ANALYSIS OF A CABLE NETWORK WITH NEGATIVE GAUSSIAN CURVATURE

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1. INTRODUCTION

On comparison with other kinds of cable roofs, one with negative Gaussian curvature is found to possess a number of architectural and statical advantages. However, the application of this type of cable roof is justified from an economic standpoint only if the edge beams, resisting the horizontal forces of the cable net, are designed expediently. The bending moment free-edge beam, often mentioned in literature, is reasonable only with a certain combination of loads. As a rule, the live load induces an increase or diminution of bending moments in the edge beam. Nevertheless, the edge beam is often considered to be absolutely stiff when analysis is made of the inner forces and deflections of the network. This supposition, however, leads to over-rating of the tensile forces in the carrying (concave) cables, and to under-rating of the forces in the stretching (convex) cables and the roof deflections. It is consequently essential that consideration is given at the design stage to the co-operation of the cable network with the edge beam.

The methods of analysis developed at the Tallinn Polytechnic Institute cenable computation of the inner forces in the cable roof, and its deflections, with due account being taken of the horizontal deflections of the edge beam induced by the forces of the cable network. In this, however the cable network is regarded as being a geometrically non-linear elastic system with limited strains (ε << 1) The edge beam is assumed to be a linear elastic system. The method of analysis described below is applicable by employment of the both the discrete computing scheme, and the continuous scheme.

2. INITIAL FORM OF THE PRESTRESSED CABLE NETWORK

As a rule, the initial form of the roof is taken as a well-known surface that is easy to describe mathematically . A number of them automatically satisfy the equilibrium conditions of a prestressed cable network without external loads. The accomplishment of such surfaces often proves rather troublesome, since in this case the cables must be connected to each other in the nodes during the prestressing procedure. Moreover, it is often difficult to choose the correct shape of the edge beam that is statically suitable for a given surface of the roof, and at the same time acceptable from the aspect of erection. In many cases, it is necessary to be content with a non-plane curved edge beam which may prove rather difficult to carry into effect. Consequently, a great deal of interest is attached to determination of the surface for a roof in which the shape of the edge beam is given beforehand, and the nodes of the cable net are themselves in equilibrium under the prestressing forces of the cables. Of the possible variants of the surfaces, the most interesting are those which form themselves by the mutual free sliding of the cables one against the other.

With the system of a freely sliding cable network, the prestressing force is the same along the entire length of each cable, and the equilibrium conditions of the node i, k (fig 1) are expressed by the following vector-equation:

$$S_{oi}\left(\frac{\bar{r}_{i,k+1}-\bar{r}_{i,k}}{a_{i,k}}+\frac{\bar{r}_{i,k-1}-\bar{r}_{i,k}}{a_{i,k-1}}\right)+T_{ok}\left(\frac{\bar{r}_{i+1,k}-\bar{r}_{i,k}}{b_{i,k}}+\frac{\bar{r}_{i-1,k}-\bar{r}_{i,k}}{b_{i-1,k}}\right)=0. \quad (1)$$

where S_{oi} - the force in the i-th carrying cable from prestressing only;

T_{ok} - the corresponding force in the k-th stretching cable;

$r_{i,k}$ - the radius-vector of the node i,k;

$a_{i,k}$ - the length of the k-th section of the i-th carrying cable;

$b_{i,k}$ - the length of the i-th section of the k-th stretching cable.

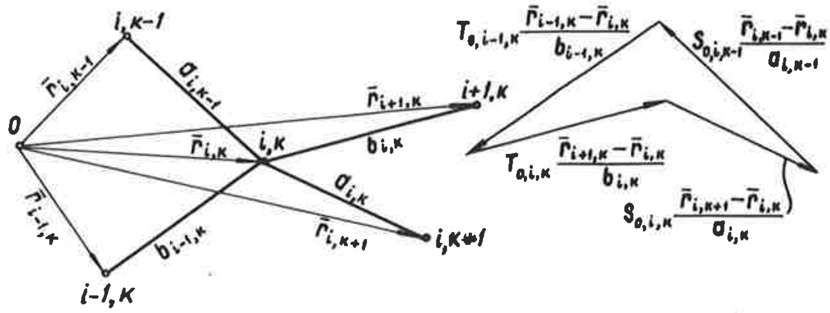

Fig. 1. Node i,k and cable forces acting on it.

The three scalar-equations that follow correspond to equation (1):

$$S_{oi}\left(\frac{X_{i,k+1}-X_{i,k}}{a_{i,k}}+\frac{X_{i,k-1}-X_{i,k}}{a_{i,k-1}}\right)+T_{ok}\left(\frac{X_{i+1,k}-X_{i,k}}{b_{i,k}}+\frac{X_{i-1,k}-X_{i,k}}{b_{i-1,k}}\right)=0, \quad (2)$$

$$S_{oi}\left(\frac{Y_{i,k+1}-Y_{i,k}}{a_{i,k}}+\frac{Y_{i,k-1}-Y_{i,k}}{a_{i,k-1}}\right)+T_{ok}\left(\frac{Y_{i+1,k}-Y_{i,k}}{b_{i,k}}+\frac{Y_{i-1,k}-Y_{i,k}}{b_{i-1,k}}\right)=0, \quad (3)$$

$$S_{oi}\left(\frac{Z_{i,k+1}-Z_{i,k}}{a_{i,k}}+\frac{Z_{i,k-1}-Z_{i,k}}{a_{i,k-1}}\right)+T_{ok}\left(\frac{Z_{i+1,k}-Z_{i,k}}{b_{i,k}}+\frac{Z_{i-1,k}-Z_{i,k}}{b_{i-1,k}}\right)=0, \quad (4)$$

from which it is possible directly to find co-ordinates X_{ik}; $Y_{i,k}$; $Z_{i,k}$ of the net nodes, when the prestressing forces of the cables S_{oi}; T_{ok} and the co-ordinates of the contour nodes are given. When the cable net is orthegonal on plan, and the cables of both families are placed at the same intervals, a and b respectively, the following equation is derived for the ordinate $Z_{i,k}$ of the nede i,k (4):

$$Z_{i,k}=\frac{\left(Z_{i,k-1}+Z_{i,k+1}\right)+\frac{H_{ok}a}{G_{oi}b}\left(Z_{i-1,k}+Z_{i+1,k}\right)}{2\left(1+\frac{H_{ok}a}{G_{oi}b}\right)}, \quad (5)$$

where a - the length of the section of carrying cables (the interval between the stretching cables);

b - the length of the section of the stretching cables (the interval between the carrying cables);

G_i - the horizontal projection of the inner force S_{oi}

H_k - the horizontal projection of the inner force T_{ok}.

To permit of (5) also being used for the nodes placed near the contour (fig.2), it is necessary for some fictitious nodes to be employed outside the contour, for which there is obtained

$$Z_{m+1,k} = \frac{Z_{o,k} - \frac{a1}{a} Z_{m,k}}{1 - \frac{a1}{a}} \quad \text{or} \quad Z_{i,n+1} = \frac{Z_{i,o} - \frac{b1}{b} Z_{i,n}}{1 - \frac{b1}{b}} . \qquad (6)$$

where $Z_{o,k}$; $Z_{i,o}$ - the ordinates of the contour nodes of the k-th stretching and i-th carrying cables respectively

$Z_{m,k}$; $Z_{i,n}$ - the ordinates of the nodes of the k-th stretching and the i-th carrying cable respectively, next to the contour nodes;

a1 ; b1 - refer to fig.2.

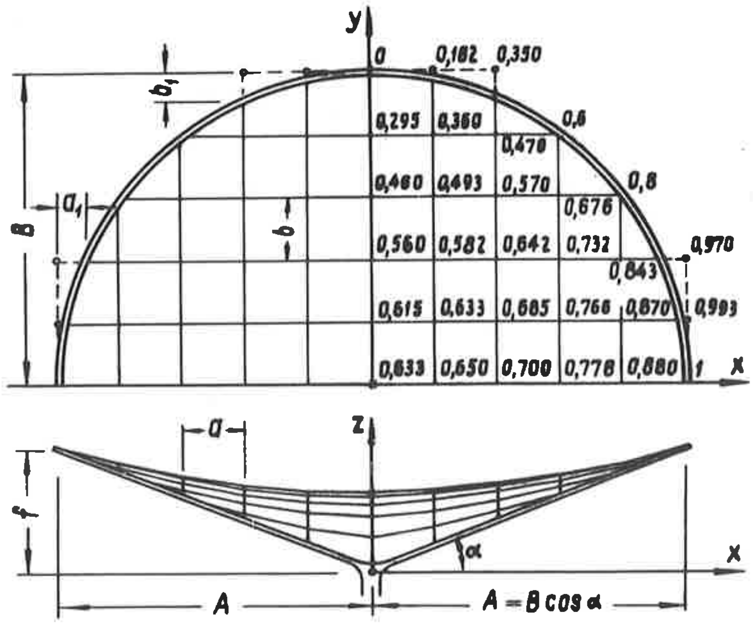

Fig. 2 Co-ordinates of the nodes of a cable network.

Fig. 2 provides an example in which the co-ordinates of a cable net, comprising 9 carrying cables and the same number of stretching cables, have been determined by the application of equations (5) and (6). In this case, the edge beam has been formed by two plane half-circle arches.

3. INNER FORCES AND DEFLECTIONS OF A CABLE NETWORK WITH A LIMITED NUMBER OF CABLES

For determination of the inner forces and deflections, in general it becomes necessary to solve a system of equilibrium equations and equations of geometrical compatibility. The lengths of the sections of cables, and their horizontal projections, have to be found beforehand by means of the formulae given in chapter 2. The deflections of the contour nodes are to be found as linear functions of the inner forces of the cables, with the help of the corresponding influence lines. The total number of equations equals five times the number of nodes.

With orthogonal cable networks, solution of the system is markedly simplified. It has been established by computing experience and experimental research work that it is possible, under the influence of vertical loads, to ignore in the equations those horizontal components of displacements which are perpendicular to the plane of the cable under examinations. Similarly, it can be accepted that the horizontal projections of the inner forces of the cables are the same in all sections of a cable. Furthermore it is accepted that the edge beam, consisting of two plane arches, is supported by stiff vertical columns, and can accordingly be displaced only in the horizontal direction.

In this case, the system of equations assumes the following form:

$$W_{i,k} = \frac{1}{2\left(1+\frac{H_k a}{G_i b}\right)}\left[(W_{i,k-1}+W_{i,k+1}) + \frac{H_k a}{G_i b}(W_{i-1,k}+W_{i+1,k}) + \right.$$

$$\left. + (Z_{i,k-1}-2Z_{i,k}+Z_{i,k+1}) + \frac{H_k a}{G_i b}(Z_{i-1,k}-2Z_{i,k}+Z_{i+1,k}) - \frac{P_{i,k} a}{G_i}\right], \quad (7)$$

$$G_i = G_{0i} + \frac{EF_i}{\sum_{\ell=1}^{n}\left[1+\left(\frac{Z_{i,\ell+1}-Z_{i,\ell}}{a}\right)^2\right]^{3/2}}\left[\sum_{\ell=1}^{n}\frac{W_{i,\ell+1}-W_{i,\ell}}{a}\left(\frac{Z_{i,\ell+1}-Z_{i,\ell}}{a}+\right.\right.$$

$$\left.\left. + \frac{W_{i,\ell+1}-W_{i,\ell}}{2a}\right) - \sum_{j=1}^{m}(G_j-G_{0j})\eta_{ij} - \sum_{\ell=1}^{n}(H_\ell-H_{0\ell})\eta_{i,\ell}\cos\alpha\right], \quad (8)$$

$$H_k = H_{0k} + \frac{EF_k}{\sum_{j=1}^{m}\left[1+\left(\frac{Z_{j+1,k}-Z_{j,k}}{b}\right)^2\right]^{3/2}}\left[\sum_{j=1}^{m}\frac{W_{j+1,k}-W_{j,k}}{b}\left(\frac{Z_{j+1,k}-Z_{j,k}}{b}+\right.\right.$$

$$\left.\left. + \frac{W_{j+1,k}-W_{j,k}}{2b}\right) - \sum_{j=1}^{m}(G_j-G_{0j})\frac{\eta_{k,j}}{\cos\alpha} - \sum_{\ell=1}^{n}(H_\ell-H_{0\ell})\eta_{k,\ell}\right], \quad (9)$$

where g_{0i}; g_i; H_{0k}; H_k - the respective horizontal projections of the forces of the i-th carrying cable and the k-th stretching cable before and after application of the system or external loads $P_{i,k}$;

$W_{i,k}$ - deflection of the node i, k induced by external forces $P_{i,k}$;

$Z_{i,k}$ - ordinate of the node i,k before application of the external loads;

$\eta_{i,j}$; $\eta_{i,\ell}$ - ordinates of the influence lines of the horizontal deflections of the i-th point of the edge beam in the vertical plane of j-th carrying cable, or the ℓ-th stretching cable respectively.

$\eta_{k,j}$; $\eta_{k,\ell}$ - ordinates, but for the k-th stretching cable.

α - the angle of the inclined plane of the edge beam.

The system of equations (7), (8), (9) contains one unknown quantity ($W_{i,k}$) for every node, and one unknown quantity (G_i or H_k) for every cable. The number of equations is thus considerably reduced, as compared with the exact solution. At the same time, the results obtained on analysis by the application of this greatly simplified means are wery well comparable with the results obtainable by both the exact method and experiment. Mention is also due that the solution system (7), (8), (9) can also be applied succesfully to an unorthogonal cable network.

With solution of the system of equations (7), (8), (9), it is possible to recommend the following well-converging method of successive approximation:

1) assume some approximate values of deflections $W_{i,k}$;

2) compute the preliminary approximations to inner forces G_i and

H_k with the aid of (8), (9);

3) solve the system, consisting of equations (7) with respect to deflections $W_{i,k}$, and using the G_i and H_k newly determined;

4) repeat the process until the degree of precision given earlier has been achieved.

This method of approximation can very well be applied with an electronic computer. Computing examples with a 9 × 9 cable network demonstrated that the solution of the problem took 10-15 minutes of time of computer "Minsk-22", if it had been assumed that the edge beam was stiff, and 22-25 minutes if the contour was deformable. However, when the number of cables is not very large, say with a cable net of 5 × 5, it is practicable to effect the analysis by means of the more usual arithmetical technique.

Fig. 3 provides an example of analysis of the deflections of the nodes and the inner forces of the cables in relation to the cable roof illustrated in fig. 2. This figure also indicates the results obtained in an exact model test made at the hockey stadium in Tallinn. It is apparent that the results coincide fairly well.

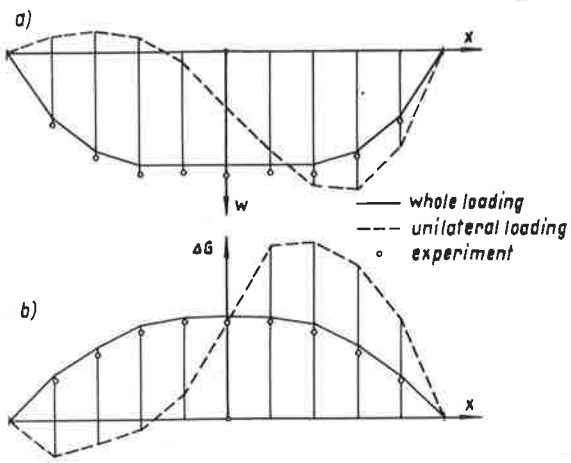

Fig. 3 Comparison of the calculated results for deflections and cable forces with experimental results.

4. ANALYSIS OF A HYPERBOLIC PARABOLOID CABLE ROOF WITH A CONTINUOUS CABLE NET

In the case of an unlimited number of cables in the solution, instead of an algebraic system of equations (7), (8), (9) there will appear a system of differential equations as follows:

$$G \frac{\partial^2(Z+W)}{\partial x^2} + H \frac{\partial^2(Z+W)}{\partial y^2} = q(x,y) \,, \tag{10}$$

that guarantees equilibrium in the vertical direction, and two equations of geometrical compatibility:

$$\int_{x_1}^{x_2} \frac{\partial u}{\partial x}\, dx = \frac{\Delta G}{E\delta_x} \int_{x_1}^{x_2} \left[1+\left(\frac{\partial z}{\partial x}\right)^2\right]^{3/2} dx - \int_{x_1}^{x_2} \frac{\partial w}{\partial x}\left(\frac{\partial z}{\partial x} + \frac{1}{2}\frac{\partial w}{\partial x}\right) dx \,, \tag{11}$$

$$\int_{y_1}^{y_2} \frac{\partial v}{\partial y}\, dy = \frac{\Delta H}{E\delta_y} \int_{y_1}^{y_2} \left[1+\left(\frac{\partial z}{\partial y}\right)^2\right]^{3/2} dy - \int_{y_1}^{y_2} \frac{\partial w}{\partial y}\left(\frac{\partial z}{\partial y} + \frac{1}{2}\frac{\partial w}{\partial y}\right) dy \,, \tag{12}$$

where δ_x; δ_y - the reduced thickness of the cable net;

u; v - the horizontal displacements of the network;

$\Delta G = G - G_0$ - the increase in forces of carrying cables on the unit of the width of the net;

$\Delta H = H - H_0$ - the corresponding increase in the stretching cables;

$x_1 = x_1(y)$; $x_2 = x_2(y)$ - the co-ordinates of the edge beam.
$y_1 = y_1(x)$; $y_2 = y_2(x)$

Note: in the composition of equations (11) and (12) use has been made of the wellknown non-linear geometrical expressions

$$\varepsilon_x = \frac{1}{1+\left(\frac{\partial z}{\partial x}\right)^2}\left[\frac{\partial u}{\partial x} + \frac{\partial w}{\partial x}\left(\frac{\partial z}{\partial x} + \frac{1}{2}\frac{\partial w}{\partial x}\right)\right] ,$$

$$\varepsilon_y = \frac{1}{1+\left(\frac{\partial z}{\partial y}\right)^2}\left[\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\left(\frac{\partial z}{\partial y} + \frac{1}{2}\frac{\partial w}{\partial y}\right)\right] ,$$

and the expressions of Hocke's law:

$$\varepsilon_x = \frac{\Delta G}{E\delta_x}\left[1+\left(\frac{\partial z}{\partial x}\right)^2\right]^{1/2} ,$$

$$\varepsilon_y = \frac{\Delta H}{E\delta_y}\left[1+\left(\frac{\partial z}{\partial y}\right)^2\right]^{1/2} .$$

The analysis of a hyperbolic paraboloid cable roof follows as an example. It has an elliptical edge beam on plan, which is supported by the vertical columns, so that the edge beam can be displaced only in the horizontal direction. In this case, it is reasonable to approximate the deflection function w of the roof in the form

$$w = \sum_{s=0,1...}\sum_{t=0,1...} w_{s,t}\left(\frac{x}{a}\right)^s\left(\frac{y}{b}\right)^t\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) , \qquad (13)$$

which automatically satisfies the boundary conditions on the entire contour ($w=0$). After substitution into equation: (11), (12) of the partial derivatives of z and w with respect to x and y they can be solved with respect to the inner forces of cables ΔG; ΔH. The forces so found are introduced into (10). Care is required to ensure that the left-hand side of eq. (10) modified in such a way corresponds most satisfactorily to the load function q(xy). This is achievable in many ways (say by the application of Bubnov-Galerkin's method). There is thus derived a system of equations with respect to the unknown parameters $w_{s,t}$. The left-hand sides of eq. (11), (12) can be expressed by the horizontal displacements of the edge beam induced by the forces of the cable net. If the load is distributed evenly over the whole roof, the horizontal loads on the edge beam are distributed almost parabolically (fig. 4). The edge beam is consequently loaded by corresponding distributed loads, as follows:

$$\Delta G = \max \Delta G\left(1 - \frac{y^2}{b^2}\right) ,$$

$$\Delta H = \max \Delta H\left(1 - \frac{x^2}{a^2}\right) ,$$

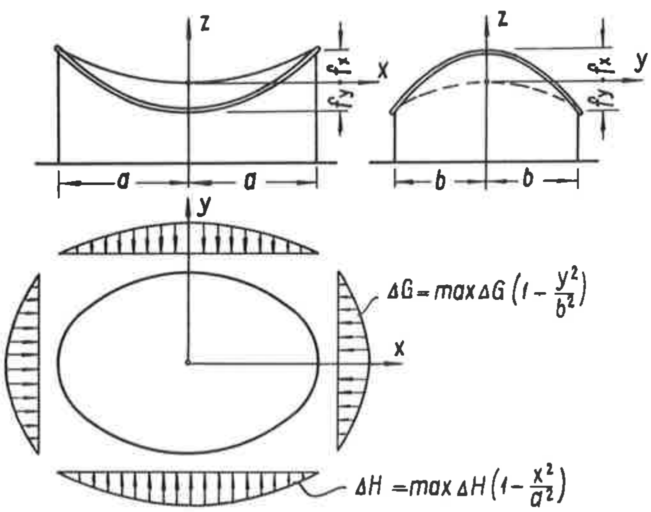

Fig. 4 Distribution of the horizontal loads on the edge beam

and the corresponding displacements of the edge beam are

$$EJ\Delta_{xx} = -\max \Delta G\left(1 - \frac{y^2}{b^2}\right)^{3/2} \left[\frac{19}{24} + \frac{1}{20}\left(1 - \frac{y^2}{b^2}\right)\right]\frac{a^{1/2}b^{7/2}}{12} , \quad (14)$$

$$EJ\Delta_{yx} = \max \Delta G\left(1 - \frac{x^2}{a^2}\right)^{3/2} \left[\frac{7}{8} - \frac{1}{20}\left(1 - \frac{x^2}{a^2}\right)\right]\frac{a^{3/2}b^{5/2}}{12} , \quad (15)$$

$$EJ\Delta_{xy} = \max \Delta H\left(1 - \frac{y^2}{b^2}\right)^{3/2} \left[\frac{7}{8} - \frac{1}{20}\left(1 - \frac{y^2}{b^2}\right)\right]\frac{a^{5/2}b^{3/2}}{12} , \quad (16)$$

$$EJ\Delta_{yy} = -\max \Delta H\left(1 - \frac{x^2}{a^2}\right)^{3/2} \left[\frac{19}{24} + \frac{1}{20}\left(1 - \frac{x^2}{a^2}\right)\right]\frac{a^{7/2}b^{1/2}}{12} . \quad (17)$$

After evaluation of the displacements of the edge beam according to formulae (14) to (17), and consideration of only one term $w_{s,t} = w_{0,0}$ in expression of the deflection function (13), the following equation is derived for solution of the problem:

$$\zeta_0^3(1+\psi+4\xi) + 3\zeta_0^2[(1-\psi\alpha)+2\xi(1-\alpha)] + 2\zeta_0[(1+\psi\alpha^2) + \xi(1-\alpha)^2 + \lambda + \lambda\xi(1+\frac{1}{\psi})] = q^*[1+\xi(1+\frac{1}{\psi})] , \quad (18)$$

where $\zeta_0 = \frac{W_{0,0}}{f_x}$; $\zeta_1 = \frac{W_{1,0}}{f_x}$ - the relative parameters of the deflection function;

$\alpha = \frac{f_y}{f_x}$; $\psi = \frac{a^4\delta_y\left(1 + \frac{5}{3}\frac{f_x^2}{a^2}\right)}{b^4\delta_x\left(1 + \frac{5}{3}\frac{f_y^2}{b^2}\right)}$ - the geometrical parameters;

$\lambda = \frac{9a^2\left(G_0 + H_0\frac{a^2}{b^2}\right)}{10E\delta_x f_x^3}\left(1 + \frac{5f_x^2}{3a^2}\right)$ - the prestressing factor;

$$q^* = \frac{9q\ a^4}{10E\delta_x f_x^3}\left(1 + \frac{5f_x^2}{3a^2}\right)$$ - the load factor;

$$\xi = \frac{5E\delta_y a^{7/2} b^{-1/2}}{72EJ(1+5f_y^2/3b^2)}$$ - the parameters of the bending stiffness of the edge beam.

The corresponding changes in the inner forces are

$$\Delta G = \frac{5E\delta_x f_x^2 \zeta_0 [(2+\zeta_0)+2\xi(1-\alpha+\zeta_0)]}{9a^2(1+5f_x^2/3a^2)[1+\xi(1+\frac{1}{\psi})]}\left(1-\frac{y^2}{b^2}\right), \tag{19}$$

$$\Delta H = \frac{-5E\delta_y f_x^2 \zeta_0 [(2\alpha-\zeta_0)-\frac{2}{\psi}\ \xi(1-\alpha+\zeta_0)]}{9b^2(1+5f_y^2/3b^2)[1+\xi(1+\frac{1}{\psi})]}\left(1-\frac{x^2}{a^2}\right). \tag{20}$$

From (18), representing an algebraic cubical equation, $\zeta_0 = W_{0,0}\ f_x$ can be evaluated by the method of successive approximation. In a analogous way, the problem can be solved when the loads are distributed unsymmetrically, for example in the direction of the carrying cables. In this case, if once again we are content with only one term in (13), the deflection parameter will be $W_{1,0}$ (that is, $s = 1;\ t = 0$) and the way of solution closely resembles that described above.

Fig. 5 illustrates the dependence of the relative deflection $W_{0,0}\ f_x$ upon the load factor q^* when the live load is on the whole surface of the roof and also when only one half of it is loaded. In this example, it is assumed that the live load (p) equals the dead load (g).

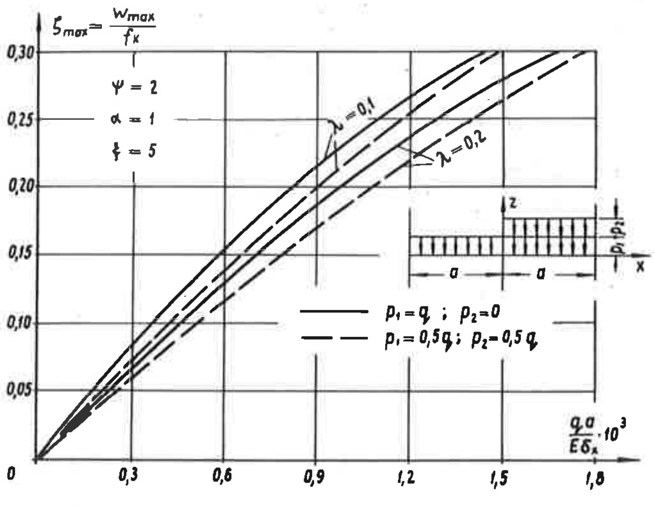

Fig. 5 Dependence of the relative deflection upon the load factor.

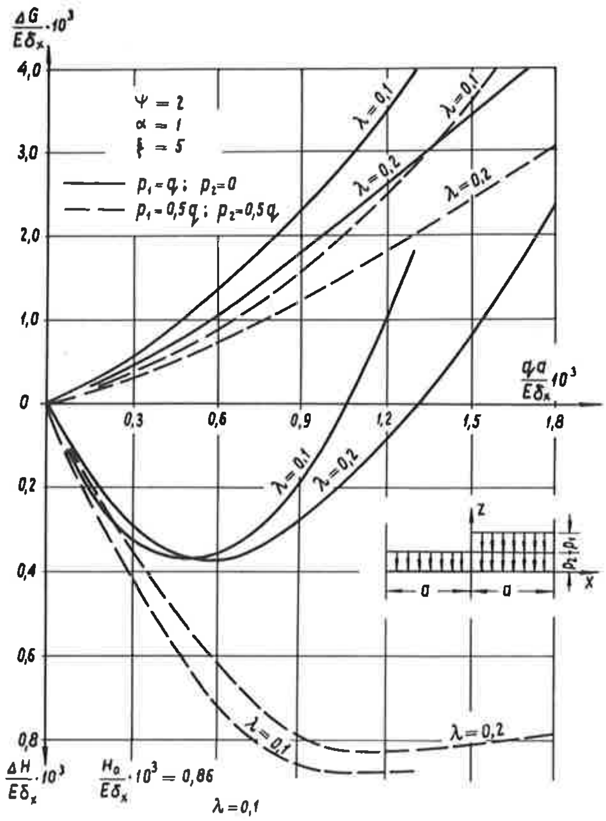

Fig. 6 Changes in the forces in the carrying and stretching cables due to loading.

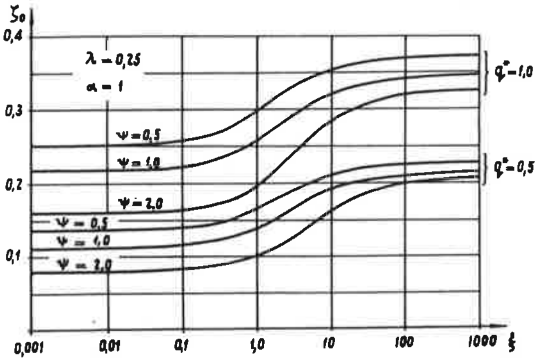

Fig. 7 Dependence of the relative deflection upon the bending stiffness of the edge beam.

The changes in the forces in the carrying and stretching cables arising from the increase in the loads, are indicated in fig. 6. It is (noteworthy that initially) the forces of the stretching cables diminish with increase in the loads. Subsequently, this diminution is retarded, and at a certain load the forces in the stretching cables begin once again to increase again. This implies that the stretching cables have recommenced work as tension members resisting the horizontal thrust of the edge beam. In this phenomenon of co-operation between the cables and the edge beam, an essential reserve is hidden, economically to raise the stiffness of the structure. In many cases, it appears more reasonable to increase the tension stiffness of the stretching cables, rather than the bending stiffness of the edge beam. A reasonable interval for the stiffness of the edge beam is illustrated in fig. 7. It is observable from this that the maximum vertical deflections with the diminution in stiffness of the edge beam asymptotically approach a certain limit, which is dependent upon the external loads and parameters of the cable net.

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