PROJECTION AND NON-PROJECTION FINITE ELEMENT METHODS - A SURVEY OF DIFFERENT APPROACHES

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ABSTRACT

Finite element methods are variational methods for solving differential equations with approximations which usually are polynomials defined with aid of a mesh dividing the region into elements of different shapes. The correct solution is found in a apace V of functions which satisfy certain continuity conditions. If the approximate solution is sought in a subspace $V_h \subset V$ the method is said to be a projection one. Some procedures for construction of projection and non-projection finite element methods are described.

1. INTRODUCTION

In structural frame analysis finite element formulations of different types have been used since long time ago - displacement-, force-, and mixed formulations. In the case of technical beam theory for prismatic linearly elastic beams the result - irrespective of the method used - can be considered as exact up to rounding-off errors. For two- and threedimensional cases in structural mechanics the finite element method is a method which gives, for a specific element mesh, only an approximate answer to the given problem - mostly stated as a differential equation with boundary conditions. It is then required that - with decreasing size of the elements - the solution should tend asymptotically towards the correct answer. Also for these cases displacement-, force-, and mixed formulations have been used.

For a study of the character of the approximation it is, however, of more interest to know if the method for a specific mesh corresponds to a minimum in potential or complementary energy or - possibly - does not correspond to a minimum of any physically interpreted functional.

In the general framework of all boundary value problems which can be solved by the finite element method the corresponding two classes will be: projection methods and non-projection methods.

2. PROJECTION METHOD

2.1. Variational formulation

The projection method can be demonstrated on a simple 2D-case- the Dirichlet problem: Find the function u defined on a polygonal region Ω in R^2 such that

$$-\Delta u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{1}$$

where $\partial\Omega$ is the boundary of Ω.

Take now a class V of continuous functions v with piecewise continuous first derivatives and which vanish on the boundary. Multiply both sides of (1) with any function $v \in V$ and integrate over the domain

$$\int_\Omega -\Delta u v dx = \int_\Omega f v dx \, , \quad dx = dx_1 dx_2 \tag{2}$$

By Green's formula and the boundary condition

$$\int_\Omega -\Delta u v dx = -\int_{\partial\Omega} \frac{\partial u}{\partial n} v ds + \int_\Omega (\nabla u)^T \nabla v dx = \int_\Omega (\nabla u)^T \nabla v dx$$

where

$$\nabla = [\partial/\partial x_1, \ \partial/\partial x_2]^T \ .$$

Then (2) can be written

$$a(u,v) = b(v) \tag{3}$$

where

$$a(u,v) = \int_\Omega (\nabla u)^T \nabla v dx$$

is a symmetrical bilinear form which is positive definite for

$$u = v \ ,$$

and

$$b(v) = \int_\Omega f v dx \ .$$

We can now give the variational formulation of the boundary value problem (1):

Find that $u \in V$ for which

$$a(u,v) = b(v) \quad \text{for all} \quad v \in V \ . \tag{4}$$

This formulation is equivalent to (1) because (4) is a consequence of (1) and (1) is a consequence of (4). The second statement is proved by using Green's formula backwards. Then it follows from (4) that

$$\int_\Omega (\Delta u + f) v dx = 0 \quad \text{for all} \quad v \in V$$

from which (1) follows.

We observe the important property of the variational formulation that the highest derivates appearing are of lower order than in the differential formulation. This broadens the field of admissible functions substantially.

2.2. Minimum property

The solution u to (1), (4) corresponds to a minimum of a certain functional:

$$F(v) = a(v,v) - 2b(v) \ , \qquad v \in V \ .$$

Calculate

$$F(v) - F(u) = a(v,v) - 2b(v) - a(u,u) + 2b(u) =$$

$$= a(v-u,v-u) + 2a(v,u) - 2a(u,u) - 2b(v) + 2b(u) =$$

$$= a(v-u,v-u) + 2[a(u,v-u) - b(v-u)] \ .$$

From (3) it follows that the last term is zero, so

$$F(v) - F(u) = a(v-u,v-u) \geq 0$$

and

$$F(u) \leq F(v) \quad \text{for all} \quad v \in V \, . \tag{6}$$

For some types of physical problems a linear programming procedure based on this property has been found efficient. Furthermore, in many cases (5) can be physically interpreted. It can be used for evaluation of approximate solutions.

2.3. Finite element approximation by use of variational formulation

In the projection method an approximation is sought in a finite-dimensional subspace $V_h \subset V$.

For any given basis $\{\varphi_i\}$ of N functions in a finite-dimensional space V_h a function $v_h \in V_h$ can be written

$$v_h(x) = \sum_{j=1}^{N} v_h^j \varphi_j(x) \tag{7}$$

where $\{v_h^j\}$ is a unique set of parameters. For definition of a finite element basis the region is divided into elements which meet along sides, edges, and at nodes. A typical member of a finite element basis takes the value one at just one node and is non-zero only in the elements incident to that node. For more details see [1], [2], [3].

In the projection method it is thus required that the functions φ_j fulfil the continuity conditions for the functions in the space V, exemplified in section 2.1.

The variational formulation of the approximate problem is:

Find that $u_h \in V_h \subset V$ for which

$$a(u_h, v_h) = b(v_h) \quad \text{for all} \quad v_h \in V_h \, . \tag{8}$$

The solution is obtained by writing (8) for all members of the basis

$\{\varphi_j\}$. In that way N independent linear equations are obtained:

$$a(u_h,\varphi_k) = b(\varphi_k) \ , \qquad k = 1,\ldots,N \ .$$

Write here

$$u_h = \sum_{j=1}^{N} u_h^j \varphi_j(x)$$

so

$$\sum_{j=1}^{N} a(\varphi_j,\varphi_k)u_h^j = b(\varphi_k) \ , \qquad k = 1,\ldots,N \ . \tag{9}$$

This is a symmetric, positive definite set of linear equations. It follows that a unique solution exists.

2.4. Properties of finite element solution by projection method

A special case of (4) is

$$a(u,v_h) = b(v_h) \quad \text{for all} \quad v_h \in V_h \subset V \ .$$

Subtraction from (8) gives

$$a(u_h-u,v_h) = 0 \quad \text{for all} \quad v_h \in V_h \ . \tag{10}$$

If $a(u,v)$ is regarded as a scalar product the "error" u_h-u can be said to be orthogonal to V_h or equivalently, u_h be said to be the orthogonal projection of u on V_h.

The "length" or norm of the "error" u_h-u is defined in the usual way as

$$|||u_h-u||| = [a(u_h-u,u_h-u)]^{1/2} \ . \tag{11}$$

From the orthogonal property should follow that this length is the shortest to V_h which means that any other point $v_h \in V_h$ gives a larger length.

Let us calculate the difference

$$|||v_h-u|||^2 - |||u_h-u|||^2 = a(v_h-u,v_h-u) - a(u_h-u,u_h-u) =$$

$$= [a(v_h,v_h) - 2a(u,v_h)] - [a(u_h,u_h) - 2a(u,u_h)] =$$

$$= [a(v_h,v_h) - 2b(v_h)] - [a(u_h,u_h) - 2b(u_h)] =$$

$$= F(v_h) - F(u_h) \geq 0 \quad \text{(see (6))} .$$

It follows that

$$|||u_h-u||| \leq |||v_h-u||| \text{ , for all } v_h \in V_h . \tag{12}$$

In this sense the approximation u_h is "best". The right-hand member can be estimated in a simple way by choosing v_h as the interpolate of u which is that v_h which takes the values of u at the nodes. For such estimations see [4], [5], [6].

2.5. Direct projection

The projection property can be used for a direct deduction of the approximating set of linear equations. This has been proposed by Oden [7].

Let us first study the problem of finding the projection $Pu \in V_h$ of a function $u \in V$, where $V = L_2(\Omega)$. If $\{\varphi_j\}$ is a basis in V_h any function u_h in V_h can be written

$$u_h = \sum_{j=1}^{N} u_h^j \varphi_j(x) . \tag{13}$$

The coefficients u_h^j can be calculated as Fourier coefficients. Then the dual basis $\{\varphi^k\}$ to $\{\varphi_j\}$ is needed. It is calculated from

$$\int_\Omega \varphi_j \varphi^k dx = \delta_j^k . \tag{14}$$

In our case φ_j are FEM-functions which are local. However, the functions φ^k are in general not local.

The coefficients u_h^j in (13) are now calculated by multiplying each side by φ^k and integrating:

$$\int_\Omega u_h \varphi^k dx = \int_\Omega (\Sigma u_h^j \varphi_j)\varphi^k dx = u_h^k \quad . \tag{15}$$

The <u>projection</u> $Pu \in V_h$ of $u \in V$ will now be defined as

$$Pu = \sum_{j=1}^{N} u_h^j \varphi_j(x) \tag{16}$$

where

$$u_h^k = \int_\Omega u \varphi^k dx \quad . \tag{17}$$

The set of dual functions $\{\varphi^k\}$ to $\{\varphi_j\}$ are thus required here for calculation of the coefficient u_h^j. The dual functions are calculated with the "fundamental matrix" C from

$$\sum_{j=1}^{N} C_{ij}\varphi^j = \varphi_i \quad , \qquad i = 1,\ldots,N \tag{18}$$

where

$$C_{ij} = \int_\Omega \varphi_i \varphi_j dx \quad , \qquad i,j = 1,\ldots,N \; .$$

Alternatively, the coefficients u_h^j can be calculated from the condition that the error should be orthogonal to V_h, c.f. (10) in the L_2-sense.

Then

$$\int_\Omega (Pu-u)\varphi_i = 0 \quad , \qquad i = 1,\ldots,N \; . \tag{19}$$

With Pu from (16) this is

$$\int_\Omega (\Sigma u_h^j \varphi_j)\varphi_i dx = \int_\Omega u \varphi_i dx \quad , \qquad i = 1,\ldots,N$$

or

$$C_{ij}u_h^j = \int_\Omega u\varphi_i dx \ , \quad i = 1,\ldots,N \tag{20}$$

from which u_h^j can be calculated without calculation of the dual functions.

This projection method can be used for smoothening of discontinuous functions which often appear as results from finite element analyses. Thus, stress components in elasticity often are discontinuous as they are deduced from displacements which are assumed to have only piecewise continuous first derivatives.

The matrix [C] and the right hand vector of (20) can be assembled in usual way from element matrices $[C^e]$. For the "constant strain" triangle it is, for example

$$[C^e] = (A/12)\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{21}$$

where A is the area of the element. For a two-dimensional problem in elasticity the cost for solving (20) is about 1/8 of the cost for solving the set of equations for the structure. Since the matrix [C] is the same for all variables to be smoothened the extra cost for each new variable can be neglected, see [22].

Let us now return to the problem of deriving the approximating equations by direct projection.

Take as example the Dirichlet problem. Project each side of (1) on $L_2(\Omega)$ by (13), (17) with $\{\varphi_j\}$ and $\{\varphi^k\}$ exchanged:

$$\sum_{j=1}^{N} (\int_\Omega -\Delta u\varphi_j dx)\varphi^j = \sum_{j=1}^{N} (\int_\Omega f\varphi_j dx)\varphi^j \ .$$

Use Green's formula and the boundary condition:

$$\sum_{j=1}^{N} a(u,\varphi_j)\varphi^j = \sum_{j=1}^{N} b(\varphi_j)\varphi^j \; .$$

Introduce for u the approximation $u_h \in V_h$ according to (13):

$$\sum_{j,k=1}^{N} a(\varphi_k,\varphi_j)u_h^k\varphi^j = \sum_{j=1}^{N} b(\varphi_j)\varphi^j \; .$$

This is valid for arbitrary φ^j so corresponding coefficients are equal. This results in

$$\sum_{j=1}^{N} a(\varphi_j,\varphi_k)u_h^j = b(\varphi_k) \; , \quad k = 1,\ldots,N \tag{22}$$

which is the set (9) obtained in another way: Equation (9) was obtained by approximation of the "weak formulation" while (22) was obtained by direct projection of the differential equation and equalization of coefficients. It can be shown that u_h is a projection on V_h in energy, e.g. $a(.,.)$, see [22].

2.6. Deduction by diagram

In many physical problems the governing differential equation is deduced from a sequence of lower order differential transformations. Conversely, the resulting differential equation can be split into so called canonical form.

For example, in linear elasticity, Navier's and Beltrami-Michell's equations are deduced from equilibrium and compatibility equations. The theory of elasticity is built up from two symmetric tensor fields, the stress field $\sigma_{ij}(x)$ and the strain field $\varepsilon_{ij}(x)$. Each of them can for our purpose be regarded as an element of a six-dimensional local vector space. Between the two spaces an isomorphism, Hooke's law S^d, is defined.

For the components of any vector $\{\varepsilon\}$ compatibility conditions are prescribed:

$$\tilde{\nabla}_2^T\{\varepsilon\} = 0 \tag{23}$$

where $\tilde{\nabla}_2^T$ is a 3×6 differential matrix operator containing second order derivatives.

Between the components of any vector $\{\sigma\}$ equilibrium conditions are prescribed:

$$-\tilde{\nabla}^T\{\sigma\} = \{U\} \tag{24}$$

where $\tilde{\nabla}^T$ is a 3×6 differential matrix operator containing first order derivatives and $\{U\}$ is a three-dimensional vector of prescribed volume forces.

The solution to (23) can be expressed in strain potentials called displacements $\{u\}$, three-dimensional vectors:

$$\{\varepsilon\} = \tilde{\nabla}\{u\} \tag{25}$$

where $\tilde{\nabla}$ is the transpose of $\tilde{\nabla}^T$.

Similarly the solution to the homogeneous part $\{\sigma_0\}$ of (24) can be expressed in stress potentials $\{\chi\}$, three-dimensional vectors:

$$\{\sigma_0\} = \tilde{\nabla}_2\{\chi\} \tag{26}$$

where $\tilde{\nabla}_2$ is the transpose of $\tilde{\nabla}_2^T$.

In this way two exact sequences of transformations are obtained

$$\begin{array}{ccccc} \{\chi\} & \xrightarrow{\tilde{\nabla}_2} & \{\sigma\} & \xrightarrow{-\tilde{\nabla}^T} & \{U\} \\ & & \uparrow S^d & & \\ \{0\} & \xleftarrow{\tilde{\nabla}_2^T} & \{\varepsilon\} & \xleftarrow{\tilde{\nabla}} & \{u\} \end{array} \tag{27}$$

By combination of the left transformations Beltrami-Michell's equations are obtained

$$\tilde{\nabla}_2^{\,T}(S^d)^{-1}\tilde{\nabla}_2\{\chi\} = 0 \qquad (28)$$

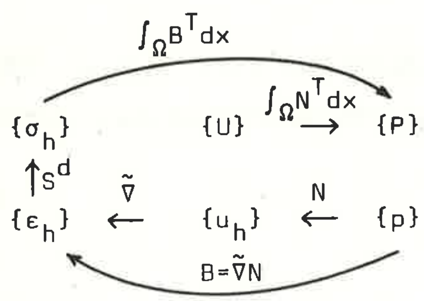
and by combination of the right transformations Navier's equations are obtained

$$-\tilde{\nabla}^T S^d \tilde{\nabla}\{u\} = \{U\} \ .$$

Let us now construct a projection approximation to Navier's equations of the second order by use of a diagram. This means that we seek an approximate solution vector $\{u_h\}$ in a finite-dimensional subspace $V_h \times V_h \times V_h \subset V \times V \times V$ where $V_h \subset V$ and the exact solution $\{u\} \in V \times V \times V$. Relative to a finite element basis $\{\varphi_j\}$ the approximation $\{u_h\}$ can be expressed in the node values u_h^j. We collect these node values in a column matrix $\{p\}$ and write

$$\{u_h\} = [N]\{p\} \ . \qquad (30)$$

We then use the right hand side of diagram (27)



(31)

By this we obtain <u>consistently</u> deduced $\{\sigma_h\}$. The equilibrium equations for the stresses are generally not fulfilled neither inside nor between the elements. (There are, however, methods, by which also the equilibrium equations are fulfilled inside the elements, see below).

If the vector of node forces associated to {p} are denoted by {P} the invariance of virtual work gives

$$\int_\Omega \{\sigma_h\}^T\{\varepsilon_h\}dx = \{P\}^T\{p\} \ . \tag{32}$$

According to the diagram (31)

$$\{\varepsilon_h\} = \tilde{\nabla}\,[N]\{p\} = [B]\{p\}$$

so

$$(\int_\Omega \{\sigma_h\}^T[B]dx)\{p\} = \{P\}^T\{p\} \tag{33}$$

and

$$\{P\} = \int_\Omega [B]^T\{\sigma_h\}dx \ . \tag{34}$$

This consequence of the invariance of virtual work has by Asplund [18] been called "Clebsch's theorem".

By using again the invariance of virtual work now on the right hand part of (31)

$$\int_\Omega \{U\}^T\{u_h\}dx = \{P\}^T\{p\} \tag{35}$$

is obtained. Here $\{u_h\}$ can be expressed in {p} so

$$(\int_\Omega \{U\}^T[N]dx)\{p\} = \{P\}^T\{p\} \tag{36}$$

and

$$\{P\} = \int_\Omega [N]^T\{U\}dx \ . \tag{37}$$

We observe that combination of (32) and (35) yields the approximation of the "weak formulation" of the differential equation so the resulting equations are the same as obtained by the earlier presented procedures. This "deduction by diagrams" using "Clebsch's theorem" has, according to the author's experience, a pedagogical value.

3. NON-PROJECTION METHODS

3.1. Ways of constructing non-projection methods

Non-projection finite element methods could be classified according to their approximation properties. Such a classification should be of great importance for the understanding of the methods.

Here, however, the methods will be classified according to proposed ways of constructing them. As many methods can be constructed in more than one way the classes obtained are overlapping.

Four ways of constructing non-projection finite element methods (for which $V_h \not\subset V$) are: By

1. "Reduction" of a finite element basis in $V_h \subset V$.
2. Consistent deduction with basis functions which are not elements of V because all required continuity conditions are not fulfilled.
3. Inconsistent deduction with two or more independent finite-dimensional vector spaces.
   a) Separate fields, over the elements Ω_e are approximated indepedently, (e.g. in elasticity $\{u\}$ and $\{\sigma\}$ may be approximated independently which generally is inconsistent because $\{\varepsilon\}$ is different if calculated from $\{u\}$ or from $\{\sigma\}$) ("mixed" method).
   b) Separate fields, one over all Ω_e and one over all $\partial\Omega_e$, are approximated independently ("hybrid" method).
4. Adding new functions to a basis of $V_h \subset V$ so chosen that the

homogeneous differential equation is exactly satisfied inside element ("exact" method).

The four ways of constructing non-projection methods will be demonstrated by examples below.

3.2. Example of "reduced" element

A simple example is the "reduced bilinear element" which can be applied to second order operators. For the ordinary bilinear element a subspace $V_h \subset V$ is spanned by the functions 1, x_1, x_2, $x_1 x_2$. Over a square, Fig. 1, then any function v_h in V_h is expressed in the node values v_h^i, $i = 1,2,3,4$ by

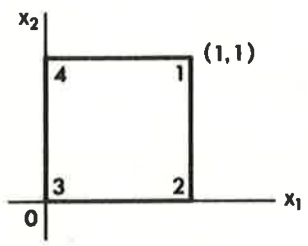


Fig. 1

$$v_h(x) = v_h^1 x_1 x_2 + v_h^2 x_1 (1-x_2) + v_h^3 (1-x_1)(1-x_2) + v_h^4 (1-x_1) x_2 \ . \quad (32)$$

Continuity between such elements is fulfilled so $v_h \in V$ and a projection method is obtained with this bilinear element.

Dirichlet's problem in variational formulation (4) is approximated according to (9). With bilinear elements the contribution from one element to the matrix $a(\varphi_j, \varphi_k)$ of (9) is

$$\begin{bmatrix} .67 & -.17 & -.33 & -.17 \\ -.17 & .67 & -.17 & -.33 \\ -.33 & -.17 & .67 & -.17 \\ -.17 & -.33 & -.17 & .67 \end{bmatrix} \qquad (33)$$

This element matrix is degenerated of rank 1. (After adding the contributions from all elements and introducing the boundary conditions the degeneracy disappears).

For same purposes it may be advantageous to use a space V_k spanned by piecewise linear polynomials. One way to obtain this is to approximate the x_1x_2-terms in (32) by a linear polynomial $f^a = a+bx_1+cx_2$. Then minimize $I = \int(x_1x_2-f^a)^2dx$ over the square with respect to a, b, c. The result will be

$$x_1x_2 \approx -.25 + .50x_1 + .50x_2 \ . \qquad (34)$$

Introduction into (32) gives

$$\begin{aligned} v_k(x) = {} & v_k^1(-.25 + .50x_1 + .50x_2) + v_k^2(.25 + .50x_1 - .50x_2) + \\ & + v_k^3(.75 - .50x_1 - .50x_2) + v_k^4(.25 - .50x_1 + .50x_2) \ . \end{aligned} \qquad (35)$$

Corresponding contribution from one element to the matrix $a(\varphi_j,\varphi_k)$ is

$$\begin{bmatrix} .50 & 0 & -.50 & 0 \\ 0 & .50 & 0 & -.50 \\ -.50 & 0 & .50 & 0 \\ 0 & -.50 & 0 & .50 \end{bmatrix} \qquad (36)$$

which is degenerated of rank 2. Still, after adding contributions from elements and regarding the boundary conditions the degeneracy disappears.

A basis function in V_k for square elements is visualized in Fig. 2.

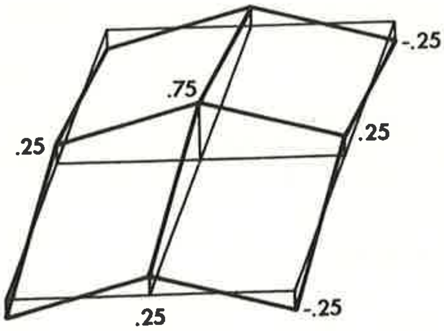


Fig. 2.

We observe that it is not a typical finite element basis function as it is not zero along the boundary of the shown patch. This means that the continuity condition for functions in V is not fulfilled so $V_k \not\subset V$. The method then is a non-projection one.

The solution, however, is convergent to the correct solution, because the error obtained by disregarding line integrals along interelement boundaries is of small order and tends to zero with decreasing element size.

The coefficients of (36) can be obtained in other ways. The simplest way is to "diagonalize" (33) by adding in each row neighbour coefficients and to place the sums in diagonals according to (36). This procedure is motivated simply by the fact that the function values which get the same coefficients in a row are "neighbour" function values.

A third way to obtain the coefficients in (36) is to calculate the coefficients of (33) by an approximative integration rule. Use here the formula $I = \int f dx \approx f(1/2,1/2)$ over the unit square, Fig.

1, which is exact for a linear polynomial but only approximative for a second-order polynomial.

Coefficients 11 and 12 in (33) are according to (22), (32), and (3)

$$a(\varphi_1,\varphi_1) = \int[(\frac{\partial\varphi_1}{\partial x_1})^2 + (\frac{\partial\varphi_1}{\partial x_2})^2]dx = \int(x_2^2 + x_1^2)dx \approx$$

$$\approx 2[x_1^2(1/2,1/2)] = 0.50 \ ,$$

$$a(\varphi_1,\varphi_2) = \int[\frac{\partial\varphi_1}{\partial x_1}\frac{\partial\varphi_2}{\partial x_1} + \frac{\partial\varphi_1}{\partial x_2}\frac{\partial\varphi_2}{\partial x_2}]dx = \int[x_2(1-x_2) + x_1(-x_1)]dx \approx$$

$$\approx x_2(1/2,1/2) - 2[x_1^2(1/2,1/2)] = 0.50 - 0.50 = 0 \ .$$

3.3. Example on consistent deduction of non-projection finite element method

Consistent deduction of so called non-conforming elements has been tried in many versions for the fourth order differential equations in plate bending. The reason is that for this case the continuity requirement for the weak formulation is that both the function and its derivatives should be continuous. This is difficult to achieve so non-conforming elements are tried.

Also for second order differential equations as in the Dirichlet problem non-conforming elements of this type have been constructed. Thus, to the bilinear basis (32) for the rectangle two functions have been added, see [9]. These are parabolas, Fig. 3, governed by for each element specific parameters. Then continuity between elements is not obtained. The line integrals around the element are, however, not needed because they almost cancel two and two along opposite sides so the sum of the line integrals tends to zero for

for decreasing element size, see [4].

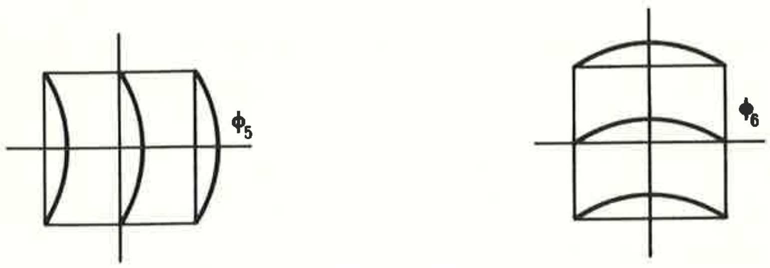


Fig. 3

Non-conforming elements of the "reduced" or "consistent" type must always be analysed in order to check if the line integrals around the elements are "nearly" zero or are "nearly" cancelling. Only then they can be used as if they were conforming, that is as if they build up a projection method.

For regular cases these elements can also be analysed by regarding the linear equations as difference equations and checking if these are consistent with the differential equations or not.

3.4. Mixed and hybrid methods

General

Typical for so called mixed and hybrid finite element methods is that the unknown function and one or more differential operators of it are approximated independently. The approximation can then be said to be inconsistent. For example, in elasticity displacements $\{u\}$ and stresses $\{\sigma\}$ may be approximated independently. Then the strains $\{\varepsilon\}$ are not consistently approximated. It takes one value if calculated from the displacements and another value if calculated from the stresses.

If all inconsistent approximations are defined over the whole domain Ω the method is said to be mixed. In this case the unknowns of the resulting linear equations are of two or more types, for example displacements and stresses.

If, instead, some approximations are defined over the whole domain and others only over the interelement boundaries, the method is said to be hybrid.

In this case only the last type of variables appears as unknowns in the resulting set of linear equations.

Mixed elements

Mixed elements have been used mostly for the fourth order differential equation of plate bending. This equation can be split into a so called canonical form. (In fact, it is deduced from this form). The canonical form can be shown by a transformation diagram

$$\begin{array}{ccc} \{M\} & \xrightarrow{\bar{\nabla}_2^T} & \{W\} \\ \uparrow S^d & & \\ \{\kappa\} & \xleftarrow{\bar{\nabla}_2} & \{w\} \end{array} \qquad \bar{\nabla}_2^T S^d \bar{\nabla}_2 \{w\} = \{W\} \tag{37}$$

where

$$\{M\} = \begin{bmatrix} M_1 \\ M_2 \\ M_{12} \end{bmatrix}, \quad \{\kappa\} = \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{bmatrix}, \quad \bar{\nabla}_2 = -\begin{bmatrix} \partial^2/\partial x_1^2 \\ \partial^2/\partial x_2^2 \\ 2\partial^2/\partial x_1 \partial x_2 \end{bmatrix} \tag{38}$$

and the relation between moments $\{M\}$ and curvatures $\{\kappa\}$ is

$$S^d = \text{const} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{39}$$

Let us assume that all variables are defined over a polygonal region

Ω and that $w = \partial w/\partial n = 0$ on $\partial\Omega$.

The fourth order differential equation (37) can be split into

$$(S^d)^{-1}\{M\} = \bar{\nabla}_2\{w\} ,$$
$$\bar{\nabla}_2{}^T\{M\} = \{W\} . \qquad (40)$$

Let now M be an element in a product space $\mathcal{M}=L_2(\Omega)\times L_2(\Omega)\times L_2(\Omega)$ and w an element in a space $\mathcal{V}$ of functions which are continuous, have continuous first derivatives and which fulfil the boundary conditions. Then scalar multiply the first equation with an element M^* in $\mathcal{M}$, multiply the second equation with an element w^* in $\mathcal{V}$ and integrate both equations over the domain. For simplicity let in (39) the constant be one and $\nu = 0$. We can now state the <u>variational formulation of the plate equation in mixed form</u>: Find that pair (M,w) in $\mathcal{M} \times \mathcal{V}$ that makes

$$E(M^*,M) = E(M^*,\bar{\nabla}_2 w), \quad \text{for all} \quad M^* \in \mathcal{M}$$
$$(\nabla_2{}^T M,w^*) = (W,w^*), \quad \text{for all} \quad w^* \in \mathcal{V} \qquad (41)$$

where

$$E(M^*,M) = \int_\Omega (M_1^* M_1 + M_2^* M_2 + 2M_{12}^* M_{12})dx ,$$

$$E(M^*,\bar{\nabla}_2 w) = -\int_\Omega \left(M_1^* \frac{\partial^2 w}{\partial x_1^2} + M_2^* \frac{\partial^2 w}{\partial x_2^2} + 2M_{12}^* \frac{\partial^2 w}{\partial x_1 \partial x_2}\right)dx ,$$

$$(\bar{\nabla}_2{}^T M,w^*) = -\int_\Omega \left(\frac{\partial^2 M_1}{\partial x_1^2} + \frac{\partial^2 M_2}{\partial x_2^2} + 2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2}\right)w^* dx ,$$

$$(W,w^*) = \int_\Omega f w^* dx .$$

By application of the divergence theorem twice and the boundary conditions to $E(M^*,\bar{\nabla}_2 w)$ it will be found that this form is equal to $(\bar{\nabla}_2{}^T M, w^*)$ so (41) is symmetric.

Let us now study a region Ω that is divided into polygonal finite elements Ω_e. For an element then the integral $E(M,\bar{\nabla}_2 w)$ over Ω_e can by the divergence theorem be transformed to line integrals, a sum and one area integral:

$$E(M,\bar{\nabla}_2 w)_{\Omega_e} = -\int_{\partial\Omega_e} M_n \frac{\partial w}{\partial n} ds + \int_{\partial\Omega_e} R_n w ds -$$

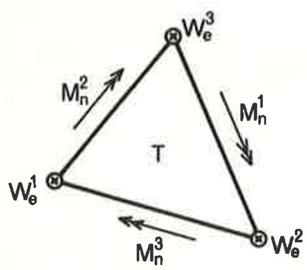
$$-\sum_i W(P_i)w(P_i) - \int_{\Omega_e} w(-\bar{\nabla}_2{}^T M)dx \ .$$

Here M_n is the normal boundary moment, R_n is the boundary Kirchhoff force, and $W(P_i)$ is the corner force at corner P_i. The sum is taken over the boundary corners.

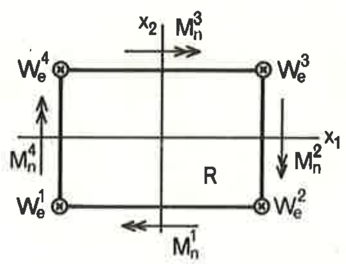
We shall find that in this case the line integrals cannot all be disregarded when constructing an approximate solution. On the other hand the area integral on right hand side of (42) can be disregarded for constant or linear M.

Two simple examples are the following:

Triangle T



Rectangle R



Spaces of finite element functions are

$\mathcal{V}_h$: w_h such that

1. w linear on T (comb. of $1, x_1, x_2$)
2. w continuous on Ω
3. w is represented by the three corner displacements

1. w bilinear on R (comb. of $1, x_1, x_2, x_1x_2$)
2. w continuous on Ω
3. w is represented by the four corner displacements

$\mathcal{M}_h$: $M_h = \{M_1, M_2, M_{12}\}$ such that

1. M_1, M_2, M_{12} constant on T
2. M_n equal between elements
3. M_h is represented by M_n

1. M_1 linear in x_1 on R
   M_2 linear in x_2 on R
   M_{12} constant on R
2. M_n equal between elements
3. M_h is represented by M_n and M_{12}

The approximate finite element solution is that $(w_h, M_h) \in \mathcal{V}_h \times \mathcal{M}_h$ for which

$$E(M_h^*, M_h) = E(M_h^*, \bar{\nabla}_2 w_h), \quad \text{for all} \quad M_h^* \in \mathcal{M}_h$$
$$(\bar{\nabla}_2{}^T M_h, w_h^*) = (W, w_h^*), \quad \text{for all} \quad w_h^* \in \mathcal{V}_h \tag{43}$$

where

$$E(M_h^*, M_h) = \int_\Omega (M_{h1}^* M_{h1} + M_{h2}^* M_{h2} + 2M_{h12}^* M_{h12}) dx \tag{44}$$

$$E(M_h^*, \bar{\nabla}_2 w_h) = \sum_j \int_{\partial\Omega_e} R_{nh}^* w_h ds - \sum_i W_h^*(P_i) w_h(P_i) \tag{45}$$

The first sum on the right hand side should be taken over all elements and the second over all nodes P_i.

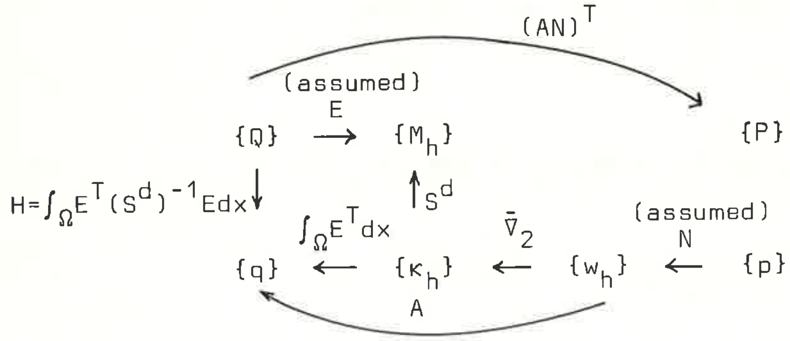
Comparing (45) with (42) it is found that the first and last term in (42) have no corresponding terms in (45). The last term is zero for the examples considered with $M = M_h$. It could be taken into account by substituting W for $\bar{\nabla}_2{}^T M$. It should then be regarde as a "particular" solution. The first term is not zero for $M_n \in \mathcal{M}_h$ and $w \in \mathcal{V}_h$ as M_{nh} is equal at both sides between two elements but $\partial w_h/\partial n$ takes different values at the two sides of such a line. Stil the solution tends to the correct one with this term disregarded.

Finally, we observe that for the <u>triangle</u> above $R_{nh} = 0$ so the equilibrium conditions between the elements are fulfilled. For zero surface load then all equilibrium conditions are fulfilled.

Let now $\mathcal{H} \subset \mathcal{M}$ be the corresponding finite-dimensional space. Then $\mathcal{H}_h \subset \mathcal{H}$. The method is thus a <u>projection one</u>, so the distance between the correct M and the approximate M_h is a minimum for all M fulfilling the equilibrium conditions.

On the other hand, for the <u>rectangle</u> above $R_{nh} \neq 0$ so the method is a <u>non-projection one</u>. Still the solution converges towards the correct one.

The mixed approximate solution can be illustrated by a transfor- mation diagram



Moments $\{M_h\}$ and deflections $\{w_h\}$ are assumed as polynomials represented by parameters $\{Q\}$ and $\{p\}$. These are mostly interelement moments and node deflections. Also inner parameters can appear (see the rectangle above). The transformation matrix [A] can be obtained from (45) or can, if {q} is physically recognizable, be deduced directly. The transformation $[AN]^T$ comes from the invariance of virtual work.

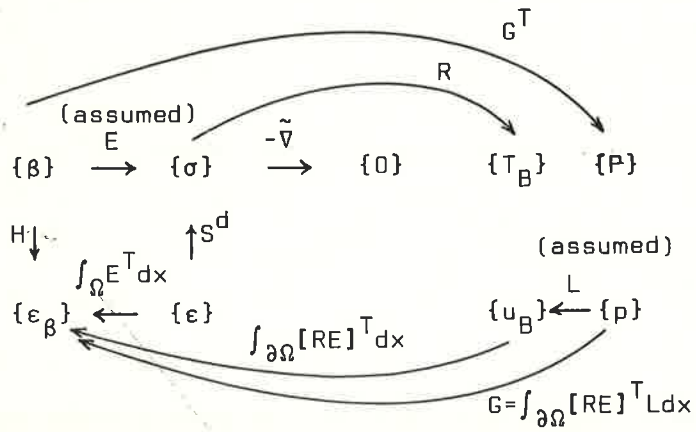
The resulting set of equations will get the following form

$$\begin{bmatrix} H & -AN \\ (AN)^T & 0 \end{bmatrix} \begin{bmatrix} Q \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ P \end{bmatrix} \tag{46}$$

which is antisymmetric and non-definite.

Hybrid elements

Another type of finite element method with inconsistent deduction is proposed by Pian [10]. It has been used for differential equations of both second and fourth order. We will here apply it to linear elasticity. First, the stresses $\{\sigma\}$ in a element are expressed by a polynomial in some parameters $\{\beta\}$ in such a way that $\tilde{\nabla}\{\sigma\} = 0$, that is, in such a way that the homogeneous equilibrium equations are satisfied inside the element, see the diagram below.



From these stresses the tractions at the boundary of the element are calculated:

$$\{T_B\} = [R]\{\sigma\} = [R][E]\{\beta\} \ . \tag{47}$$

By application twice of the invariance of virtual work the variables $\{\varepsilon_\beta\}$ dual to $\{\beta\}$ are calculated in two ways

$$\{\varepsilon_\beta\} = \int_\Omega [E]^T\{\varepsilon\}dx = \int_\Omega [E]^T[S^d]^{-1}[E]\{\beta\}dx = [H]\{\beta\} \ ,$$
$$\{\varepsilon_\beta\} = \int_{\partial\Omega_e} [RE]^T\{u_B\}dx \ . \tag{48}$$

Here $\{u_B\}$ are the variables (element boundary displacements) dual to $\{T_B\}$. These are expressed by a polynomial in the node displacements $\{p\}$

$$\{u_B\} = [L]\{p\} \tag{49}$$

so

$$\{\varepsilon_\beta\} = \int_{\partial\Omega_e} [RE]^T[L]\{p\}dx = [G]\{p\} \ . \tag{50}$$

By one further application of the invariance of virtual work the transformation dual to (50) is obtained

$$\{P\} = [G]^T\{\beta\} \ . \tag{51}$$

From the diagram the following equations in "displacement" form are obtained

$$([G]^T[H]^{-1}[G])\{p\} = \{P\} \ . \tag{52}$$

Thus a non-projection "displacement" method has been constructed. We observe that pointiwise inside the element homogeneous equilibrium equations are fulfilled. If $\{\sigma\}$ is a polynomial of first order so is also $\{\varepsilon\}$ of that order and the compatibility conditions are identically satisfied. Between the elements neither $\{u\}$ nor $\{\sigma\}$ are continuous.

A simple example is the rectangular 2D-elasticity element, Fig. 4.

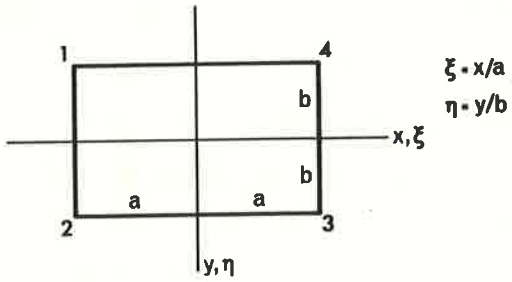


Fig. 4

With

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} 1 & \eta & 0 & 0 & 0 \\ 0 & 0 & 1 & \xi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} , \quad \{\sigma\} = [E_e]\{\beta\} \tag{53}$$

$$\tilde{\nabla} E_e \equiv 0 \tag{54}$$

so E_e is admissible.

With the 2·4 corner displacements as parameters the displacement polynomials along the edges are linear so L_e is easily established. It is a routine procedure to establish the resulting equations according to the diagram. This element happens to be the same as that described in section 3.3 which was deduced consistently, see [15].

3.5. "Exact" finite element methods

With "exact" finite element methods we mean methods with approximations which fulfil exact the homogeneous differential equations locally inside the element. For such elements then all approximations are concentrated to the interelement boundaries.

Evidently, for fourth oreder differential equations polynomials up to order three give exact methods. For second order differential equations polynomials of zero and first order give exact methods.

An example of an "exact" projection method for the fourth order biharmonic equation is the triangular "constant moment" triangle described in section 3.4. The "constant strain" triangle is an example of an "exact" projection method for any second order differential equation.

Already the bilinear approximation for the rectangle does not give an "exact" method for the 2D-Navier equations. In this case it is possible to add functions so the resulting approximation gives an "exact" method.

For the 2D-theory of elasticity diagram (31) is valid. The bilinear approximation matrix for the rectangle, Fig. 4, is

$$N_e = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \tag{55}$$

for $p_e = [u_x^1, u_y^1, u_x^2, v_y^2, u_x^3, v_y^3, u_x^4, u_y^4]^T$

$$u_h = [u_x, u_y]^T$$

where

$$4N_1 = (1-\xi)(1-\eta)$$
$$4N_2 = (1-\xi)(1+\eta)$$
$$4N_3 = (1+\xi)(1+\eta)$$
$$4N_4 = (1+\xi)(1-\eta)$$

With this approximation the method is a projection one but not an "exact" one. Is it possible to find polynomials N_{xi} and N_{yi} according to

$$N_e = \begin{bmatrix} N_1 & N_{x1} & N_2 & N_{x2} & N_3 & N_{x3} & N_4 & N_{x4} \\ N_{y1} & N_1 & N_{y2} & N_2 & N_{y3} & N_3 & N_{y4} & N_4 \end{bmatrix} \tag{56}$$

such that the homogeneous Navier's equations (29)

$$\tilde{\nabla}^T S^d \tilde{\nabla} u = \begin{bmatrix} \partial^2/\partial x^2 + [(1-\nu)/2]\partial^2/\partial y^2 & (1+\nu)\partial^2/\partial x\partial y \\ (1+\nu)\partial^2/\partial x\partial y & \partial^2/\partial y^2 + [(1-\nu)/2]\partial^2 \partial x^2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = 0 \tag{57}$$

are satisfied inside the element and such that the polynomials are zero at the nodes? If so, will the non-projection finite element method consistently deduced from (56) be convergent to the correct solution? The answer is yes.

Let us calculate N_{y1}. Then take all components of p_e equal to zero except u_x^1 which is taken equal to unity. Then $u_x = N_1$, $u_y = N_{y1}$ is substituted into (57) from which N_{y1} can be solved in terms of 4 constants α_i:

$$N_{y1} = \alpha_1(a^2\xi^2/2(1+\nu) - b^2\eta^2/4) + a_2\xi + a_3\eta + \alpha_4 - b(1+\nu)\eta^2/16a \; . \tag{58}$$

The boundary conditions are that N_{y1} is zero at the four corners. The obtained set of equations is, however, of rank 3 only so N_{y1} can be given with one arbitrary constant γ:

$$N_{y1} = \gamma a(1-\xi^2)8b + b(1-\eta^2)[1+\nu-\beta(1+\nu)]/16a \; . \qquad (59)$$

For $\gamma = 1$

$$N_{y1} = a(1-\xi^2)/8b + \nu b(1-\eta^2)/8a \qquad (60)$$

which is the same as obtained both in the example on consistent deduction in 3.3 above and in the example on a hybrid method in 3.4, see [15]. The stresses are thus in this case

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} 1 & \eta & 0 & 0 & 0 \\ 0 & 0 & 1 & \xi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} \qquad (61)$$

which was proposed already by Turner et al [16].

For $\beta = -(2+\nu)/\nu$ the following stresses are obtained

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} 1 & \xi & 0 & 0 & 0 \\ 0 & 0 & 1 & \eta & 0 \\ 0 & -\eta & 0 & -\xi & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} \qquad (62)$$

which has been proposed already by Gallagher [17].

Conclusions

The possibility of using non-projection finite element methods broadens considerably the field of admissible approximation functions. The non-projection methods often show advantages: polynomials of lower order may be used, functions like stresses may be better approximated, important modes are taken into account, the homogeneous differential equation may be locally satisfied. Important contributions to the understanding of non-projection methods have been made the last years by among others Johnson, Nitsche, Strang, Ciarlet, Brezzi, and Oden.

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