THERMAL STRESSES INDUCED BY THE POINT SOURCE IN AN ISOTROPIC CIRCULAR RING

Rakenteiden Mekaniikka
8 (1975) 2, s. 138...154
Rakenteiden Mekaniikan
Seura, Helsinki

J.S. KAJASTE-RUDNITSKI

SUMMARY

The problem has two parts, the first of which concerns determination of the temperature field, with temperature being expressed in terms of Fourier series in the circumferential direction. The unknown coefficients are functions of the radial coordinate, and are derived, with the help of Bessel functions, from the resultant ordinary differential equations. The second part concerns solution of the stresses attributable to the temperature field obtained. This is achieved by use of the thermoelastic potential, which is expressed similarly to the temperature.

1. TEMPERATURE FIELD INDUCED BY THE POINT SOURCE

Let us consider a circular ring with an inner radius of r_o, an outer radius of R and a thickness h. The equation of heat conduction is [1]

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta 2} - \kappa^2 T(r,\theta) = -\frac{1}{\lambda}W(r,\theta). \qquad (1.1)$$

Here

$\kappa^2 = \frac{2c}{\lambda h}$,

$T(r,\theta)$ = temperature in the ring (outside the ring, temperature is zero),

c = coefficient of heat transfer,

λ = coefficient of heat conduction,

$W(r,\theta)$ = rate of heat generation per unit volume.

The boundary conditions are

$$\left.\frac{\partial T}{\partial r}\right|_{\substack{r = r_o \\ r = R}} = 0 . \qquad (1.2)$$

The function $T(r,\theta)$ must be periodic, with a period of 2π. So

$$T(r,\theta) = \frac{1}{2}a_o(r) + \sum_m a_m(r)\cos m\theta + b_m(r)\sin m\theta . \qquad (1.3)$$

After substitution (1.3) into (1.1), there is derived

$$\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr} a_o) - \kappa^2 a_o(r) = -\frac{1}{\lambda}\alpha_o(r)$$

$$\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr} a_m) - \frac{m^2}{r^2} a_m(r) - \kappa^2 a_m(r) = -\frac{1}{\lambda}\alpha_m(r) \qquad (1.4)$$

$$\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr} b_m) - \frac{m^2}{r^2} b_m(r) - \kappa^2 b_m(r) = -\frac{1}{\lambda}\beta_m(r)$$

Here

$$\alpha_o(r) = \frac{1}{\pi}\int_o^{2\pi} W(r,\theta)dt ,$$

$$\alpha_m(r) = \frac{1}{\pi}\int_o^{2\pi} W(r,\theta)\cos m\theta dt ,$$

$$\beta_m(r) = \frac{1}{\pi}\int_o^{2\pi} W(r,\theta)\sin m\theta dt .$$

The boundary conditions for $a_o(r)$, $a_m(r)$, $b_m(r)$ are homogeneous.

The general solution of the first equation of (1.4) is

$$a_o(r) = AI_o(\kappa r) + BK_o(\kappa r) +$$

$$- \frac{1}{\lambda}\int_o^r \xi \; I_o(\kappa\xi)K_o(\kappa r) - I_o(\kappa r)K_o(\kappa\xi) \;\; \alpha_o(\xi)\, d\xi . \qquad (1.5)$$

Constants A and B are determinable from (1.2):

$$A = - \frac{1}{\lambda I_o'(\kappa r_o)K_o'(\kappa R) - I_o'(\kappa R)K_o'(\kappa r_o)} \quad K_o'(\kappa R)K_o'(\kappa r_o)\int_{r_o}^{R} \xi I_o(\kappa\xi)\alpha_o(\xi)d\xi +$$

$$-I_o'(\kappa R)K_o'(\kappa r_o) \int_{r_o}^{R} \xi K_o(\kappa\xi)\alpha \; (\xi)d\xi ,$$

(1.6)

$$B = - \frac{1}{\lambda \; K_o'(\kappa r_o)I_o'(\kappa R) - K_o'(\kappa R)I_o'(\kappa r_o)} \quad K_o'(\kappa R)I_o'(\kappa r_o) \int_{r_o}^{R} \xi I_o(\kappa\xi)\alpha_o(\xi)d\xi +$$

$$-I_o'(\kappa R)I_o'(\kappa r_o) \int_{r_o}^{R} \xi K_o(\kappa\xi)\alpha_o(\xi)d\xi .$$

Here $I_o(\kappa r)$ and $K_o(\kappa r)$ are the Bessel functions of an imaginary argument.

There is now taken a point source of heat generation of strength W per unit time, first by means of a spacewise constant $W(r,\theta)$ in a small volume formed by the surfaces $r = \rho$, $r = \rho + \delta$, $\theta = \phi$, $\theta = \phi + \chi$, $Z = 0$, $Z = h$: the volume is then allowed to shrink to zero, with the total strength W being kept constant. When $\delta \to 0$ and $\chi \to 0$, then

$$\int_{r_o}^{R} \xi I_o(\kappa\xi)\alpha_o(\xi)d\xi = \lim_{\substack{\delta\to 0\\ \chi\to 0}} \frac{W}{\delta\chi h\rho} \int_{\rho}^{\rho+\delta} \xi\, I_o(\kappa\xi)d\xi \int_{\phi}^{\phi+\chi} \frac{1}{\pi}d\theta = \frac{W}{\pi h} I_o(\kappa\rho),$$

$$\int_{r_o}^{R} \xi K_o(\kappa\xi)\alpha_o(\xi)d\xi = \lim_{\substack{\delta\to 0\\ \chi\to 0}} \frac{W}{\delta\chi h\rho} \int_{\rho}^{\rho+\delta} \xi\, K_o(\kappa\xi)d\xi \int_{\phi}^{\phi+\chi} \frac{1}{\pi}d\theta = \frac{W}{\pi h} K_o(\kappa\rho). \tag{1.7}$$

When $r_o \leq r < \rho$,

$$a_o(r) = -\frac{W}{\lambda\pi h} \cdot \frac{K_o'(\kappa r_o)I_o(\kappa r)-I_o'(\kappa r_o)K_o(\kappa r)}{I_o'(\kappa r_o)K_o'(\kappa R)-I_o'(\kappa R)K_o'(\kappa r_o)}$$

$$\cdot\ K_o'(\kappa R)I_o(\kappa\rho)-I_o'(\kappa R)K_o(\kappa\rho) \quad , \tag{1.8}$$

and when $\rho < r \leq R$

$$a_o(r) = -\frac{W}{\lambda\pi h} \cdot \frac{K_o'(\kappa r_o)I_o(\kappa r)-I_o'(\kappa r_o)K_o(\kappa r)}{I_o'(\ r_o)K_o'(\ R)-I_o'(\ R)K_o'(\kappa r_o)}$$

$$\cdot\ K_o'(\kappa R)I_o(\kappa\rho)-I_o'(\kappa R)K_o(\kappa\rho)\ +$$

$$-\ \frac{W}{\lambda\pi h} K_o(\kappa r)I_o(\kappa\rho)-I_o(\kappa r)K_o(\kappa\rho) \quad .$$

The general solution of the second equation of (1.4) is

$$a_m(r) = AI_m(\kappa r)+BK_m(\kappa r)+$$

$$-\frac{1}{\lambda}\int_{r_o}^{r}\xi\ I_m(\kappa\xi)K_m(\kappa r)-I_m(\kappa r)K_m(\kappa\xi)\ \alpha_m(\xi)d\xi. \tag{1.9}$$

According to (1.2), and with the following expressions being taken into account,

$$\int_o^R \xi I_m(\kappa\xi)\alpha_m(\xi)d\xi = \lim_{\substack{\delta\to 0\\ \chi\to 0}}\frac{W}{\delta\chi\rho h}\int_\phi^{\phi+\delta}\xi\ I_m(\kappa\xi)d\xi\int_\phi^{\phi+\chi}\frac{1}{\pi}\cos m\theta d\theta = \frac{W}{\pi h}I_m(\kappa\rho)\cos m\phi\ ,$$

$$\int_o^R \xi K_m(\kappa\xi)\alpha_m(\xi)d\xi = \lim_{\substack{\delta\to 0\\ \chi\to 0}}\frac{W}{\delta\chi\rho h}\int_\phi^{\phi\ \delta}\xi\ K_m(\kappa\xi)d\xi\int_\phi^{\phi+\chi}\frac{1}{\pi}\cos m\theta d\theta = \frac{W}{\pi h}K_m(\kappa\rho)\cos m\phi\ ,$$

there are derived

$$A = \frac{W}{\lambda\pi h}\cos m\phi\frac{K_m'(\kappa R)I_m(\kappa\rho)-I_m'(\kappa R)K_m(\kappa\rho)}{I_m'(\kappa R)K_m'(\kappa r_o)-I_m'(\kappa r_o)K_m'(\kappa R)}\ K_m'(\kappa r_o)\ ,$$

$$B = \frac{W}{\lambda\pi h}\cos m\phi\frac{K_m'(\kappa R)I_m(\kappa\rho)-I_m'(\kappa R)K_m(\kappa\rho)}{I_m'(\kappa r_o)K_m'(\kappa R)-I_m'(\kappa R)K_m'(\kappa r_o)}\ I_m'(\kappa r_o)\ . \tag{1.10}$$

Similar expressions can be written for $b_m(r)$.

When $r_o \leq r < \rho$,

$$T(r,\theta) = \frac{W}{2\pi\lambda h}\ \frac{K_o'(\kappa r_o)I_o(\kappa r)-I_o'(\kappa r_o)K_o(\kappa r)}{I_o'(\kappa r_o)K_o'(\kappa R)-I_o'(\kappa R)K_o'(\kappa R_o)}\left[K_o'(\kappa R)I_o(\kappa\rho)-I_o'(\kappa R)K_o(\kappa\rho)\right]-$$

$$-\frac{W}{\lambda\pi h}\sum_m\frac{K_m'(\kappa r_o)I_m(\kappa r)-I_m'(\kappa r_o)K_m(\kappa r)}{I_m'(\kappa R)K_m'(\kappa r_o)-I_m'(\kappa r_o)K_m'(\kappa R)}\left[K_m'(\kappa R)I_m(\kappa R)-I_m'(\kappa\rho)K_m(\kappa\rho)\right]\cos m(\phi-\theta). \tag{1.11}$$

And when $\rho < r \leq R$,

$$T(r,\theta) = \frac{W}{2\pi\lambda h}\frac{K_o'(\kappa r_o)I_o(\kappa r)-I_o'(\kappa r_o)K_o(\kappa r)}{I_o'(\kappa r_o)K_o'(\kappa R)-I_o'(\kappa R)K_o'(\kappa r_o)}\left[K_o'(\kappa R)I_o(\kappa\rho)-I_o'(\kappa R)K_o(\kappa\rho)\right]+$$

$$+\frac{W}{2\pi\lambda h}\left[K_o(\kappa r)I_o(\kappa\rho)-I_o(\kappa r)K_o(\kappa\rho)\right]- \tag{1.12}$$

$$-\frac{W}{\pi\lambda h}\sum_m\frac{K_m'(\kappa r_o)I_m(\kappa r)-I_m'(\kappa r_o)K_m(\kappa r)}{I_m'(\kappa R)K_m'(\kappa r_o)-I_m'(\kappa r_o)K_m'(\kappa R)}\left[K_m'(\kappa R)I_m(\kappa\rho)-I_m'(\kappa RK_m(\kappa\rho)\right]$$

$$\cos m(\phi-\theta)+\frac{W}{\pi\lambda h}\sum_m\left[I_m(\kappa\rho)K_m(\kappa r)-I_m(\kappa r)K_m(\kappa\rho)\right]\cos m(\phi-\theta) .$$

$T(r,\theta)$ is a continuous function although its derivative $\frac{\partial}{\partial r}T(r,\theta)$ has a first order discontinuity at $r = \rho$(see fig. 1). At that point, formulae (1.11) and (1.12) give similar values for temperature.

The series given in (1.11) and (1.12) converges slowly by reason of the inhomogeneity of (1.4) in fact, if Bessel functions are presented as functions of their order m,it may be shown that the general term of series (1.12) does not decrease more rapidly than $\frac{1}{m}$. Thus the convergence of (1.12) must be improved, and if this is to be achieved then some series which is the solution of homogeneous equation (1.4) has to be subtracted from (1.12), homogeneous boundary conditions (1.2) have to be satisfied. This will be the Green function of equation (1.4). Consequently instead of (1.11) and (1.12), we can write

$$f_m(r\rho) = a_m(r\rho)-g_m(r\rho)$$

The Green function $g_m(r\rho)$ may be expanded in absolutely and uniformly convergent series along the fundamental functions of equation (1.4), with boundary conditions (1.2), and may be presented in the bilinear form

$$g_m(r\rho) = \sum_m \frac{Y_m(r)Y_m(\rho)}{\nu_m}$$

Here,

$I_m(r)I_m(\rho)$ are fundamental functions, and

ν_m characteristic values.

Thus for (1.4) ,

$$Y''+\frac{1}{r}Y'-(\frac{m^2}{r^2} + \nu^2K^2)Y = 0 ,$$

with the boundary conditions

$$Y'\Big|_{r = r_o}^{r = R} = 0 ,$$

the fundamental functions are

$$Y_m = C\ I_m(\kappa\gamma r)K'_m(\kappa\gamma R)+K_m(\kappa\gamma r)I'_m(\kappa\gamma R)$$

Y_m are the roots of equation

$$I'_m(\kappa\gamma r_o)K'_m(\kappa\gamma R)-I'_m(\kappa\gamma R)K'_m(\kappa\gamma r_o) = 0 .$$

Constant C can be defined from the condition of normalization

$$\int_{r_o}^{R} K^2Y_m^2(r)dr = 1 .$$

in similarity to $a_m(r\rho)$, the Green function $g_m(r\rho)$ is continuous, and its derivative has a first order discontinuity of the same point $r = \rho$.

2. THERMAL STRESSES INDUCED BY THE TEMPERATURE FIELD

For solution of the thermoelastic equations, thermoelastic potential 1 can be introduced in accordance with equation

$$\Delta\psi(r,\theta) = (1+\mu)\ T(r,\theta) \tag{2.1}$$

The stresses will then be derived from

$$\sigma_{rr} = -\ 2\ G(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})\psi\ ,$$

$$\sigma_{\theta\theta} = -\ 2\ G\ \frac{\partial^2}{\partial r^2}\psi\ , \tag{2.2}$$

$$\sigma_{r\theta} = 2\ G\ \frac{\partial}{\partial r}(\frac{1}{r}\ \frac{\partial}{\partial\theta}\psi)\ .$$

Here ,

μ = Poisson's ratio,

α = the coefficient of thermal expansion, and

G = the modulus of rigidity .

The normal stress σ_{rr}, and shearing stress $\sigma_{r\theta}$, are zero at edges $r = r_o$ and $r = R$, and thus, in view of formulae (2.2), we derive

$$(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})\psi\Bigg|_{\substack{r = r_o \\ r = R}} = 0,\quad \frac{\partial}{\partial r}(\frac{1}{r}\ \frac{\partial}{\partial\theta}\psi)\Bigg|_{\substack{r = r_o \\ r = R}} = 0 \tag{2.3}$$

Equation (2.1) may be presented in the following form:

$$\frac{\partial^2}{\partial r^2}\psi(r,\theta) + \frac{1}{r}\frac{\partial}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\psi(r,\theta) = (1+\mu)\alpha T(r,\theta). \quad (2.4)$$

$\psi(r,\theta)$ must be periodic, with a period of 2π. Thus

$$\psi(r,\theta) = \frac{1}{2}c_o(r) + \sum_n \left[c_n(r)\cos n\theta + d_n(r)\sin n\theta \; . \right] \quad (2.5)$$

After the substitution of (2.5) into (2.4), there are derived

$$\frac{d^2}{dr^2}c_o(r) + \frac{1}{r}\frac{d}{dr}c_o(r) = (1+\mu)\alpha\bar{\alpha}_o(r) \; ,$$

$$\frac{d^2}{dr^2}c_n(r) + \frac{1}{r}\frac{d}{dr}c_n(r) - \frac{n^2}{r^2}c_n(r) = (1+\mu)\alpha\bar{\alpha}_n(r) \; , \quad (2.6)$$

$$\frac{d^2}{dr^2}d_n + \frac{1}{r}\frac{d}{dr}d_n(r) - \frac{n^2}{r^2}d_n(r) = (1+\mu)\alpha\bar{\beta}_n(r) \; ,$$

in which:

$$c_o(r) = \frac{1}{\pi}\int_o^{2\pi}\psi(r,\theta)d\theta \; ,$$

$$c_n(r) = \frac{1}{\pi}\int_o^{2\pi}\psi(r,\theta)\cos n\theta d\theta \; ,$$

$$d_n(r) = \frac{1}{\pi}\int_o^{2\pi}\psi(r,\theta)\sin n\theta d\theta \; ,$$

$$\bar{\alpha}_o(r) = \frac{1}{\pi}\int_o^{2}\psi(r,\theta)d\theta \; ,$$

$$\bar{\alpha}_n(r) = \frac{1}{\pi}\int_o^{2\pi}\psi(r,\theta)\cos n\theta d\theta \; ,$$

$$\bar{\beta}_n(r) = \frac{1}{\pi}\int_o^{2\pi}\psi(r,\theta)\mathrm{inn}\theta d\theta \; .$$

The general solution of the first equation of (2.6), with the boundary conditions

$$\left.\frac{d}{dr}C_o(r)\right|_{\substack{r\,=\,r_o \\ r\,=\,R}} = 0$$

is

$$C_o(r) = C\ln r + D + (1+\mu)\alpha\ln r \int_{r_o}^{R} \xi\, \overline{\alpha}(\xi)d\,\xi \; + \qquad (2.7)$$

$$- (1+\mu)\alpha \int_{r_o}^{r} \xi\; \ln\xi\overline{\alpha}_o(\xi)d\xi \; .$$

From the boundary conditions, $C = 0$, when $r = r_o$. Furthermore,

$$\int_{r_o}^{R} \xi\; \overline{\alpha}_o\; (\xi)d\; \xi = 0$$

when $r = R$. If the last equation is not fulfilled, the constant pressure at rim $r = R$ is easily removable. Now

$$C_o(r) = D + (1+\mu)\alpha\ln r \int_{r_o}^{r} \xi\alpha_o(\xi)d\xi - (1+\mu)\alpha \int_{r_o}^{r} \xi\ln\, \xi\alpha\; (\xi)d\xi \; ,$$

$$\frac{d}{dr}C_o(r) = (1+\mu)\alpha\, \frac{1}{r} \int_{r_o}^{r} \xi\; \alpha_o(\xi)d\xi \; , \qquad (2.8)$$

$$\frac{d^2}{dr^2}C_o(r) = -(1+\mu)\alpha\frac{1}{r^2} \int_{r_o}^{r} \xi\; \alpha_o(\xi)d\xi + (1+\mu)\alpha\overline{\alpha}_o(r).$$

Here, $\overline{\alpha}_o(r)$ is defined by (1.8), and the integrals in (2.8) are

$$\int_{r_o}^{r} \xi\, I_o(\kappa\xi)d\,\xi = \frac{1}{\kappa 2}\Big[\kappa r I_1(\kappa r)\Big]_{r_o}^{r} ,$$

$$\int_{r_o}^{r} \xi\, K_o(\kappa\xi)d\,\xi = -\frac{1}{\kappa 2}\Big[\kappa r K_1(\kappa r)\Big]_{r_o}^{r} .$$

The stresses induced by this term of the series may be defined by (2.2).

Consideration is now given to the second equation (2.6)

$$\frac{d^2}{dr^2}C_n(r) + \frac{1}{r}\frac{d}{dr}C_n(r) - \frac{n^2}{r^2}C_n(r) = (1+\mu)\alpha\,\bar{\alpha}_n(r), \qquad (2.9)$$

with the boundary conditions

$$\underbrace{\frac{1}{r}\frac{d}{dr}C_n(r) - \frac{n^2}{r^2}C_n(r)}_{\sigma_{rr}}\Bigg|_{\substack{r = r_o\\ r = R}} = 0,$$

$$\underbrace{\frac{1}{r}\frac{d}{dr}C_n(r) - \frac{1}{r^2}C_n(r)}_{\sigma_{r\theta}}\Bigg|_{\substack{r = r_o\\ r = R}} = 0 \qquad (2.10)$$

The general solution of (2.9) is

$$C_n(r) = Er^n + Fr^{-n} + \frac{r^n}{2n}\int_{r_o}^{r}\frac{(1+\mu)\alpha}{n-1}\,\bar{\alpha}_n(\xi)d\,\xi +$$

$$- \frac{r^{-n}}{2n}\int_{r_o}^{r}\frac{(1+\mu)\alpha}{\xi^{-n-1}}\,\bar{\alpha}_n(\xi)d\,\xi . \qquad (2.11)$$

It is evident that E and F of (2.11) can not be found from (2.10). be found from (2.10). The normal and shear stresses at the rim define the combined stress

$$p^2 = \sigma^2 + \tau^2 .$$

If the combined stress at the rims $r = r_o$, $r = R$ equals zero,

$$p \Big|_{\substack{r = r_o \\ r = R}} = 0$$

then the components

$$\left.\begin{matrix}\sigma \\ \tau\end{matrix}\right\}_{\substack{r = r_o \\ r = R}} = 0 \; .$$

The constants E and F are consequently determinable from the non-linear boundary conditions

$$\sigma^2 + \tau^2 \Big|_{\substack{r = r_o \\ r = R}} = 0 \qquad (2.12)$$

Conditions (2.12) present a system of second order equations, of which the first presents a real point (the result of intersection of two imaginary lines), and the second a real ellipse. However, it can not be demostrated in advance that this point lies on the countour of the ellipse, and the system has two real solutions, corresponding to the physical sense of the problem. Instead of general solution (2.11) the simple particular solution can be applied:

$$\overline{C}_n(r) = \frac{r^n}{2n} \int_{r_o}^{r} \frac{(1+\mu)\alpha}{\xi^{n-1}} \overline{\alpha}_n(\xi) d\xi - \frac{r^{-n}}{2n} \int_{r_o}^{r} \frac{(1+\mu)\alpha}{\xi^{-n-1}} \overline{\alpha}_n(\xi) d\xi \qquad (2.13)$$

Here $\overline{\alpha}_n(\xi)$ is defined as (1.11) and (1.12)

The expression is the same for $b_n(r)$ $d_n(r)$.

In this case, and in accordance with (2.2) and (2.5), the normal and shear stresses appear at rims $r = r_o$ and $r = R$, $r = r_o$;

$$r = r_o;\quad \sigma_{r_o r_o} = 0;\quad \sigma_{\theta r_o} = 0 \tag{2.14}$$

$$r = R;\quad \sigma_{RR} = -2G\left[\frac{1}{R}\bar{C}'_n(R) - \frac{n^2}{R^2}\bar{C}'_n(R)\right](\cos n\theta + \sin n\theta);$$

Or

$$\sigma_{\theta R} = 2G\left[-\frac{n}{R}\bar{C}'_n(R) + \frac{n^2}{R^2}\bar{C}'_n(R)\right](\sin n\theta - \cos n\theta);$$

$$\sigma_{RR} = \underbrace{-2G\left[-\frac{R^{n-2}}{2}(n-1)\int_{r_o}^{r}\frac{(1+\mu)\alpha}{\xi^{n-1}}\bar{\alpha}_n(\xi)d\xi + \frac{R^{-n-2}}{2}(n+1)\int_{r_o}^{r}\frac{(1+\mu)\alpha}{\xi^{-n-1}}\bar{\alpha}_n(\xi)d\xi\right]}_{M_n}(\cos n\theta + \sin n\theta) \tag{2.15}$$

$$\sigma_{\theta R} = \underbrace{2G\left[-\frac{R^{n-2}}{2}(n-1)\int_{r_o}^{R}\frac{(1+\mu)\alpha}{\xi^{n-1}}\alpha_n(\xi)d\xi - \frac{R^{-n-2}}{2}(n+1)\int_{r_o}^{R}\frac{(1+\mu)\alpha}{\xi^{-n-1}}\bar{\alpha}_n(\xi)d\xi\right]}_{N_n}(\sin n\theta - \cos n\theta) \tag{2.16}$$

Then

$$\sigma_{RR} = M_n(\cos n\theta + \sin n\theta),$$
$$\sigma_{R\theta} = N_n(\sin n\theta - \cos n\theta)$$

Thus, if these stresses can be removed from the rims, the problem is solved.

On the outer rim of the ring, there is now assumed a system of normal and tangent loads

$$p = -\sigma_{RR};\quad q = -\sigma_{\theta R} \tag{2.17}$$

If there is now taken a biharmonic function $\phi_{(r,\theta)}$ which statisfies

$$D(r,\theta)\left[D(r,\theta)\phi\,(r,\theta)\right] = 0 \tag{2.18}$$

$$D(r,\theta) \equiv \left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)$$

in which $\phi(r,\theta)$ must also be periodic, with a period of 2π

Thus $$\phi(r,\theta) = U_o(r) + \sum_n \left[U_n(r)\cos n\theta + V_n(r)\sin n\theta\right] \tag{2.19}$$

Stresses can be found

$$\sigma_{rr} = \frac{1}{r}\frac{\partial}{\partial r}\phi(r,\theta) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi(r,\theta);$$

$$\sigma_{\theta\theta} = \frac{\partial^2}{\partial r^2}\phi(r,\theta); \tag{2.20}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) .$$

On rim $r = R$, stresses σ_{RR} and $\sigma_{R\theta}$ must attain the values of (2.17). If all of thes is fulfilled, there can be written:

$$\sigma_{ij} = \sigma_{ij}(\psi) + \sigma_{ij}(\phi) \tag{2.21}$$

The first term of (2.21) is derived from (2.2), (2.5), (2.8) and (2.13). There are now determined the stresses defined by $\phi(r,\theta)$ with the boundary conditions (2.17).

After the substitution of (2.19) into (2.18) we have

$$\begin{aligned} &D(r,\theta)\left[D(r,\theta)\,U_o(r)\right] = 0;\\ &D(r,\theta)\left\{D(r,\theta)\left[U_n(r)\cos n\theta\right]\right\} = 0;\\ &D(r,\theta)\left\{D(r,\theta)\left[V_n(r)\sin n\theta\right]\right\} = 0. \end{aligned} \tag{2.22}$$

The boundary conditions are

$$\left.\frac{d}{dr} U_o(r)\right|_{\substack{r = r_o \\ r = R}} = 0; \tag{2.23}$$

$$\frac{1}{r}\frac{d}{dr}U_n(r)(\cos n\theta+\sin n\theta) - \frac{n^2}{r^2}U_n(r)(\cos n\theta+\sin n\theta) = \begin{cases} 0 \;;\; r = r_o \\ p \;;\; r = R \end{cases} \tag{2.24}$$

$$\frac{n}{r}\frac{d}{dr}U_n(r)(\sin n\theta-\cos n\theta) - \frac{n}{r^2}U_n(r)(\sin n\theta-\cos n\theta) = \begin{cases} 0 \;;\; r = r_o \\ g \;;\; r = R \end{cases} \tag{2.25}$$

It is evident from (2.22) that $U_n(r) \equiv V_n(r)$

The general solution of the first equation of (2.22) takes

$$U_o(r) = A\ln r + B + C\,\frac{r^2}{4}(\ln r-1) + Dr\left(\frac{r\ln r}{2} - \ln r - 1\right) \tag{2.26}$$

From (2.23) there is obtained A=C=D=0.So $U_o(r)=B$.

The general solution of the second equation of (2.22) is

$$U_n(r) = Cr^n + Dr^{-n} + \frac{A}{A(n+1)}r^{n+2} - \frac{B}{A(n-1)}r^{-n+2} \tag{2.27}$$

The constants C,D,A,B are determinable from the boundary conditions

$$\frac{1}{r}\frac{d}{dr}U_n(r) - \frac{n^2}{r^2}U_n(r) = \begin{cases} 0 \;;\; r = r_o \\ -M_n ;\; r = R \end{cases}$$
$$\frac{n}{r}\frac{d}{dr}U_n(r) - \frac{n}{r^2}U_n(r) = \begin{cases} 0 \;;\; r = r_o \\ -N_n ;\; r = R \end{cases} \tag{2.28}$$

Thus the problem is solved in general although the expressions for

$\sigma_{ij}(\psi)$ and $\sigma_{ij}(\phi)$ are extremely cumbersome.

For example, from (2.5) there can be written the expression for the normal stress induced by the terms $C_n(r)$ and $d_n(r)$.

$$\sigma_{rr}(\psi) = G(1+\mu)\alpha \frac{W}{\lambda rh} \frac{Bn}{\kappa r} \left\{ \frac{n(n-1)}{\kappa r_o} \left[K_n(\kappa r_o) I_{n-1}(\kappa r) + I_n(\kappa r_o) K_{n-1}(\kappa r) \right] + \right.$$

$$+ (n-1)\left[K_{n-1}(\kappa r_o) I_{n-1}(\kappa r) - I_{n-1}(\kappa r_o) K_{n-1}(\kappa r) \right] - n(n-1)\frac{r^{n-1}}{r_o^{\,n-1}} +$$

$$+ \frac{n(n+1)}{\kappa r_o}\left[K_n(\kappa r_o) I_{n+1}(\kappa r) + I_n(\kappa r_o) K_{n+1}(\kappa r) \right] -$$

$$\left. - (n+1) \left[K_{n+1}(\kappa r_o) I_{n+1}(\kappa r_o) K_{n+1}(\kappa r) \right] - n(n+1)\frac{r_o^{\,n+1}}{r^{n+1}} \right\} (\cos n\theta + \sin n\theta);$$

$$\rho < r \leq R$$

$$\sigma_{rr}(\psi) = G(1+\mu)\alpha \frac{W}{\pi\lambda h} \frac{B_n}{\kappa r} \left\{ - \frac{n(1-n)}{\kappa r_o} \left[K_n(\kappa r_o)_{n-1}(\kappa r) + I_n(\kappa r_o) K_{n-1}(\kappa r) \right] + \right.$$

$$+ \frac{n(1+n)}{r_o}\left[K_n(\kappa r_o) I_{n+1}(\kappa r) + I_n(\kappa r_o) K_{n+1}(\kappa r) \right] -$$

$$- (1-n) \left[K_{n-1}(\kappa r_o) I_{n-1}(\kappa r) - I_{n-1}(\kappa r_o) K_{n-1}(\kappa r) \right] -$$

$$- (1+n) \left[K_{n+1}(\kappa r_o) I_{n+1}(\kappa r) - I_{n+1}(\kappa r_o) K_{n+1}(\kappa r) \right] +$$

$$\left. + n(1-n) \frac{r^{n-1}}{r_o^{\,n-1}} - n(1+n) \frac{r_o^{\,n+1}}{r^{n+1}} \right\} (\cos n\theta + \sin n\theta) -$$

$$- G(1+\mu)\alpha \frac{W}{\pi\lambda h} \frac{1}{\kappa r} \left\{ -(1-n) \left[I_n(\kappa\rho) K_{n-1}(\kappa r) + K_n(\kappa\rho) I_{n-1}(\kappa r) \right] - \right.$$

$$- (1+n)\left[I_n(\kappa\rho) K_{n+1}(\kappa r) + K_n(\kappa\rho) I_{n+1}(\kappa r) \right] +$$

$$+ (1-n)\left[I_n(\kappa\rho) K_{n-1}(\kappa r_o) + K_n(\kappa\rho) I_{n-1}(\kappa r_o) \right] \frac{r^{n-1}}{r_o^{\,n-1}} +$$

$$\left. + (1\_n)\left[I_n(\kappa\rho) K_{n+1}(\kappa r_o) + I_{n+1}(\kappa r_o) K_n(\kappa\rho) \right] \frac{r_o^{\,n+1}}{r^{n+1}} \right\} (\cos n\theta + \sin n\theta);$$

$$B_n = \frac{K_n'(\kappa R) I_n(\kappa \rho) - I_n'(\kappa R) K_n(\kappa \rho)}{I_n'(\kappa R) K_n'(\kappa r_o) - I_n'(\kappa r_o) K_n'(\kappa R)} , \quad m = n \; ; \; \phi = 0$$

Stresses $\sigma_{ij}(\phi)$ have a first order discontinuity at point $r = \rho$, but stresses $\sigma_{ij}(\phi)$ are continuous (Fig. 1).

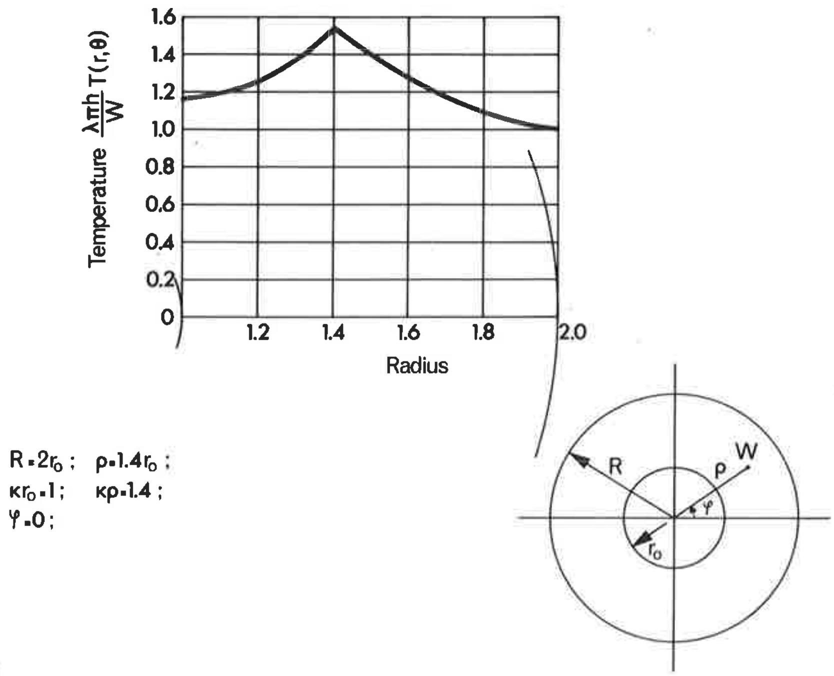

Fig. 1.

REFERENCES

1. Melan, E., Parcus, H., Wärmespannungen infolge stationarer temperaturfelder. Wien, Springer-Verlag, 1953.

J.S. Kajaste-Rudnitski, tech.lic., Helsinki University of Technology, Otaniemi, Finland