ANALYTICAL SOLUTION FOR EIGENVALUES OF A ROTATING TIMOSHENKO BEAM

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SUMMARY: In this paper the eigenvalue equation for a rotating hinged-hinged beam has been derived. The effects of gyration, shear deflection and rotary inertia, which are essential for thick beams, have been considered. When the cross-coupling between these effects is taken into account the number of eigenfrequencies is doubled.

INTRODUCTION

The one-dimensional beam theory is a simplification of the general three-dimensional elasticity theory. In the classical Euler-Bernoulli beam theory only the deflection due to bending is considered. When this theory is used for dynamical calculations, the eigenfrequencies will generally be over-estimated.

A better approximation is obtained when the deflection due to shear deformation and rotary inertia also are considered. This improved beam-theory is called Timoshenko-theory.

A number of investigations, where the Timoshenko beam theory is used for developing finite element programs, have been published /1-5/.

Dimentberg /6/ has derived analytically the eigenvalue equation for a rotating hinged-hinged beam. The rotary inertia and shear deflection were considered.

By studying the analytical solution, after introduction of some non-dimensional parameters, one can better understand the influence of shear deflection rotary inertia and the cross-coupling of these phenomena.

EQUATIONS OF MOTION FOR A BEAM WITH ROTARY INERTIA

Dimentberg /6, pp. 92/ has given the following equations of motion for a circular beam, taking the rotary inertia into account:

$$\frac{\partial T_x}{\partial s} = \rho A \frac{\partial^2 u_x}{\partial t^2} , \qquad (1)$$

$$EI \frac{\partial^3 u_x}{\partial s^3} + T_x = \rho I \frac{\partial}{\partial s} \left(\frac{\partial^2 u_x}{\partial t^2}\right) + 2\Omega \rho I \frac{\partial}{\partial s} \left(\frac{\partial u_y}{\partial t}\right), \tag{2}$$

where

A	=	area [m^2],
E	=	Young's modulus [N/m^2],
I	=	moment of inertia of the cross-section area about an axis normal to the plane of bending [m^4],
s	=	coordinate along the axis [m],
t	=	time [s],
T_x	=	shearing force [N],
u_x, u_y	=	radial deflections in a non-rotating coordinate system [m],
ρ	=	density [kg/m^3],
Ω	=	speed of revolution [rad/s].

Equations (1) and (2) concern the translational and rotational motion respectively of an infinitesimally thin shaft segment (Figure 1).

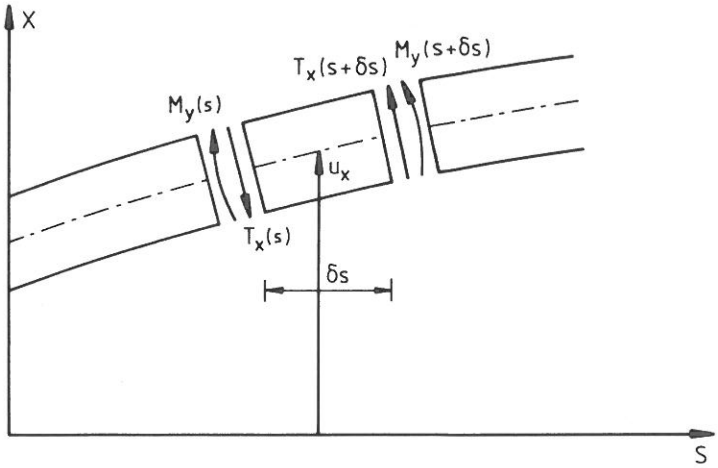

Figure 1. An infinitesimal shaft segment.

For explanation of the terms appearing in the equations one sees that for the segment

resultant force	=	$T_x(s + \delta s) - T_x(s)$,
acceleration	=	$\frac{\partial^2 u_x}{\partial t^2}$
mass	=	$\rho A \ \delta s$,
resultant moment	=	$[T_x(s + \delta s) - T_x(s)] \ \delta s/2 + M_y(s + \delta s) - M_y(s)$,

angular acceleration $= \frac{\partial^2}{\partial t^2} (\frac{\partial u_x}{\partial s})$,

diametral mass moment of inertia $= \rho I \, \delta s$,

polar mass moment of inertia $= 2\rho I \, \delta s$,

moment due to gyral action $= 2\rho I \, \Omega \frac{\partial}{\partial t} (\frac{\partial u_y}{\partial s})$.

From the elementary beam theory it is known that

$$M_y(s) = EI \frac{\partial^2 u_x}{\partial s^2} .$$

By taking the corresponding equations in the y-direction, eliminating the shearing forces and introducing the complex quantity

$$z = u_x + iu_y \tag{3}$$

one obtains

$$EI \, z^{IV} - \rho I(\ddot{z}'' - 2i \, \Omega \dot{z}'') + \rho A \, \ddot{z} = 0, \tag{4}$$

where $\dot{z}$ is the time derivative and z' the length derivative.

For a hinged-hinged beam with length L the boundary conditions are:

$$z(0) = z(L) = 0,$$

$$z''(0) = z''(L) = 0. \tag{5}$$

The common trial

$$z = Z \sin \frac{\pi n s}{L} e^{i\omega t}, \tag{6}$$

in which n refers to the n:th eigenvalue, gives

$$\left[1 + \frac{\pi^2 n^2 I}{AL^2} (1 - 2 \frac{\Omega}{\omega}) \right] \omega^2 - \frac{EI}{\rho A} \frac{\pi^4 n^4}{L^4} = 0. \tag{7}$$

By varying the whirling speed ratio (Ω/ω) the different critical speeds are obtained:

$\Omega/\omega = 1$ critical speeds for forward precession,

$\Omega/\omega = 0$ eigenfrequencies non-rotating beam,

$\Omega/\omega = -1$ critical speeds for backward precession.

By introducing the non-dimensional parameters μ and r'

$$\mu = (\frac{\omega_c}{\omega})^2, \quad \omega_c = \frac{\pi^2}{L^2} \sqrt{\frac{EI}{\rho A}}, \quad r' = \frac{1}{L} \sqrt{\frac{I}{A}} \tag{8}$$

equation (7) can be written in the following non-dimensional form:

$$\mu = \left[1 + (\pi n r')^2 \left(1 - \frac{2\Omega}{\omega}\right)\right] \frac{1}{n^4} \; . \tag{9}$$

If the non-dimensional radius of gyration r' is neglected, the eigenfrequencies of an Euler-Bernoulli beam are obtained /7, pp. 218-221/:

$$\omega = \frac{\omega_c}{\sqrt{\mu}} = n^2 \omega_c \qquad n = 1,2, \ldots\ldots\ldots$$

EQUATION OF MOTION FOR A TIMOSHENKO BEAM

In the previous section the eigenvalue equation for a hinged-hinged beam was derived taking the rotary inertia into account. Now also the deflection due to shear will be considered. In equation (2) it was assumed that

$$\frac{\partial u_x}{\partial s} = \Theta_y , \tag{10}$$

where Θ_y is the rotation of the cross-section about the y-axis.

If the deflection due to shear is considered the expression (10) is modified to (Figure 2):

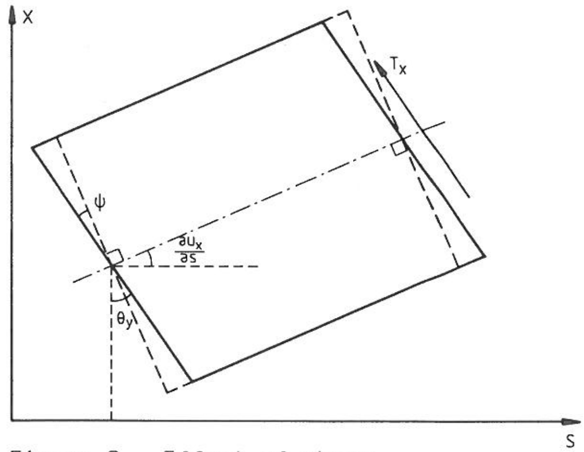

Figure 2. Effect of shear.

$$\frac{\partial u_x}{\partial s} = \Theta_y + \frac{T_x}{kGA} , \tag{10'}$$

in which k is the shear coefficient and G the shear modulus.

Equation (2) is now changed to

$$EI \frac{\partial^2 \Theta_y}{\partial s^2} + T_x = \rho I \frac{\partial^2}{\partial t^2} (\Theta_y) + 2\Omega\rho I \frac{\partial}{\partial t} (\Theta_x) . \tag{2'}$$

After multiplying (2') with the operator

$$\frac{1}{\rho A}\frac{\partial^2}{\partial s^2} - \frac{1}{kGA}\frac{\partial^2}{\partial t^2}$$

and making use of (1) and (10') the shearing force is eliminated. By using the trial function

$$\Theta = \Theta_o \cos\frac{n\pi s}{L} e^{i\omega t}$$

one obtains the eigenvalue equation for a hinged-hinged beam:

$$\frac{\rho I}{GAk}\left[1 - 2\frac{\Omega}{\omega}\right]\omega^4 - \left[1 + \frac{I}{A}\left(\frac{n\pi}{L}\right)^2\left(1 - 2\frac{\Omega}{\omega} + \frac{EI}{GAk}\cdot\left(\frac{n\pi}{L}\right)^2\right)\right]\omega^2$$

$$+ \frac{EI}{\rho A}\left(\frac{\pi n}{L}\right)^4 = 0. \qquad (11)$$

By using the non-dimensional parameters μ and r' defined by equation (8) and the well-known relationship $E = 2(1 + \nu)G$ the non-dimensional equation

$$\mu^2 - \{1 + (\pi n r')^2\left[1 - 2\frac{\Omega}{\omega} + \frac{2(1+\nu)}{k}\right]\}\frac{\mu}{n^4} + 2\frac{(1+\nu)}{k}\cdot$$

$$\left[1 - 2\frac{\Omega}{\omega}\right]\cdot\left(\frac{\pi r'}{n}\right)^4 = 0 \qquad (12)$$

is obtained.

We have now derived an equation of second degree in μ. When the shear deflection was neglected a first degree equation (9) was obtained. This is due to the fact that by taking the shear deflection into account an extra set of degrees of freedom is introduced. The beam can deflect as a sinus wave in principally two different ways (Figure 3):

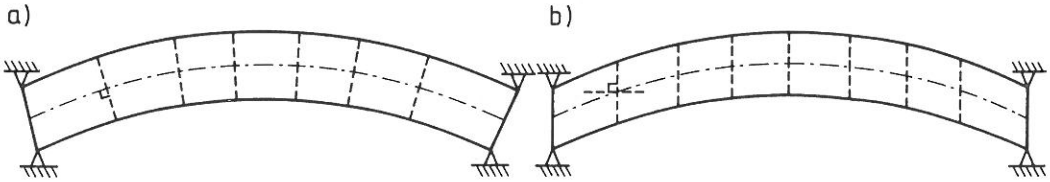

Figure 3. (a) Bending mode, (b) shear mode.

For each value of n, two eigenfrequencies are obtained. Normally the lower frequency corresponds essentially to the bending mode and the higher to the shear mode.

One can interpret the terms of the eigenvalue equation (12) in the following way:

μ^2 -

$\{1$ + bending deflection,

$(\pi n r'^2)\left[1 - 2\frac{\Omega}{\omega}\right]$ rotary inertia,

$+ (\pi n r'^2)\frac{2(1+\nu)}{k}\} \cdot$ shear deflection,

$\frac{\mu}{n^4}$ +

$\frac{2(1+\nu)}{k}\left[1 - \frac{2\Omega}{\omega}\right](\frac{\pi r'}{n})^4$ coupling between rotary inertia and shear deflection.

When the coupling between rotary inertia and shear deflection is neglected only one trivial solution for each n is obtained. The other solution $\mu = 0$, corresponds to an infinite eigenvalue.

FINITE ELEMENT APPROACH

A number of papers have been published, concerning finite elements where the Timoshenko effects (rotary inertia and shear deflection) have been considered. A review was given by Thomas & Abbas /4/.

A beam element used for the classical Euler-Bernoulli theory has four degrees of freedom in each bending plane. It was proved that the number of degrees of freedom is doubled when the coupling between rotational inertia and shear deflection is taken into account. A proper Timoshenko beam element should therefore have eight degrees of freedom. Such an element has been developed by Thomas & Abbas /4/.

By neglecting the coupling between rotary inertia and shear deflection, but still taking both phenomena separately into account, the number of degrees of freedom is reduced to four. This is indicated by the structure of equation (12). Such an element has been developed by Egle /2/. In a recent paper by Nelson /11/ a finite element for a rotating shaft was presented.

THE SHEAR COEFFICIENT

Different expressions for the shear coefficient k have been published by several authors /8-10/. Timoshenko /8/ originally proposed $k = 2/3$ for a rectangular beam and $k = 3/4$ for a beam with circular cross section. These values have been found to be too low. The value $k \simeq 0.85 - 0.90$ is a more realistic estimation. An experimental study has been performed by Kaneko /12/. A recent paper has been presented by Stephen /13/.

EXAMPLE 1

Eigenfrequencies and critical speeds have been calculated for a hinged-hinged beam with the following non-dimensional parameters: $\nu = 0.3$, $k = 0.85$, $r' = 0.008$. The following results were obtained:

	Eigenfrequency ω/ω_c			
	Timoshenko theory			Euler-Bernoulli Theory
Mode	Backward critical	Non-rotating beam	Forward critical	
1	0.9981	0.9987	0.9993	1
2	3.970	3.980	3.990	4
3	8.850	8.898	8.947	9
4	15.684	15.684	15.833	16

The eigenfrequencies are expressed in non-dimensional form ω/ω_c, where

$$\omega_c = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho A}}, \quad \frac{\omega_c}{\omega} = \frac{1}{\sqrt{\mu}} .$$

μ is obtained from equation (12).

It is to be noted that for the cases of backward precession and non-rotating beam, a second eigenfrequency was obtained for each n. These eigenfrequencies correspond to the shear deflection modes. The orders of magnitude are: non-rotating beam $\omega/\omega_c \simeq 900$, backward precession $\omega/\omega_c \simeq 500$.

When the cross-coupling between rotary inertia and shear deflection was neglected the following results were obtained:

	Eigenfrequency ω/ω_c		
	Timoshenko theory without cross-coupling		
Mode	Backward critical	Non-rotating beam	Forward critical
1	0.9981	0.9987	0.9994
2	3.970	3.980	3.990
3	8.849	8.898	8.948
4	15.532	15.682	15.836

EXAMPLE 2

The same calculations as in the first example have been carried out for a ten times thicker beam: $\nu = 0.3$, $k = 0.85$, $r' = 0.08$. The following results were obtained from equation (12):

	Eigenfrequency ω/ω_c			
	Timoshenko theory			Euler-Bernoulli theory
Mode	Backward critical	Non-rotating beam	Forward critical	
1	0.8589	0.8957	0.9363	1
2	2.653	2.884	3.124	4
3	4.726	5.218	5.628	9
4	6.897	7.363	8.118	16

The Timoshenko effects now have a considerable effect. Also the cross-coupling term in equation (11) is not negligible in this case:

	Eigenfrequency ω/ω_c		
	Timoshenko theory without cross-coupling		
Mode	Backward critical	Non-rotating beam	Forward critical
1	0.8504	0.8922	0.9407
2	2.514	2.811	3.244
3	4.269	4.949	6.109
4	5.495	7.084	9.116

It is seen that the eigenfrequencies for non-rotating beam are underestimated, but the forward critical speeds are overestimated, when the cross-coupling term is neglected.

SUMMARY AND CONCLUSIONS

By using beam theory a three-dimensional elastic problem is reduced into an one-dimensional problem. In the Euler-Bernoulli beam theory only the displacement due to bending is considered. When this theory is used for dynamical calculations, too high eigenfrequencies are generally obtained. The Timoshenko beam theory takes also the shear deflection and rotary inertia into account. These effects are not negligible for thick beams.

In this paper an analytical solution for a rotating beam given by Dimentberg is improved by introducing some non-dimensional parameters. By studying this solution one can understand the influence of shear deflection, rotary inertia and the cross-coupling of these effects.

The analytical solution can also be used for testing computer programs.

NOTATION

A	cross-sectional area [m^2]
D	diameter of beam [m]
E	Young's modulus [N/m^2]
G	shear modulus [N/m^2]
I	moment of inertia of the cross-sectional area about a principal axis normal to the plane of bending [m^4]
k	shear coefficient
L	length of beam [m]
M	bending moment [Nm]
r'	non-dimensional radius of gyration
s	length coordinate [m]
t	time [s]
T_x	shearing force [N]
u_x, u_y	radial deflection [m]
ν	Poisson ratio
ρ	density [kg/m^3]
Θ_y	rotation of cross-section about the y-axis [rad]
Ω	speed of revolution [rad/s]
ω	eigenfrequency [rad/s]
ω_c	lowest eigenfrequency of a hinged-hinged beam [rad/s]

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