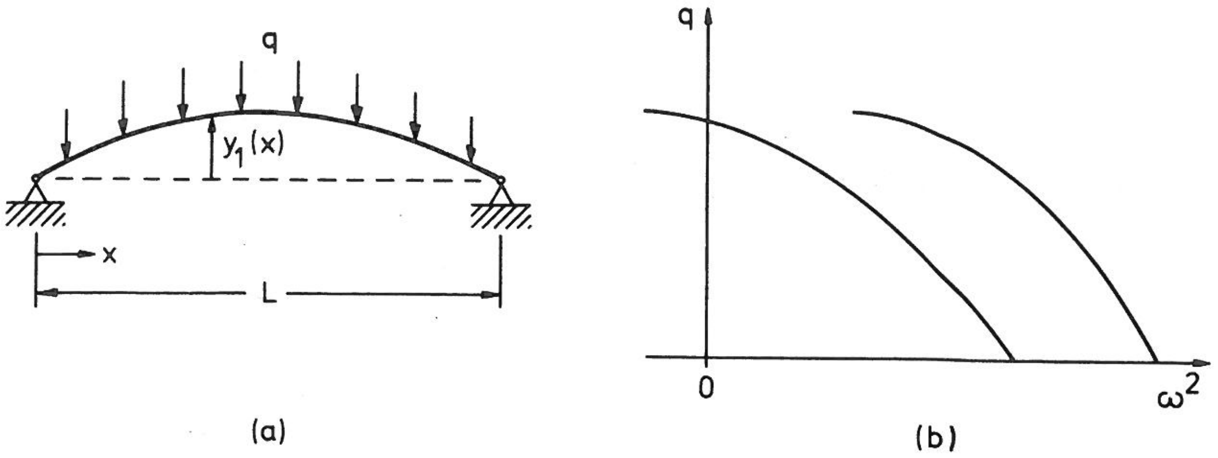

Fig. 8. Shallow arch. (a) Equilibrium configuration under distributed load. (b) Characteristic curves.

equation is

$$EIy'''' + \frac{EA}{2L} y'' \int_0^L [(y_0')^2 - (y_1')^2]dx - \frac{EA}{L} y_1'' \int_0^L y' y_1' dx - m\omega^2 y = 0, \tag{15}$$

with $y = 0$ and $EIy'' = 0$ at the ends of the arch. For small values of q, all roots ω^2 are real and positive, as shown in Figure 8 (b). If one of the roots becomes negative, the equilibrium state $y_1(x)$ becomes unstable and the arch suddenly deflects into an inverted configuration (i.e., snap-through occurs).

DISCRETIZED EQUATIONS

In most cases, it is not possible to obtain the characteristic equation analytically as in (9). Approximate methods usually must be applied, such as the Galerkin, Rayleigh-Ritz, finite difference, or finite element method. Consider the types of problems discussed previously in which a trivial equilibrium state exists and small vibrations are investigated (i.e., all examples except the arch). An approximate analysis leads to a homogeneous matrix equation, and the approximate characteristic equation is obtained by setting the determinant of the coefficient matrix equal to zero. This equation often can be written in the following general form, if damping is included /4/, /7/:

$$|K + \eta S - \eta^2 A + i\omega(D + \eta G) - \omega^2 M| = 0. \tag{16}$$

In (16), η (or η^2) is the loading parameter, with η^2 corresponding to P, p, v, Ω^2, or u^2 in Figures 1-7. K is called the stiffness matrix of the unloaded structure, S represents certain damping terms in rotating systems, A is the loading matrix, D is the usual damping matrix, G is a gyroscopic matrix corresponding to Coriolis terms, and M is the mass or inertia matrix. K and M are symmetric and positive definite. S and G are skew-symmetric. A is symmetric if the load is conservative, and asymmetric otherwise. D is symmetric and positive semi-definite. If damping affects all possible motions, D is positive definite and the system is said to be completely dissipative /7/. In that case, critical values of the loading parameter can be determined by the Routh-Hurwitz conditions.

If S = 0, D = 0, G = 0, and A is symmetric, the system is conservative and nongyroscopic. The column with vertical tip load is an example of this class. Instability occurs if $Re(\omega^2) \leq 0$ for one or more of the roots.

If S = 0, D = 0, G = 0, and A is asymmetric, the system is called circulatory /7/. The load is nonconservative and depends on the system's displacements but not on velocity, as is the case for the examples corresponding to Figures 2-4. The system is stable if all roots ω^2 are real and positive. Instability may be of the divergence type, with a zero frequency at the critical value of the loading parameter, or of the flutter type, with two frequencies coalescing at the critical value.

If S = 0, D = 0, and A is symmetric, the system is conservative and the force corresponding to G, which does no work during motion, is called gyroscopic /7/. The rotating shaft and simply supported pipe in Figures 5 and 6 are examples of this class /4/. As the loading parameter is increased, instability of the divergence type occurs first, which then may be followed by further intervals of stability (gyroscopic stabilization) and possibly by flutter instability.

If S = 0, G = 0, and A is symmetric, then damping is included in the analysis of a nongyroscopic system which would otherwise be conservative. The inclusion of damping here does not affect the critical load, which corresponds to a zero frequency /4/, /7/.

If S = 0, G = 0, and A is asymmetric, i.e., damping is added to a circulatory system, the onset of divergence is again the same as if D = 0. However, a critical value of the loading parameter for flutter instability may be decreased or increased by the presence of damping /8/. Therefore, it is important to include damping in the analysis of a nonconservative system which can exhibit flutter.

If S = 0 and A is symmetric, the system corresponds to a conservative gyroscopic one to which a certain type of damping has been added in the analysis. Initial instability is of the divergence type and the corresponding critical value is unaffected by the inclusion of damping. If D is positive definite, the system is unstable at all higher values of the loading parameter (i.e., there is no gyroscopic stabilization) /7/.

The matrix S may be nonzero for systems in which the coordinate system is rotating but there is damping which is proportional to the absolute velocity. In that case, the damping alters critical values for both divergence and flutter instability /4/.

CONCLUDING REMARKS

The stability of equilibrium states of elastic systems can, and often must, be investigated by means of a dynamic analysis. Small vibrations about the equilibrium state are considered, and the vibration frequencies ω are determined. When the characteristic equation, which relates the frequencies to the loading parameter, is a function of ω^2, then it is convenient to plot the loading parameter as a function of ω^2. The resulting characteristic curves may have various forms, depending on the types of loads acting on the system.

For small values of the loading parameter, the system is stable, and for each vibration mode the corresponding frequency ω either is real or is complex with positive imaginary part. The transition to instability may occur in several ways. At the onset of divergence instability, ω is zero for a mode. For one type of flutter instability, the real frequencies of two modes coalesce and then become a complex pair. For single-mode flutter, the imaginary part of a complex frequency passes through zero to negative values.

These features have been illustrated by some examples involving one-dimensional structures. Classifications of types of loads have been given in terms of a discretized set of equations, applicable to certain classes of systems which possess a trivial (undeflected) equilibrium state for all values of the loading parameter. Loads which depend explicitly on time have not been considered, nor have multiple loading parameters.

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