Some corrections to the lecture notes "Variational Methods of Mechanics, Spring 1999"

Page 5-9. Last equation (6) should begin as $Q_j^{\mathrm{I}} =$.

Page 5-14. In formula (j) the symbol q should be replaced by the symbol s.

Page 5-23. In formula (8) the dash in the term $-\delta V'$ should be removed and transferred to the term $(-\delta V)$.

Page 10-4. In formula (7') the unit vector multiplying term $\partial z/\partial t$ should be $\boldsymbol{k}$.

Page 11-5. In the second formula (2") the last term should be $\tau_{yz}\boldsymbol{k}$.

Page 11-7. In formula (9) the right-hand side should be ${}^{0}n_j \boldsymbol{T}^j$.

Page 13-17. In formula (d) the second element in the column vector on the left-hand side should be $\tilde{\theta}$.

Page 13-19. In formula (k) the last two elements in the column matrix should be q and r, respectively.

Page 14-6. Formula (d) and the text following it should read

$$[M]_{\mathrm{s}} = \frac{\hat{\varepsilon}_h m h^2}{(12\hat{\varepsilon}_h+1)}\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} + \frac{I_p}{(12\hat{\varepsilon}_h+1)}\frac{1}{6}\begin{bmatrix} 0 & 3 & 0 & 3 \\ 0 & 2h & 0 & h \\ 0 & -3 & 0 & -3 \\ 0 & h & 0 & 2h \end{bmatrix}. \qquad \text{(d)}$$

The matrices are not symmetric. It would be interesting to consider the effect of using only the symmetric parts. However, ...

Page 15-5. Formula (5) should begin $\delta' W =$.

Page 15-10. In formulas (e) the symbol T should be replaced by the symbol K and in the last formula the differentiation should be with respect to y.

Page 15-23. In formulas (22) and (23) the term $\{\dot{y}\}_j$ should be replaced by the term $\Delta\{\dot{y}\}_j$.

Page D-14, line 8 from bottom. The term $\partial f_{\mathrm{P}}/\partial x_i$ should be replaced by the term $\partial f/\partial x_i$.

Page D-23. In formula (1) the square root in the integrand should end befote the factor $\mathrm{d}x$.

Page D-42. In formula (h) the latter term $n_x \partial T/\partial x$ inside the parentheses in the last integral should be replaced by the term $n_y \partial T/\partial y$.

Page D-64. The paragraph in the middle of the page should end with the formula $K_{ji} = K_{ij}$.

VARIATIONAL METHODS OF MECHANICS

Spring 1999

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(These lecture notes can be used in the examination of Mat-5.160 Mekaniikan variaatioperiaatteita)

CONTENTS (30.3.1999)

PART II VIRTUAL WORK IN SOLID MECHANICS

PREFACE

The *principle of virtual work* (virtuaalisen työn periaate) is probably the most useful and the most misunderstood principle of mechanics. Shortly, the principle contains the equations of equilibrium in statics or the equations of motion in dynamics of a mechanical system *as one scalar equation*, however complicated the system. The principle is very old but it has an enormous number of applications. During the last decades its practical importance has been further emphasized by the wide use of the *finite element method* (elementtimenetelmä) in solid mechanics. The discrete equations of the finite element method are usually generated via the principle of virtual work.

General mathematics literature dealing with the finite element method is constantly involved with scalar equations called *weak forms* (heikko muoto) or *weak formulations* (heikko formulaatio) or *variational formulations* (variaatio-formulaatio). The principle of virtual work is in fact a weak form. Thus getting familiar and understanding this principle opens the door for understanding weak forms in other physical situations, for instance in heat transfer and fluid flow.

It is often stated that the virtual displacements are infinitesimal and that they take place infinitely fast. Why must they be infinitesimal? If they take place infinitely fast, does it mean that the corresponding accelerations and thus inertia forces are also infinitely large?

The principle of virtual work is usually stated in the form: For a system in equilibrium the virtual work of all the forces acting on the system is zero. The so-called Fourier's inequality, however, says that if the virtual work of the forces for any virtual displacement is zero or negative, the system is in equilibrium. How should one understand this?

Virtual work is often denoted by the symbol δW. In variational calculus the δ-symbol is quite generally employed for the first variation of a quantity. Virtual work δW does not, however, mean the variation of work.

This kinds of problems and even the rather exotic name of the principle often trouble the student and easily make the principle to seem to have some mystic properties. This text is an attempt to remove all this mystique and to show the vast applicability of the principle in engineering mechanics.

The presentation is aimed mainly at applications in solid mechanics. However, we start with particle mechanics. This is because

> Particle mechanics introduces in a simple form concepts and manipulations which are rather directly transformed to be used similarly in continuum mechanics.

> Important approximate discrete methods in continuum mechanics — such as the finite element method — produce equations similar to those in particle mechanics.

PART I

VIRTUAL WORK IN PARTICLE MECHANICS

CHAPTER 1

INTRODUCTION

One way to classify mechanics is to do it through the material models used. The most usual models are the particle model and the continuum model.

Particle (partikkeli, massapiste, ainepiste, hiukkanen) corresponds to the concept of point in geometry equipped with mass. The latter is a positive constant for each particle.

In a *continuum* (kontinuumi) the material is assumed to be distributed continuously in space.

Neither of these models corresponds exactly to reality but they both have important application areas. Depending on the model used, we speak about *particle mechanics* (partikkelimekaniikka) and *continuum mechanics* (kontinuumimekaniikka).

To construct the theory needed to solve problems in mechanics we need roughly the following ingredients:

1. Mathematics
2. Axioms of mechanics
3. Kinematics
4. Constitutive relations

Most of the mathematics needed are assumed to be familiar to the reader. Certain concepts of variational calculus are explained later in some detail.

Axioms of mechanics or *natural laws* (aksiooma, luonnonlaki) give the most important governing equations. For pedagogical reasons, we state different axioms for the particle model and for the continuum model.

The continuum axioms are considered to be valid irrespective of the material of the body considered, say, fluid or solid. *Constitutive relations* (konstitutiivinen yhteys) are needed to describe more or less realistically the detailed material

response of the material under study due to say stress and temperature changes. Examples: Hooke's law, ideal gas law.

To represent the axioms and the constitutive laws mathematically we need preparatory geometrial concepts concerning displacements and motion in general, called kinematics.

It is important to emphasize the difference between the concepts of kinematics and kinetics.

Kinematics (kinematiikka) studies motion as a geometric phenomenon without considering the cause of the motion.

Kinetics (kinetiikka) studies motion taking into account the cause of the motion: the forces. Kinetics is often also called *dynamics* (dynamiikka) and as its special case we obtain *statics* (statiikka). Kinetics can be considered as the end product obtained by combining the four ingredients listed on page 1-1. In more detail, the cause of motion can be something else than forces, for instance, say, some heat input to a system.

CHAPTER 2

AXIOMS OF PARTICLE MECHANICS

We consider *system of particles* (partikkelisysteemi), that is, an entity consisting of several particles. The results we obtain are thus valid as special cases for just one particle and for *rigid bodies* (jäykkä kappale) or for systems of rigid bodies. In a rigid body we assume the distances of the particles forming the body to remain constant irrespective of the external excitation. We call the mechanics dealing with systems of particles as *particle mechanics* (partikkelimekaniikka).

The axioms of particle mechanics, used here, are, Synge and Griffiths (1959):

(1) *Law of motion* (liikelaki): A particle of mass m, subject to a force $\boldsymbol{F}$, moves according to equation

$$\boxed{\boldsymbol{F} = m\boldsymbol{a},} \tag{1}$$

where $\boldsymbol{a}$ is the acceleration of the particle (Figure 2.1).

Figure 2.1 Force, mass and acceleration.

(2) *Law of action and reaction* (voiman ja vastavoiman laki): When two particles exert forces on each other, these forces are equal in magnitude and opposite in sense and act along the line joining the particles.

Using the notation of Figure 2.2, we obtain the formula

$$\boxed{\boldsymbol{f}_{ji} = -\boldsymbol{f}_{ij},} \tag{2}$$

which does not however totally imply the law — this is just the information about opposite force vectors — and we must additionally tell information about the line of action.

Figure 2.2 Two forces.

(3) *Law of the parallelogram of forces* (voimien suunnikaslaki): When two forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ act on a particle, they are together equivalent to a single force

$$\boxed{\boldsymbol{F} = \boldsymbol{P} + \boldsymbol{Q},} \tag{3}$$

the vector sum being defined by the parallelogram construction (Figure 2.3).

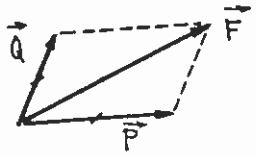


Figure 2.3 Forces $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{F}$.

Remark 2.1. The law of motion is valid only in a *Newtonian frame of reference* or an *inertial frame* (inertiaalikehys). Synge and Griffith (1959, p. 27). See also the same reference for a more general form of the law of motion than the one presented here. □

REFERENCE

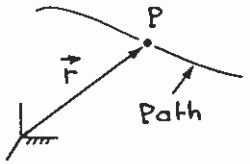
Synge, J. L. and B. A. Griffith, (1959): *Principles of Mechanics*, 3rd. ed., McGraw-Hill, New York.

CHAPTER 3

KINEMATICS

3.1 GENERAL

The most important concepts of particle kinematics are the *path* (rata), the *position* (asema), the *velocity* (nopeus) and the *acceleration* (kiihtyvyys) (Figure 3.1):



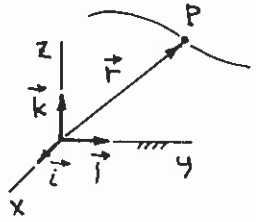
Position	$\boldsymbol{r} = \boldsymbol{r}(t)$
Velocity	$\boldsymbol{v} = \dot{\boldsymbol{r}}$
Acceleration	$\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}}$

(1)

Figure 3.1 Kinematical concepts.

The derivative of a quantity $(\cdot)$ with respect to time t is denoted shortly with $\dot{(\cdot)} \equiv \mathrm{d}(\cdot)/\mathrm{d}t$. The units of the quantities of Figure 3.1 are $[r] = \mathrm{m}$, $[v] = \mathrm{m/s}$, $[a] = \mathrm{m/s^2}$.

In rectangular cartesian xyz-coordinates we have

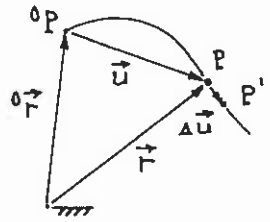


$$\begin{aligned} \boldsymbol{r} &= r_x\boldsymbol{i} + r_y\boldsymbol{j} + r_z\boldsymbol{k} \\ &= x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, \\ \boldsymbol{v} &= v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k} \\ &= \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}, \\ \boldsymbol{a} &= a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k} \\ &= \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}. \end{aligned} \tag{2}$$

Figure 3.2 Rectangular cartesian coordinate system.

The unit base vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ along the coordinate axes x, y, z form in this order a right-handed orthogonal system.

The values of velocity and acceleration do not change if the position vector of a particle P is measured instead of the origin with respect to any fixed point in the coordinate system. In solid mechanics the equation $\boldsymbol{r} = \boldsymbol{r}(t)$ for the path of the particle and the velocity and acceleration are represented usually instead of formulas (2) as follows (Figure 3.3):



$$\begin{aligned} \boldsymbol{r} &= {}^0\boldsymbol{r} + \boldsymbol{u}(t), \\ \boldsymbol{u} &= u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k} \\ &= u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}, \\ \boldsymbol{v} &= \dot{\boldsymbol{u}} = \dot{u}\boldsymbol{i} + \dot{v}\boldsymbol{j} + \dot{w}\boldsymbol{k}, \\ \boldsymbol{a} &= \ddot{\boldsymbol{u}} = \ddot{u}\boldsymbol{i} + \ddot{v}\boldsymbol{j} + \ddot{w}\boldsymbol{k}. \end{aligned} \tag{3}$$

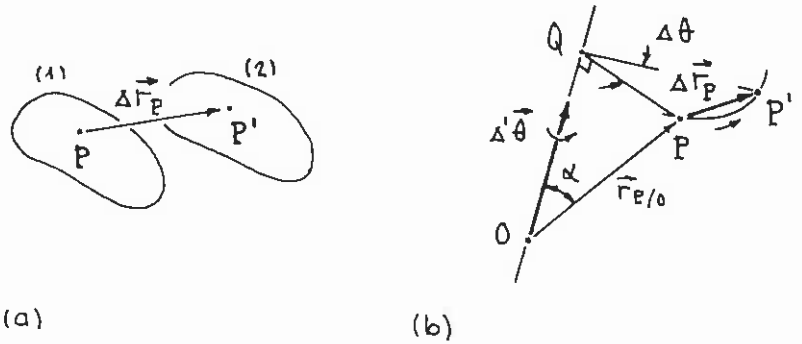
Figure 3.3 Displacement $\boldsymbol{u}$.

Here $\boldsymbol{u}$ ($[\boldsymbol{u}] = \mathrm{m/s}$) is the displacement of the particle with respect to an *initial* or *reference state* (alkutila, referenssitila) (indicated with a forward superscript 0) selected for the particle. The initial state is usually selected to be the state of the body at the moment of time $t = 0$ when the forces under study do not yet act.

The change of displacement $\overline{\mathrm{PP}'} = \Delta\boldsymbol{r} = \Delta\boldsymbol{u}$ shown in Figure 3.3 is also often called simply displacement. Thus the exact meaning of the concept is only clear after the initial and final state connected to the displacement are mentioned.

3.2 RIGID BODY MOTION

To describe the motion of a rigid body the concepts of translation and rotation are in use.



(a) (b)

Figure 3.4 (a) Translation. (b) Rotation.

Translation (translaatio, etenemisliike) is the motion of a rigid body in which the orientation of the body does not change. Each particle of the body obtains the same displacement. By the word translation we often mean also in addition to the motion the corresponding displacement. Thus the displacement $\overrightarrow{PP'} = \Delta r_P = \Delta u_P$ (Figure 3.4 (a)) of an arbitrary particle P of a rigid body gives all necessary information about the translation.

Rotation about a fixed line — the rotation axis — (rotaatio eli pyörimisliike eli kiertoliike kiinteän suoran ympäri) is the motion of a rigid body where the particles on the rotation axis do not move. In Figure 3.4 (b) the rotation about an axis passing through points O and Q is considered. A particle in initial state at point P moves along a circular path (radius $|r_{P/O}|\sin\alpha$) to the point P′. The notation $r_{P/O}$ refers to the position vector of P with respect to O. By the concept of *angular displacement* or often also *rotation* (kulmasiirtymä, rotaatio) we mean the directed line segment $\Delta'\boldsymbol{\theta}$ ($[\Delta'\boldsymbol{\theta}] = \text{rad}$) whose magnitude is the value of the rotation angle $\Delta\theta$ and the direction is along OQ and the sense is according to the direction of movement of a right-handed screw in the corresponding rotation.

Quantity $\Delta'\boldsymbol{\theta}$ is not a vector. If we for instance perform two consecutive rotations about two rotation axes having different directions and going through point O, it is easy to see that the final position of the rigid body depends on the order of performing the rotations. This means that the rotations do not obey the commutative law of vector summation and rotation cannot thus be considered as a vector. It can be shown, however, that the smaller the rotations, the less the final position depends on the order of performing the rotations. *Infinitesimal or differential rotation $\mathrm{d}'\boldsymbol{\theta}$ can thus be considered as a vector.* With the help of Figure 3.4 (b) it is rather easy to show remembering the meaning of the vector cross product that the infinitesimal or differential displacement of a particle situated at point P due to $\mathrm{d}'\boldsymbol{\theta}$ is

$$\mathrm{d}r_P = \mathrm{d}u_P = \mathrm{d}'\boldsymbol{\theta} \times r_{P/O}\,. \tag{1}$$

In this formula O may be any point on the rotation axis. With finite rotation the formula for the displacement of an arbitrary point becomes considerably more complicated.

According to the *theorem of Chasles* (Figure 3.5)

> The general displacement of a rigid body is equivalent to a displacement which is constructed by first performing a translation and then a rotation about the base point. (2)

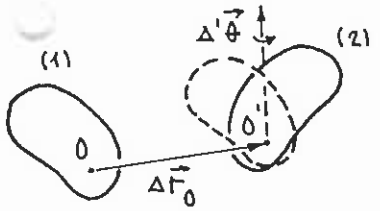


Figure 3.5 Displacement of a rigid body.

The *base point* (siirtopiste) is an arbitrary particle of the rigid body (in Figure 3.5 particle O) by which the translation is thought to be performed so that the particle moves from the initial position (1) to the final position (2). The translation vector $\Delta r_O = \Delta u_O$ depends in general on the selection of the base point but the amount of rotation needed $\Delta'\boldsymbol{\theta}$ is independent of it.

The theorem of Chasles can be stated also in the form where the rotation is performed first and the translation later or they are both performed simultaneously.

Under a small time increment the motion of a rigid body is small and especially the rotation needed in Chasles' theorem is also small and with differential time increment $\mathrm{d}t$ it becomes a differential rotation vector $\mathrm{d}'\boldsymbol{\theta}$. By adding to the expression (1) due to rotation the differential translation $\mathrm{d}r_O = \mathrm{d}u_O$ of the base point O we arrive at the formula

$$\boxed{\mathrm{d}r_P = \mathrm{d}r_O + \mathrm{d}'\boldsymbol{\theta} \times r_{P/O}} \tag{3}$$

or

$$\boxed{\mathrm{d}u_P = \mathrm{d}u_O + \mathrm{d}'\boldsymbol{\theta} \times r_{P/O}\,.} \tag{3'}$$

Thus when we know the differential displacement of one particle of a rigid body and its differential angular displacement we can calculate the differential displacement of any other particle of the rigid body by this formula.

The formula is used quite often as an accurate approximate one for small finite displacements:

$$\Delta r_P \approx \Delta r_O + \Delta'\boldsymbol{\theta} \times r_{P/O}\,. \tag{4}$$

The *angular velocity* (kulmanopeus) $\boldsymbol{\omega}$ ($[\boldsymbol{\omega}] = \text{rad/s}$) of a rigid body is defined at each moment of time by

$$\boxed{\boldsymbol{\omega} = \lim_{\Delta t \to 0} \frac{\Delta'\boldsymbol{\theta}}{\Delta t} = \frac{\mathrm{d}'\boldsymbol{\theta}}{\mathrm{d}t}} \tag{5}$$

where $d'\boldsymbol{\theta}$ is the differential rotation corresponding to the differential time increment dt.

By dividing equation (3) by dt we arrive at the formula

$$\boxed{\boldsymbol{v}_P = \boldsymbol{v}_O + \boldsymbol{\omega} \times \boldsymbol{r}_{P/O}} \tag{6}$$

concerning the velocities of the particles of a rigid body.

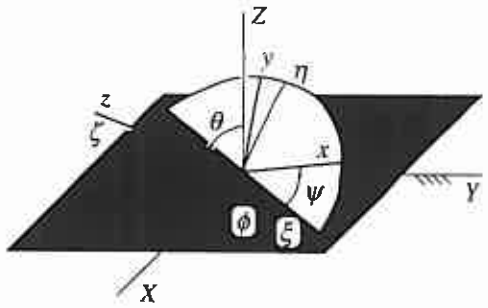
The *angular acceleration* (kulmakiihtyvyys) $\boldsymbol{\alpha}$ ($[\boldsymbol{\alpha}] = \text{rad/s}^2$) of a rigid body is defined by

$$\boxed{\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}.} \tag{7}$$

Here we have no problems in the definition as the vector $\boldsymbol{\omega}(t)$ is in principle available at each moment of time.

Remark 3.1. The angular displacement (rotation) and the angular velocity in the fully three-dimensional case are concepts which often cause some confusion as they are not analogous ones with the concepts of displacement and velocity of a point. The difference lies in the fact that a point always has a position vector $\boldsymbol{r}(t)$ but unfortunately the orientation of a rigid body cannot be given by some kind of "orientation vector " $\boldsymbol{\theta}$. (In plane motion this is possible and the situation is much simpler.) The notations Δ' and d' equipped with dashes are connected here to this. Let us consider an arbitrary function f of time: $f = f(t)$. By the the notations Δf and df we usually mean the change $\Delta f = f(t + \Delta t) - f(t)$ and differential change $df = f(t + dt) - f(t)$ of f during the the time changes Δt and dt. But if we have no function of time — as here in connection with orientation — we indicate it with the dash. Other sometimes used notations in this connections in the literature are $\bar{d}$ or $đ$. The orientation of a rigid body in space in the fully three-dimensional case can be given with the help of three independent (in the plane case with only one) scalar quantities, perhaps the most usual being the *Euler angles* (Eulerin kulmat). □

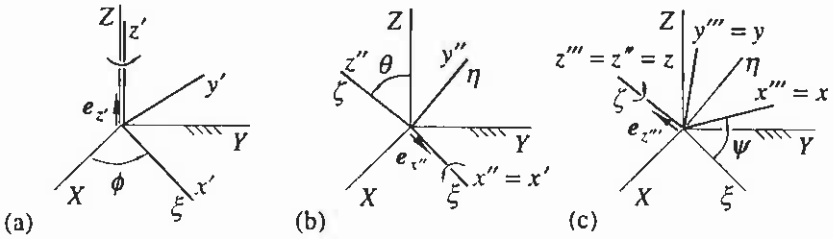
The Euler angles $\phi(t)$, $\theta(t)$, $\psi(t)$ are indicated in Figure 3.6. We need two coordinate systems. The XYZ-system is called the *earth-fixed coordinate system* (kiinteä koordinaatisto) and the xyz-system which is fixed with respect to the rigid body is called the *body-fixed coordinate system* (kappale-koordinaatisto). To describe the orientation of the body it is enough here to assume for simplicity of presentation that the origins of the two coordinate systems coincide.



Kuva 3.6 Euler angles.

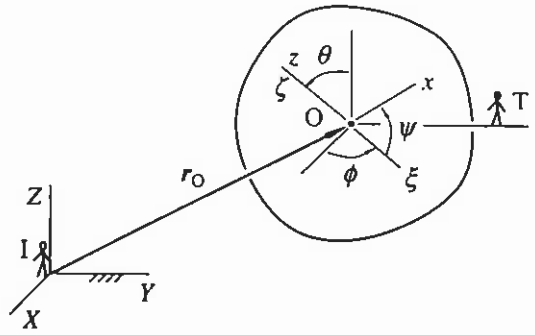
The plane defined by the x- and y- axes intersects the darkened plane defined by the X- and Y-axes along the so-called *line of nodes* (solmuviiva). The ξ-axis of an auxiliary $\xi\eta\zeta$-coordinate system is put to coincide with the line of nodes. The ζ-axis of the auxililiary coordinate system is put to coincide with the z-axis so that thus also the xy- and $\xi\eta$-planes coincide. Quantity ϕ is the angle between the ξ- and X-axes, θ is the angle between the ζ- (z-) and Z-axes and ψ is the angle between the x- and ξ-axes.

Let us assume that the xyz-coordinate system coincides originally with the XYZ-coordinate system. Figures 3.7 (a), (b) ja (c) show how the Euler angles can be interpreted as three consecutive rotations ϕ, θ, ψ about the Z-, ξ-, $\zeta(z)$-axes. These rotations in particular in this order move the body to its final position.



Kuva 3.7 Three consecutive rotations.

The signs of the Euler angles are determined by the directions of the rotations needed. If the rotation is directed along the positive Z- ξ- or ζ-axis direction, the angles are positive and vice versa. In the case shown in Figure 3.7 the angles are positive. Thus if one particle of a rigid body is fixed, the position or orientation of the body as a function of time is known when three functions $\phi(t)$, $\theta(t)$ ja $\psi(t)$ are given.



Kuva 3.8 General description of the position of a rigid body.

In the general case the position of a rigid body is determined by the coordinates X_O, Y_O, Z_O of an arbitrary base point O and by the Euler angles as described in Figure 3.8. Thus six functions $X_O(t)$, $Y_O(t)$, $Z_O(t)$, $\phi(t)$, $\theta(t)$, $\psi(t)$ are needed. This kind of description is used for instance for studying the motion of a ship or aeroplane considered as a rigid body.

Remark 3.2. The notations and sign rules employed in connection with Euler angles vary in a very annoying manner in the literature. Here we have followed those conventions employed for instance in Goldstein (1971) and Symon (1971). These are rather common in treatises on classical mechanics. The conventional notations used, say, in vehicle mechanics unfortunately usually differ from the ones described above. The essential point, however, is to comprehend the lines of thought in the derivation of the expression for the angular velocity given in the following after which it is easy to derive or check the formulas obtained in some alternative representation. □

Remark 3.3. In connectioin with classical Euler angles the z-axis has a specific role as two rotations are taken about it in the procedure shown in Figure 3.7. If the angle θ is zero or 180°, the construction breaks down since just the sum of ϕ and ψ is known but not separately the parts ϕ and ψ. The problem can be evaded for instance in a numerical solution so that when the z- and Z-axes become near enough each other in direction, the coordinates are given new symbols so that the old x-coordinate becomes a new y-coordinate etc. □

Let us consider Figures 3.6 and 3.7 and the change of the position of the body during a time increment Δt from the position given by the angles ϕ, θ, ψ to the position given by the angles $\phi+\Delta\phi$, $\theta+\Delta\theta$, $\psi+\Delta\psi$. Figure 3.7 shows the unit vectors $\boldsymbol{e}_{z'} = \boldsymbol{K}$, $\boldsymbol{e}_{x''} = \boldsymbol{e}_\xi$, $\boldsymbol{e}_{z'''} = \boldsymbol{k}$ along each of the corresponding rotation axes. If just angle ϕ would change, the angular diplacement would be $\Delta\phi\boldsymbol{e}_{z'}$. If just angle θ would change, the angular displacement would be $\Delta\theta\boldsymbol{e}_{x''}$. Finally, if just angle ψ would change, the angular displacement would be $\Delta\psi\boldsymbol{e}_{z'''}$. The

actual angular displacement $\Delta'\boldsymbol{\beta}$ (we have not used the conventional symbol $\Delta'\boldsymbol{\theta}$, to avoid possible confusion with the rotation $\Delta\theta\boldsymbol{e}_{x''}$) which takes the body from the position ϕ, θ, ψ to the position $\phi+\Delta\phi$, $\theta+\Delta\theta$, $\psi+\Delta\psi$ is not the vector sum of the quantities $\Delta\phi\boldsymbol{e}_{z'}$, $\Delta\theta\boldsymbol{e}_{x''}$ ja $\Delta\psi\boldsymbol{e}_{z'''}$ when the rotations are finite. However, when the time increment Δt gets smaller, the rotations get smaller and smaller and the equation

$$\Delta'\boldsymbol{\beta} = \Delta\phi\boldsymbol{e}_{z'} + \Delta\theta\boldsymbol{e}_{x''} + \Delta\psi\boldsymbol{e}_{z'''} \tag{8}$$

becomes more and more accurate as the vector summation rule is valid for infinitesimal rotations. Thus in the limit exactly

$$\begin{aligned}\boldsymbol{\omega} &= \lim_{\Delta t\to 0}\frac{\Delta'\boldsymbol{\beta}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\phi\boldsymbol{e}_{z'} + \Delta\theta\boldsymbol{e}_{x''} + \Delta\psi\boldsymbol{e}_{z'''}}{\Delta t}\\ &= \lim_{\Delta t\to 0}\left(\frac{\Delta\phi}{\Delta t}\boldsymbol{e}_{z'} + \frac{\Delta\theta}{\Delta t}\boldsymbol{e}_{x''} + \frac{\Delta\psi}{\Delta t}\boldsymbol{e}_{z'''}\right)\end{aligned} \tag{9}$$

or

$$\boxed{\boldsymbol{\omega} = \dot\phi\boldsymbol{e}_{z'} + \dot\theta\boldsymbol{e}_{x''} + \dot\psi\boldsymbol{e}_{z'''}.} \tag{10}$$

The angular velocity expression (10) is not directly in a form suitable for practice as the the unit vectors on the right-hand side are not of a one and same coordinate system. Usually we want to have expressions given in the body-fixed coordinate system. The relations

$$\begin{aligned}\boldsymbol{e}_{z'} &= \sin\theta\sin\psi\boldsymbol{i} + \sin\theta\cos\psi\boldsymbol{j} + \cos\theta\boldsymbol{k},\\ \boldsymbol{e}_{x''} &= \cos\psi\boldsymbol{i} - \sin\psi\boldsymbol{j},\\ \boldsymbol{e}_{z'''} &= \boldsymbol{k},\end{aligned} \tag{11}$$

can be derived using Figure 3.7. When these are substituted in (10), there is obtained

$$\boxed{\boldsymbol{\omega} = \omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k},} \tag{12}$$

where

$$\boxed{\begin{aligned}\omega_x &= \sin\theta\sin\psi\cdot\dot\phi + \cos\psi\cdot\dot\theta,\\ \omega_y &= \sin\theta\cos\psi\cdot\dot\phi - \sin\psi\cdot\dot\theta,\\ \omega_z &= \cos\theta\cdot\dot\phi + \dot\psi.\end{aligned}} \tag{13}$$

In matrix form

$$\begin{Bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{Bmatrix}=\begin{bmatrix}\sin\theta\cdot\sin\psi & \cos\psi & 0\\ \sin\theta\cdot\cos\psi & -\sin\psi & 0\\ \cos\theta & 0 & 1\end{bmatrix}\begin{Bmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{Bmatrix} \tag{14}$$

or shortly

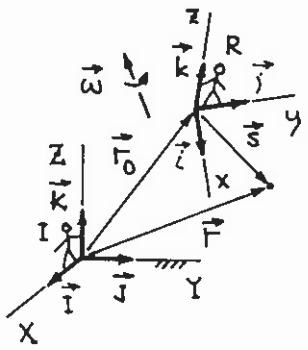
$$\{\omega\}_{xyz}=[B(\gamma)]\{\dot\gamma\}. \tag{15}$$

The argument γ in the $[B]$-matrix emphasizes that the elements of the matrix are expressed in the Euler angles.

3.3 RELATIVE MOTION

The two coordinate systems in Figure 3.9 are in motion with respect to each other. It can be shown that the relationship between the acceleration $\boldsymbol{a}$ and $\boldsymbol{a}_r$ measured by the observers I and R, respectively, is

$$\boxed{\boldsymbol{a}=\boldsymbol{a}_r+\boldsymbol{a}_O+\boldsymbol{\alpha}\times\boldsymbol{s}+\\ +\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{s})+2\boldsymbol{\omega}\times\boldsymbol{v}_r,} \tag{1}$$



where

$$\boldsymbol{a}=\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_I=\ddot X\boldsymbol{I}+\ddot Y\boldsymbol{J}+\ddot Z\boldsymbol{K},$$

$$\boldsymbol{a}_r=\left(\frac{\mathrm{d}^2\boldsymbol{s}}{\mathrm{d}t^2}\right)_R=\ddot x\boldsymbol{i}+\ddot y\boldsymbol{j}+\ddot z\boldsymbol{k},$$

$$\boldsymbol{a}_O=\left(\frac{\mathrm{d}^2\boldsymbol{r}_O}{\mathrm{d}t^2}\right)_I=\ddot X_O\boldsymbol{I}+\ddot Y_O\boldsymbol{J}+\ddot Z_O\boldsymbol{K}, \tag{2}$$

$$\boldsymbol{v}_r=\left(\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t}\right)_R=\dot x\boldsymbol{i}+\dot y\boldsymbol{j}+\dot z\boldsymbol{k}.$$

Figure 3.9 Two coordinate systems.

The content of most of the notations is obvious. Quantity $\boldsymbol{\omega}(\boldsymbol{\omega}_{R/I})$ is the angular velocity of the xyz-coordinate system with respect to the XYZ-coordinate system. Quantity $\boldsymbol{\alpha}(\boldsymbol{\alpha}_{R/I})=(\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t)_I$ is the corresponding angular acceleration. Formula (1) has use for instance when motion is considered in a non-inertial coordinate system.

It should be noticed that the construction of a coordinate system demands in practice a more or less — preferably a more — rigid body. The angular velocity $\boldsymbol{\omega}$ is in fact the angular velocity of a rigid body to which the xyz-coordinates are fixed.

3.4 REFERENCES

Goldstein, H. (1971): *Classical mechanics*, Addison-Wesley, Reading, Massachusetts.

Symon, K. R. (1971): *Mechanics*, third edition, Addison-Wesley, Reading, Massachusetts.

CHAPTER 4

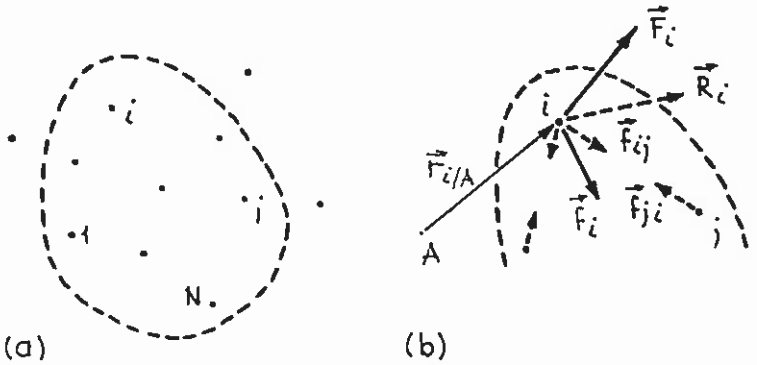
KINETICS

4.1 GENERAL

We consider certain general concepts some of which actually belong to kinematics but which have been included here nearer to their applications.

4.1.1 Particle system

The term *body* or *material body* (kappale) in mechanics is the general name given for a system under study consisting of *certain material particles*. Thus for instance a given amount of water is in the language of mechanics a body. In particle mechanics, the synonym for a body is a *particle system* or shortly a *system*.



F_i = the resultant of the external forces acting on particle i

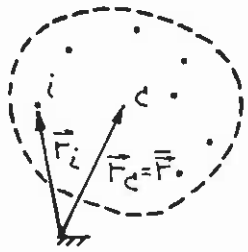
$f_i = \sum_{j=1,\, j\neq i}^{N} f_{ij}$ = the resultant of internal forces acting on particle i

$R_i = F_i + f_i$ = the resultant of the forces acting on particle i

Figure 4.1 (a) Particle system. (b) Forces acting on a particle system.

Some notations used in the numbering of the particles and in naming the forces are given in Figure 4.1. In the following the sum over the particles $(1,2,\cdots,N)$ of the system is often denoted just with the summation symbol Σ without the limits.

The concept of *center of mass* (massakeskiö) appears in many of the formulas for particle systems Often the name *center of gravity* (painopiste) is also used. The center of mass is defined as a point C, whose position vector $r_C \equiv \bar{r}$ is determined by the formula



$$\boxed{m\bar{r} = \sum m_i r_i} \qquad (1)$$

or

$$\begin{aligned} m\bar{x} &= \sum m_i x_i\,, \\ m\bar{y} &= \sum m_i y_i\,, \\ m\bar{z} &= \sum m_i z_i\,, \end{aligned} \qquad (1')$$

Figure 4.2 Center of mass.

where m_i is the mass of particle i and where

$$\boxed{m = \sum m_i} \qquad (2)$$

is the total mass of the system. It can be shown that the position of C with respect to the body does not depend on the choice of the coordinate system.

4.1.2 Constraints and generalized coordinates

The particles of a system are usually not fully free to move but some kind of *kinemetical conditions* or shortly *constraints* (rajoite) decrease the possibility of motion and we then speak about *constrained* or *guided motion* (sidottu liike). (A planetary system would be an example of an exceptional case where no kinematical constraints are present.) References Gantmacher (1975), Goldstein (1971), Rosenberg (1977) contain good presentation of constraints. We follow here mainly Goldstein.

Constraints can be classified in different ways. If a constraint can be expressed as an equation connecting coordinates of the particles of the system and possibly explicitly the time having the form

$$f(r_1, r_2, \cdots, t) = 0, \qquad (3)$$

the constraint is called *holonomic* (holonominen rajoite). If this is not the case, the constraint is called *nonholonomic* (epäholonominen rajoite).

A rigid body is produced by the constraints

$$(\mathbf{r}_j - \mathbf{r}_i)\cdot(\mathbf{r}_j - \mathbf{r}_i) - s_{ij}^2 = 0, \tag{4}$$

where the meaning of the notations is obvious. These are examples of holonomic constraints.

A constraint is called *scleronomic* (skleronominen rajoite) if time does not appear in it explicitly; if time appears in it, the constraint is called *rheonomic* (reonominen rajoite). Constraint (4) is thus holonomic and scleronomic. If a particle is constrained to move on a moving surface we have a holonomic and rheonomic constraint.

Constraints containing velocities of the particles of the system are examples of nonholonomic constraints. Similarly constraints in the form of inequalities — these constraints are called sometimes *unilateral constraints* (toispuoleinen rajoite) — are nonholonomic. For instance, a particle (gas molecule) bouncing inside a container is subjected to nonholonomic constraints. We consider in the following only *bilateral constraints* (kaksipuolinen rajoite), that is, constraints in the form of equality constraints.

Constraints change the problems of mechanics in two ways. First, the kinematical quantities in the equations of motion of a system are not all independent as they have to satisfy the constraints. Second, new unknown forces, so called constraint forces, discussed in more detail in Section 4.1.3 emerge.

Let us consider first the case where we have k scalar holonomic constraints of the form (3) or using for the sake of argument rectangular cartesian coordinates:

$$f_i(x_1, y_1, z_1, x_2, \cdots, z_N, t) = 0, \quad i = 1,2,\cdots,k. \tag{5}$$

Without constraints, the position or *configuration* (konfiguraatio) of the system consisting of N particles would be given in three dimensions by $3N$ quantities: $x_1, y_1, z_1, x_2, \cdots, z_N$. Using the constraint equations (5), we can eliminate k of the coordinates and expresss them in the remaining $3N-k=n$ coordinates which can be then considered as independent quantities that define the configuration of the system. However, we need not use the original rectangular coordinates for this purpose. It is enough to find n independent quantities, denoted conventionally by the symbols $q_1, q_2, \cdots, q_n$ and called *generalized coordinates* (yleistetty koordinaatti), by which the old coordinates or the position vectors of the system can be given in the form

$$\boxed{\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \cdots, q_n, t),} \qquad i = 1,2,\cdots,N \tag{6}$$

or

$$\begin{aligned} x_i &= x_i(q_1, q_2, \cdots, q_n, t), \\ y_i &= y_i(q_1, q_2, \cdots, q_n, t), \quad i = 1,2,\cdots,N \\ z_i &= z_i(q_1, q_2, \cdots, q_n, t). \end{aligned} \tag{6'}$$

Now the holonomic constraints (5) are taken care of automatically through the representation (6). Usually the generalized coordinates can be selected directly by inspection of the problem and there is no need to actually perform any eliminations. The generalized coordinates are functions of time: $q = q(t)$. In the scleronomic case time t disappears explicitly from the argument list. This is in practice the most usual case. The number n is called the *number of degrees of freedom* (vapausasteiden lukumäärä). This is a fixed number for each holonomic system; the generalized coordinates, on the contrary, can be selected in principle in infinitely many ways.

Remark 4.1. In solid continuum mechanics, when the displacement formulation is applied in connection with the finite element method, we end up with a representation of the type (6) giving the (approximate) configuration of the body in terms of a finite number of parameters called often *nodal parameters* (solmuparametri) (or as commented in Remark D.20 also not very approriately as degrees of freedom). From the classical mechanics point of view, however, the nodal parameters are just one application example of generalized coordinates. □

Especially in structural mechanics, where the values of the coordinates do not vary very much, displacements are considered instead position vectors as is indicated in formulas (3.1.3). Thus, the equivalents of (6) and (6') are

$$\boxed{\mathbf{u}_i = \mathbf{u}_i(q_1, q_2, \cdots, q_n, t),} \qquad i = 1,2,\cdots,N \tag{7}$$

or

$$\begin{aligned} u_i &= u_i(q_1, q_2, \cdots, q_n, t), \\ v_i &= v_i(q_1, q_2, \cdots, q_n, t), \quad i = 1,2,\cdots,N \\ w_i &= w_i(q_1, q_2, \cdots, q_n, t). \end{aligned} \tag{7'}$$

If form (6) is known, form (7) follows immediately — and the other way round — as

$$\mathbf{u}_i = \mathbf{r}_i(q_1, q_2, \cdots, q_n, t) - {}^0\mathbf{r}_i = \mathbf{u}_i(q_1, q_2, \cdots, q_n, t). \tag{8}$$

Generalized coordinates are used also in connection with rigid bodies. As the number of particles in a rigid body considered as a continuum is infinite, we cannot in practice identify the particles by numbering. For this purpose we equip a rigid body with body fixed coordinates, say a, b, c, after which the values of

the coordinates fix a particle (compare the Lagrangian representation in Section 10.2 and Example 4.2). Formulas (7) and (7'), for example, would look then

$$u = u(a,b,c;q_1,q_2,\cdots,q_n,t), \tag{9}$$

and

$$\begin{aligned} u &= u(a,b,c;q_1,q_2,\cdots,q_n,t),\\ v &= v(a,b,c;q_1,q_2,\cdots,q_n,t),\\ w &= w(a,b,c;q_1,q_2,\cdots,q_n,t). \end{aligned} \tag{9'}$$

Often, however, the parameters a,b,c are not indicated in the list of variables.

Example 4.1. We consider the extremely simple demonstration case of Figure (a), a particle moving in plane motion in the xy-plane on an inclined plane of a wedge in given motion.

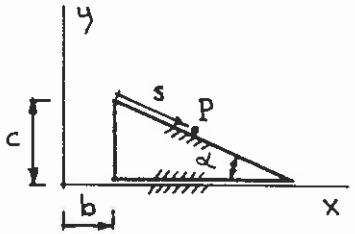


Figure (a)

The position of the wedge is given by the measure b in the form

$$b = \hat{b}\sin\omega t \tag{a}$$

where $\hat{b}$ and ω are constants. The wedge thus performs harmonic motion with amplitude $\hat{b}$ and angular frequency ω. The system is here just the particle. One constraint is that the particle is in plane motion:

$$z = 0. \tag{b}$$

The condition that the particle remains on the inclined plane can be expressed as

$$y = c - (x-b)\tan\alpha \tag{c}$$

or

$$y + (x - \hat{b}\sin\omega t)\tan\alpha - c = 0. \tag{d}$$

The two constraints (b) and (d) are holonomic, see equation (5). The former is scleronomic and the latter rheonomic. The original number of degrees of freedom of the system $3N = 3\cdot 1 = 3$ is decreased to $3-2=1$ and we have a system with one degree of freedom where for instance x is the generalized coordinate. At a quick glance, one could consider measure b as an additional generalized coordinate. This would indeed be the case if the motion of the wedge would be free but here it is in a given motion.

The equivalents of formulas (6) and (6') are

$$r = xi + [c - (x - \hat{b}\sin\omega t)\tan\alpha]j \tag{e}$$

and

$$\begin{aligned} x &= x,\\ y &= c - (x - \hat{b}\sin\omega t)\tan\alpha. \end{aligned} \tag{e'}$$

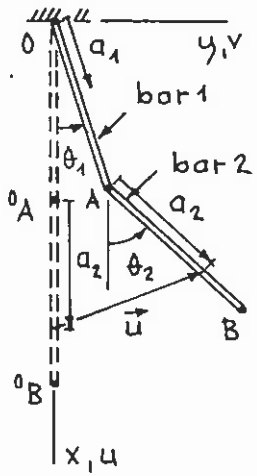
The measure s, for example, indicated in the figure, is clearly one alternative possibility to choose the generalized coordinate. The equivalents of formulas (6) and (6') are now

$$r = (s\cos\alpha + \hat{b}\sin\omega t)i + (c - s\sin\alpha)j \tag{f}$$

and

$$\begin{aligned} x &= s\cos\alpha + \hat{b}\sin\omega t,\\ y &= c - s\sin\alpha. \end{aligned} \tag{f'}$$

Example 4.2. The system shown in Figure (a) consists of two slender rigid bars pinned at O and pin-connected at A and constrained to plane motion in the xy-plane.



By direct inspection, quantities $\theta_1 = q_1$ and $\theta_2 = q_2$ are, for example, suitable generalized coordinates for this problem. We have for bar 1

$$\begin{aligned} x &= a_1\cos\theta_1,\\ y &= a_1\sin\theta_1, \end{aligned} \tag{a}$$

and for bar 2

$$\begin{aligned} x &= l\cos\theta_1 + a_2\cos\theta_2,\\ y &= l\sin\theta_1 + a_2\sin\theta_2, \end{aligned} \tag{b}$$

as the equivalents of (6'). Here l is length of the bars and a_1 and a_2 are the chosen local coordinates identifying the particles.

We have a holonomic scleronomic two degree of freedom system. As other generalized coordinates, we could take say y_A and y_B or y_A and θ_2. If point O would have given motion, the system would become rheonomic.

Figure (a)

If the dashed position of the system is taken as the reference state, the equivalents of (9') would become for bar 1

$$\begin{aligned} u &= a_1\cos\theta_1 - a_1,\\ v &= a_1\sin\theta_1, \end{aligned} \tag{c}$$

and for bar 2

$$\begin{aligned} u &= l\cos\theta_1 + a_2\cos\theta_2 - l - a_2,\\ v &= l\sin\theta_1 + a_2\sin\theta_2. \end{aligned} \tag{d}$$

Using generalized coordinates, the conventional nonholonomic constraint equations in the literature are of the form

$$\sum_{j=1}^{n} a_j \dot{q}_j + b = 0. \tag{10}$$

The time derivatives of the generalized coordinates, *generalized velocities* (yleistetty nopeus), appear linearly. The factors a and b may be functions of generalized coordinates and time.

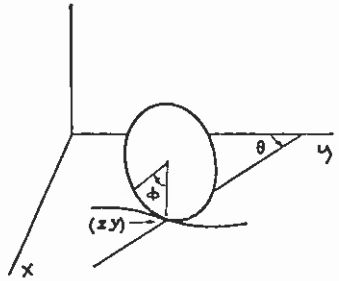


Figure 4.3 Vertical disk rolling on a horizontal plane.

Figure 4.3 gives an example of a nonholonomic constraint, Goldstein (1971). A disk rolls without slipping on a horizontal xy-plane so that the plane of the disk is always vertical. The generalized coordinates giving the configuration of the system (the disk) could be the ordinary cartesian coordinates x and y of the center of the disk, the angle ϕ between a fixed material radius line from the center and the vertical (rotation angle) and the angle θ between the axis of the disk and the y-axis. Thus four quantities are enough to fix the configuration. From the rolling condition (the point of contact has zero velocity) follows

$$v = a\dot{\phi} \tag{11}$$

for the scalar velocity of the center of the disk, where a is the radius of the disk. The velocity of the center is in the perpendicular direction to the axis of the disk:

$$\begin{aligned} \dot{x} &= v\cos\theta, \\ \dot{y} &= v\sin\theta, \end{aligned} \tag{12}$$

or

$$\begin{aligned} \dot{x} - a\cos\theta\cdot\dot{\phi} &= 0, \\ \dot{y} - a\sin\theta\cdot\dot{\phi} &= 0. \end{aligned} \tag{13}$$

We have obtained two nonholonomic constraints of the type (10).

Nonholonomic constraints cannot be employed for further elimination of kinematical variables.

Remark 4.2. We can denote right from the start all our kinematical quantities describing the configuration of a system using the symbols $q_1, q_2, \cdots$ do they satisfy the holonomic constraints or not. In some situation it may be in fact convenient not to try to satisfy all the holonomic constraints in advance. A holonomic constraint is then of the form

$$f(q_1, q_2, \cdots, t) = 0. \tag{14}$$

Differentiation this once with respect to time gives using chain differentiation

$$\frac{\partial f}{\partial q_1}\dot{q}_1 + \frac{\partial f}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial f}{\partial t} = 0. \tag{15}$$

This of now of the same type as the nonholonomic constraint (10). However, (15) can be integrated back to give (14) but for a truly nonholonomic constraint it is not possible to transform it by integrations to a holonomic form. □

4.1.3 Classification of forces

The forces acting on a system can be classified in many ways depending on the purpose.

(1) External and internal forces. A force acting on a particle of a system is called *internal force* (sisäinen voima), if it is generated by another particle of the system; if this is not the case, the force is called *external force* (ulkoinen voima).

It is easy to prove the following simple but extremely important results (see Figure 4.1)

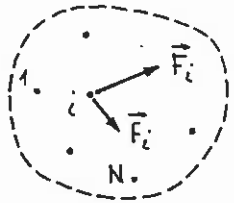
$$\boxed{\begin{aligned} \textstyle\sum f_i &= \mathbf{0}, \\ \textstyle\sum r_{i/A} \times f_i &= \mathbf{0} \end{aligned}} \tag{16}$$

or the resultant of the internal forces of a system is zero and the moment of the internal forces of the system with respect to an arbitrary point is zero, that is, the internal forces form a *zero force system* (nollavoimasysteemi).

Gravitational forces and contact forces from particles outside the system under study are examples of external forces. The same type of forces are internal if the particles generating them belong to the system.

4.2 EQUATIONS OF MOTION

The *equations of motion* (liikeyhtälöt) of a general particle system are



$$\boxed{F_i + f_i = m_i a_i\,,} \quad i = 1,2,\cdots,N. \qquad (1)$$

Figure 4.11 Forces acting on particle i.

The external force F_i is in general a function of the position and velocity of the particle and explicitly of time: $F_i = F_i(r_i, v_i, t)$. The pairwise internal forces f_{ij} and f_{ji} are in general functions of the distance s, its change of rate $\dot{s}$ and explicitly of time. Force f_i is thus in general a function of positions and velocities of all the particles of the system and of time: $f_i = f_i(r_1, r_2, \cdots, r_N, v_1, v_2, \cdots, v_N, t)$. As an example we may consider a truss modelled as particle system with all its mass concentrated at the joints which are the particles. Each bar connecting two joints gives a particle pair interacting with each other through the bar forces. These may be given say by linear springs and linear dampers.

Each equation (1) is a vector equation so in three dimensions we have a system of $3N$ scalar second order ordinary differential equations with time as the independent variable. The solution $r_1(t), r_2(t), \cdots, r_N(t)$ or $u_1(t), u_2(t), \cdots, u_N(t)$ contains $6N$ integration constants which are usually determined from the initial conditions: $r_1(t_0), v_1(t_0), \cdots, v_N(t_0)$ or $u_1(t_0), v_1(t_0), \cdots, v_N(t_0)$ are given. To find a closed form solution is clearly usually not possible. Further, the motion is usually constrained and unknown constraint forces emerge.

In a *static* (staattinen) or in a so called *quasistatic* (kvasistaattinen) case — this is a case, where the motion may depend on time but the motion is so slow that the term ma in the equations of motion may be neglected in comparison with the other terms — equations (1) transform to *equations of equilibrium* (tasapainoyhtälö)

$$\boxed{F_i + f_i = 0.} \quad i = 1,2,\cdots,N. \qquad (2)$$

These are usually algebraic equations with respect to the unknowns $r_1, r_2, \cdots, r_N$ or $u_1, u_2, \cdots, u_N$ and thus easier to solve than (1).

All problems of particle mechanics can be solved in principle — if at all solvable — using the equations of motion (1). In is, however, conventional in

mechanics to operate with additional quantities such as momentum, work, kinetic energy, virtual work, etc. and with so-called principles or theorems concerning them. It must be emphasized that no new axioms are taken into use. The principles derived can often be employed very effectively to solve certain problems. The same results would finally be obtained starting always from the three basic axioms. The value of the derived principles is that as the manipulations needed have been derived in a general form the treatment in specific applications gets shorter.

4.3 MOMENTUM

The *momentum* (liikemäärä) p ($[p] = \mathrm{kg\,m/s}$) of a particle is defined as

$$\boxed{p = mv.} \qquad (1)$$

The momentum p of a *particle system* is defined as the sum of the momentums of its particles:

$$\boxed{p = \sum m_i v_i\,.} \qquad (2)$$

In the following manipulation, the equations of motion

$$F_i + f_i = m_i \dot{v}_i \qquad (3)$$

are added together to produce one vector equation:

$$\sum F_i + \sum f_i = \sum m_i \dot{v}_i\,,$$

$$\sum F_i = \frac{\mathrm{d}}{\mathrm{d}t} \sum m_i v_i\,,$$

$$\boxed{F = \dot{p}} \qquad (4)$$

or the resultant of the external forces acting on a system is equal to rate of change of the momentum of the system.

This is called the *principle of balance of momentum* (liikemäärän taseen periaate). In the steps to (4), use have been made of the fact that the internal forces form a zero force system and that the masses of the particles of the system are constants.

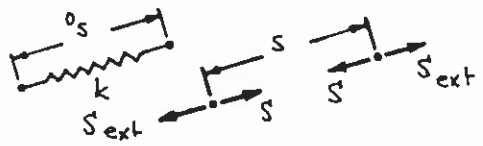
By differentiating formula (4.1.1) concerning the definition of the center of mass of a system, it is seen that the momentum of a system can be given in the form ($\bar{v} \equiv v_C$)

The division of forces into external and internal ones is the most important classification basis in mechanics. This becomes apparent especially in the principles of momentum and moment of momentum. In the following, the reference to terms connected to external and internal forces will be denoted, if needed for clarity, by the subscripts (or superscripts) ext and int, respectively.

(2) Constitutive forces and constraint forces. When a system has kinematical constraints, each constraint must be associated with a corresponding *constraint force* (force in a generalized sense) (rajoitevoima, reaktiovoima, pakkovoima, tukireaktio) which guides the motion or the state of rest of the system to such that the constraint is satisfied. Constraint forces differ from the other forces of the system in the respect that the values of them cannot be evaluated from some *constitutive relations* (konstitutiivinen yhteys) but they must be determined — if at all possible — from the equations of motion or equilibrium, when the accelerations and the other forces acting on the particles of the system have been found. We shall call other forces than constraint forces acting on the system *constitutive forces* (konstitutiivinen voima). (This terminology is not in general use.)

The force of gravity mg according to the model of constant gravitational field is perhaps the simplest example of a constitutive force. If a particle of mass m is brought to the field, this force act on the particle independent of the motion of the particle.

As a second example, let us consider the force according to the model of a *linear spring* (lineaarinen jousi). The force S acting between the two end particles of the spring is assumed to be given by the constitutive relation



$$\boxed{S = k(s - {}^0s).} \qquad (17)$$

Figure 4.4 Linear spring.

Here k ($[k] = \mathrm{N/m}$) is the *spring constant* (jousivakio) and $s - {}^0s$ is the *extension* (venymä) of the spring. When the positions of the particles are known, the extension and thus the value of S can be calculated from (17). However, if we introduce the rigid body model by putting the constraint: the distance s between the end points of the spring is constant $= {}^0s$, we cannot any more calculate the value of S from (17). The constraint can be thought to be generated by letting

the value of the spring constant to tend to infinity, meaning that the extension tends to zero, and formula (17) gives the indeterminate value $\infty \cdot 0$ for S.

Constraint forces are always connected to the exaggerated rigidities of some of the models used in macroscopic mechanics, which do not correspond exactly to reality. For example, the case of a rigid ball rolling along an inclined rigid plane does not exist. More exactly, near the area of the apparent contact point, the neighbouring molecules of the ball and the plane interact with certain constitutive forces of electromagnetic nature. The practical assumption of rigid bodies means that we have to model this interaction with an unknown constraint force.

Because of the specific nature of the constraint forces, it is often useful to try to formulate the equations of mechanics so that these forces do not appear in them. This can be accomplished systematically by using the principle of virtual work.

(3) Conservative and nonconservative forces. A force acting on a particle of a system is called *conservative force* (konservatiivinen voima), if the work done by it on the system depends only on the initial and end configurations of the system and not on the path of the system between these configurations; if this is not the case, the force is called *nonconservative force* (ulkoinen voima).

If a force $\boldsymbol{F}(\boldsymbol{r})$ is generated via a *force field* (voimakenttä), the force is conservative, if there exists a function $V(\boldsymbol{r})$ of only the point of action $\boldsymbol{r}$ of the force such that

$$\boxed{\boldsymbol{F} = -\nabla V,} \qquad (18)$$

where ∇V is the gradient of V. Function V is called the *potential energy* (potentiaalienergia) ($[V] = \mathrm{J}$) of the force. The minus sign in the formula is not essential but it is the usual convention. In rectangular cartesian coordinates, where the dependence of V on position can be expressed also as $V(x, y, z)$, the scalar form of (18) is

$$\boxed{F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}.} \qquad (19)$$

As is seen from (18), an arbitrary constant can be added to the expression of the potential energy without changing the value of the force. The point, where the value of the potential energy is fixed — usually the value zero is taken — is called *datum* (vertailupiste).

If we operate according to the convention of structural mechanics employing instead of the position vector $\boldsymbol{r}$ the displacement $\boldsymbol{u}$:

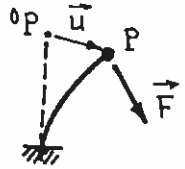
$$r = {}^0r + u, \tag{20}$$

or

$$x = {}^0x + u, \quad y = {}^0y + v, \quad z = {}^0z + w. \tag{20'}$$

The potential energy $V(r)$ is after substitution of (20) a function of u: $V = V(u) = V(u, v, w)$. The roles of quantities x, y, z are replaced by u, v, w. For instance, formulas (19) transform to

$$\boxed{F_x = -\frac{\partial V}{\partial u}, \quad F_y = -\frac{\partial V}{\partial v}, \quad F_z = -\frac{\partial V}{\partial w}.} \tag{21}$$

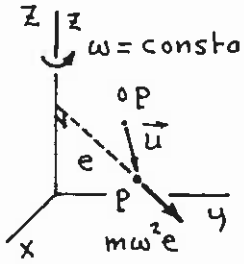


(1) Constant force F. In more detail a force with constant magnitude and direction. Potential energy

$$V = -(F_x u + F_y v + F_z w). \tag{22}$$

(2) Centrifugal force $F = -m\omega \times (\omega \times s)$ or

$$\begin{aligned} F_x &= m\omega^2({}^0x + u) \approx m\omega^2\,{}^0x, \\ F_y &= m\omega^2({}^0y + v) \approx m\omega^2\,{}^0y, \end{aligned} \tag{23}$$

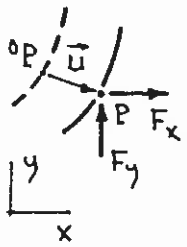


where the approximate forms are valid in the small displacement theory. Potential energy

$$\begin{aligned} V &= -m\omega^2({}^0x u + \frac{1}{2}u^2 + {}^0y v + \frac{1}{2}v^2) \\ &\approx -m\omega^2({}^0x u + {}^0y v). \end{aligned} \tag{24}$$

(3) Linear reaction force F or in plane case

$$\begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = -\begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{25}$$



or

$$\{F\} = -[k]\{u\}, \tag{25'}$$

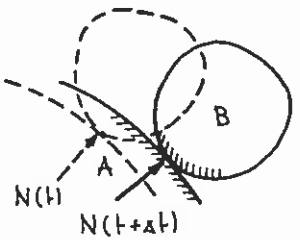
where $[k]$ is symmetric. Potential energy

$$V = \frac{1}{2}\{u\}^{\mathrm{T}}[k]\{u\}. \tag{26}$$

Figure 4.5 Conservative forces.

Figure 4.5 contains the expressions for three common conservative forces using displacement formulation and taking as datums the positions in the reference state. The extension of case (3) to three dimensions is obvious.

Systems contain often (macroscopic) forces, which cannot be given via a force field but can only be defined in connection of bodies. Typical examples of these are the constraint forces. This kind of forces are conservative if the work done by them is always zero because then the work done by them clearly does not depend on the path taken by the system. Constraint forces are usually conservative. However, if the constraints depend explicitly on time — rheonomic case — the work done by a constraint force needs not to vanish so that in such a case the constraint forces are not conservative. Figure 4.6 gives an example of this.

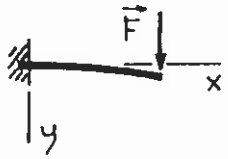


A rigid body B — here the system under study — rolls or slides along a rigid body A, which is in a given motion. The work done by the constraint force N on body B is not in general zero if body A is in motion.

Figure 4.6 Nonconservative constraint force.

Figure 4.7 contains more examples of nonconservative forces. Forces depending explicitly on time are always nonconservative. Forces depending on velocity — such as friction forces and resistance forces — are almost always nonconservative. So-called *gyroscopic forces* (gyroskooppinen voima) which are always perpendicular to the velocity are the only conservative forces depending on velocity since the work done by them is zero in real motion. The most usual example of a gyroscopic force is the Coriolis force $-2m\omega \times v_r$ appearing in a coordinate system in rotation with respect to an inertial frame (see equation (42)).

(1) Harmonic force



$$F = \hat{F}\sin\omega t\, j. \tag{27}$$

The amplitude $\hat{F}$ and the angular frequency ω are constants.

(2) Follower force, slave force $\boldsymbol{F}$ (seuraajavoima). The direction of the force is always in the direction of the bar axis at the bar end.

(3) Resistance force

$$R = \alpha v \tag{28}$$

or

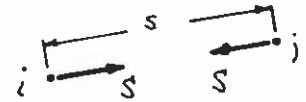
$$R = \beta v^2. \tag{29}$$

The coefficients α and β are positive quantities and v is the speed of the particle.

Figure 4.7 Nonconservative forces.

We have considered above mainly conservative and nonconservative forces acting on a particle. The same treatment is naturally valid also with respect to an external force acting on a particle of a particle system. The internal pairwise forces demand further consideration.

The pairwise forces f_{ij} and f_{ji} are conservative, if there exists a function $V_{ij}(s)$ — so-called *internal potential energy* (sisäinen potentiaalienergia) — depending only on the distance s between the points i and j so that



$$\boxed{S = \frac{\mathrm{d}V_{ij}}{\mathrm{d}s},} \tag{30}$$

Figure 4.8 Conservative internal force S.

where S is the scalar value of the pairwise forces so that a positive value corresponds to "pull". This means in other words that *the forces are conservative, if force S is a function of only of the distance s*, because the

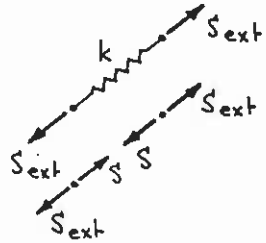
potential energy function V_{ij} is then obtained as the integral function of $S(s)$. To simplify the notations, S and s have been left without indices referring to points i and j. If the positive sign of force S is defined so that "push" is positive, formula (30) must be equipped with a minus sign.

If S is the constraint force associated with rigid body model ($s = {}^0s$), forces f_{ij} and f_{ji} are clearly conservative, because the total work done by them is always zero (see Example 4.3).

Figure 4.9 gives the internal potential energy expressions for a linear spring and for the gravitational forces between two particles. The datum, where the potential energy is zero, is taken to be for the linear spring its *natural length* (lepopituus) ${}^0s$ — the length in which the spring force is zero — and for the gravitational case the state $s = \infty$.

(1) Linear spring. Force

$$S = k(s - {}^0s), \tag{31}$$

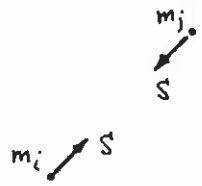


where $s - {}^0s$ is the extension of the spring and k is the spring constant. The internal potential energy

$$\boxed{V = \frac{1}{2}k(s - {}^0s)^2.} \tag{32}$$

(2) Gravitational force between two particles. Force

$$S = \gamma\frac{m_i m_j}{s^2}, \tag{33}$$



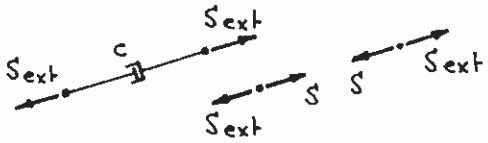
where γ is the constant of gravitation. The internal potential energy

$$\boxed{V = -\gamma\frac{m_i m_j}{s}.} \tag{34}$$

Figure 4.9 Conservative internal forces.

In structural mechanics the internal potential energy is often called *strain energy* or *elastic energy* (muodonmuutosenergia, kimmoenergia).

So-called *linear damper* (lineaarinen vaimennin), where



$$\boxed{S = c\dot{s}} \tag{35}$$

Figure 4.10 Linear damper.

and where $\dot{s} \coloneqq \mathrm{d}(s - {}^{0}s)/\mathrm{d}t$ is the *extension rate* (venymänopeus) and c ($[c] = \mathrm{N/(m/s)}$) is called the *coefficient of viscous damping* (vaimennusvakio) is an example of a nonconservative internal force. It should be added that the conventional models of a spring or a damper can be considered as a system of two massless particles where the symbolic figure of a spring or a cylinder piston pair gives the type of interaction force between the particles. This has been emphasized in the figures by denoting certain forces with the subscript ext ($S_{\mathrm{ext}} = S$).

Conservative (nonconcervative) forces and quantities associated with them will be denoted, if needed for clarity, with superscripts c (n). Conservative forces have useful properties in connection of energy principles and the principle of virtual work.

(4) Monogenic and polygenic forces. The fact that time is considered "freezed" in the principle of virtual work means that we can define a more general type of force than the conservative force — called *monogenic force* (monogeeninen voima), Lanczos (1974) — having from the point of view of the principle of virtual work the same useful properties as the conservative force. An external force $\boldsymbol{F}(\boldsymbol{r},t)$ or a pairwise internal force $S(s,t)$ is monogenic, if there exists function $V'(\boldsymbol{r},t)$ or $V'_{ij}(s,t)$ so that

$$\boxed{\boldsymbol{F} = -\nabla V'} \tag{36}$$

or

$$\boxed{S = \frac{\partial V'_{ij}}{\partial s}.} \tag{37}$$

Monogenic force differs from a conservative force in the respect that it may depend explicitly on time. We shall call function V' in lack of anything better as "time dependent potential energy". If a force is not monogenic, it is called *polygenic* (epämonogeeninen). For instance, the harmonic force (27) is monogenic and its time dependent potential energy is

$$V' = -\hat{F}\sin\omega t \cdot v. \tag{38}$$

(5) Real forces and apparent forces. A force acting on a particle will be called here a *real force* (todellinen voima), if the force is generated by another particle, that is, if the force has a pairwise opposite force according to the law of action and reaction. Often quantities having the dimension of a force, called *apparent forces* (näennäisvoima) are defined in mechanics purely computationally. These forces are not generated by any mass and they have no pairwise opposite forces. From the point of view of a system, apparent forces must be considered as external. As examples, we may mention the gravitational force as a real force and inertia force and the Coriolis force as apparent forces. Apparent forces will be denoted with superscript A. However, especially in connection of inertia forces the superscript I will be used.

The concept of inertia force is considered in more detail in Section 4.6. The concept of apparent force emerges naturally when the equations of motion are written in a non-inertial coordinate system. Consider Figure 3.9 with the XYZ-coordinate system as an inertial frame and the xyz-coordinate system as a non-inertial one. If a real force $\boldsymbol{F}$ acts on particle P, the equation of motion is

$$\boldsymbol{F} = m\boldsymbol{a} \tag{39}$$

or employing equation (3.3.1)

$$\boldsymbol{F} = m(\boldsymbol{a}_{\mathrm{r}} + \boldsymbol{a}_{\mathrm{O}} + \boldsymbol{\alpha}\times\boldsymbol{s} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{s}) + 2\boldsymbol{\omega}\times\boldsymbol{v}_{\mathrm{r}}) \tag{40}$$

or finally

$$\boxed{\boldsymbol{F} + \boldsymbol{F}^{\mathrm{A}} = m\boldsymbol{a}_{\mathrm{r}},} \tag{41}$$

where we define the apparent force

$$\boxed{\boldsymbol{F}^{\mathrm{A}} = -m\boldsymbol{a}_{\mathrm{O}} - m\boldsymbol{\alpha}\times\boldsymbol{s} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{s}) - 2m\boldsymbol{\omega}\times\boldsymbol{v}_{\mathrm{r}}.} \tag{42}$$

In some applications it is convenient to operate in non-inertial frames and then in forming the equations of motion (41) we must add the apparent forces to the real ones.

$$\boxed{p = m\bar{v}.} \tag{5}$$

When this is substituted into (4), there follows the *equation of motion of the center of mass* (massakeskiön liikelaki) ($\bar{a} \equiv a_C$)

$$\boxed{F = m\bar{a}} \tag{6}$$

or the center of mass of a system moves as a particle having the total mass of the system and to which the resultant of the external forces is acting.

4.4 MOMENT OF MOMENTUM

4.4.1 Principle of balance of moment of momentum

The *moment of momentum* or angular momentum (liikemäärämomentti, kulmaliikemäärä) L ($[L] = \mathrm{kg\,m^2/s}$) of a particle with respect to the origin is defined as

$$\boxed{L = r \times mv.} \tag{1}$$

The moment of momentum L of a *particle system* (with respect to the origin) is defined as the sum of the moments of momentum of its particles:

$$\boxed{L = \sum r_i \times m_i v_i.} \tag{2}$$

The equation of motion

$$F_i + f_i = m_i \dot{v}_i \tag{3}$$

for each particle i is manipulated by multiplying (vector product) from the left by the position vector and by adding all the resulting equations together to produce one vector equation:

$$\sum r_i \times F_i + \sum r_i \times f_i = \sum r_i \times m_i \dot{v}_i,$$

$$\sum r_i \times F_i = \frac{\mathrm{d}}{\mathrm{d}t} \sum r_i \times m_i v_i - \sum \dot{r}_i \times m_i v_i,$$

$$\sum r_i \times F_i = \frac{\mathrm{d}}{\mathrm{d}t} \sum r_i \times m_i v_i,$$

$$\boxed{M = \dot{L}} \tag{4}$$

or the moment of the external forces acting on a system is equal to the rate of change of the moment of momentum of the system.

This is called the *principle of the balance of the moment of momentum* (liikemäärämomentin taseen periaate). In the steps to (4), use have again been made of the fact that the internal forces form a zero force system, that the masses of the particles of the system are constants and that $\dot{r}_i = v_i$ and $m_i v_i$ are parallel vectors so that their vector product is zero.

In this principle the moment and the moment of momentum must be evaluated with respect to a given point, called *reference point* or *moment point* (referenssipiste, momenttipiste). Equation (4) was derived keeping the origin as the reference point but it is easy to prove that the principle is valid if the reference point is an arbitrary *fixed* point A, in which case

$$M = \sum r_{i/A} \times F_i, \qquad L = \sum r_{i/A} \times m_i v_i \tag{5}$$

or if the reference point coincides continually with the center of mass C, in which case

$$M = \sum r_{i/C} \times F_i, \qquad L = \sum r_{i/C} \times m_i v_i. \tag{6}$$

If necessary for clarity, M and L can be equipped with a subscript such as O, A, C etc, referring to the reference point.

4.4.2 Rigid body

Due to the kinematical constraints of the rigid body model, the expression for the moment of momentum can be given specific detailed forms.

The rigid body is usually considered as a continuum. The formulas valid for a discrete particle system can be written directly for a continuum using the steps

$$\boxed{\begin{array}{c} m_i \to \mathrm{d}m, \\ \Sigma \to \int. \end{array}} \tag{7}$$

To a typical particle i (mass $= m_i$) corresponds a typical continuum element (mass $= \mathrm{d}m$). To summation corresponds integration. The notation $\int f \mathrm{d}m$ in use in mechanics means in words the integral of quantity f over the mass of the body.

Thus the expression for the moment of momentum $L = \sum r_i \times m_i v_i$ for a particle system becomes

$$L = \int r \times v \, \mathrm{d}m \tag{8}$$

for a continuum.

For the evaluation of the moment of momentum it is convenient to use a xyz-coordinate system with its origin at the center of mass C. From the formula (3.2.6), the velocity of a generic point of a rigid body is ($\bar{v} \equiv v_C$)

$$v = \bar{v} + \omega \times s, \tag{9}$$

where s is the position vector from C to the continuum element:

$$s = xi + yj + zk. \tag{10}$$

When this is substituted into (8), there follows

$$\boxed{L = \bar{r} \times p + \bar{L},} \tag{11}$$

where

$$\bar{L} \equiv L_C = \bar{L}_x i + \bar{L}_y j + \bar{L}_z k \tag{12}$$

with

$$\begin{aligned} \bar{L}_x &= \bar{I}_{xx}\omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z, \\ \bar{L}_y &= -\bar{I}_{yx}\omega_x + \bar{I}_{yy}\omega_y - \bar{I}_{yz}\omega_z, \\ \bar{L}_x &= -\bar{I}_{zx}\omega_x - \bar{I}_{zy}\omega_y + \bar{I}_{zz}\omega_z. \end{aligned} \tag{13}$$

Here the well-known *moments of inertia* (hitausmomentti) and *products of inertia* (hitaustulo) appear:

$$I_{xx} = \int (y^2 + z^2)\,dm, \quad I_{xy} = I_{yx} = \int xy\,dm, \quad \ldots \quad \ldots \tag{14}$$

These expressions are used also in connection with an arbitrary rectangular cartesian coordinate system so we have not equipped the symbols with the overbar which refers here to the case where the origin of the system is at the center of mass.

The matrix equivalent to $\bar{L}$ is

$$\{\bar{L}\} = [\bar{J}]\{\omega\} \tag{15}$$

with

$$[\bar{J}] = \begin{bmatrix} \bar{I}_{xx} & -\bar{I}_{xy} & -\bar{I}_{xz} \\ -\bar{I}_{yx} & \bar{I}_{yy} & -\bar{I}_{yz} \\ -\bar{I}_{zx} & -\bar{I}_{zy} & \bar{I}_{zz} \end{bmatrix}, \quad \{\omega\} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}. \tag{16}$$

4.5 WORK AND ENERGY

4.5.1 Work done by a force

The *differential work* or elementary work (differentiaalinen työ) $d'W$ ($[d'W] = \text{Nm} = \text{J}$) done by a force F acting on a particle is defined as

$$\boxed{d'W = F \cdot dr} \tag{1}$$

where dr is the differential displacement of the *particle* during a differential time increment dt (Figure 4.12 (a)). The differential work can be expressed also as

$$d'W = P\,dt \tag{2}$$

where

$$P = F \cdot v \tag{3}$$

is called the *power* (teho) ($[P] = \text{J/s} = \text{W}$) of the force, that is, work divided by time: $P = d'W/dt$. In rectangular cartesian coordinates,

$$\begin{aligned} F &= F_x i + F_y j + F_z k, \\ dr &= dx\,i + dy\,j + dz\,k, \\ v &= v_x i + v_y j + v_z k = \dot{x} i + \dot{y} j + \dot{z} k, \end{aligned} \tag{4}$$

and the differential work and the power obtain the expressions

$$d'W = F_x dx + F_y dy + F_z dz, \tag{5}$$

$$P = F_x v_x + F_y v_y + F_z v_z, \tag{6}$$

respectively.

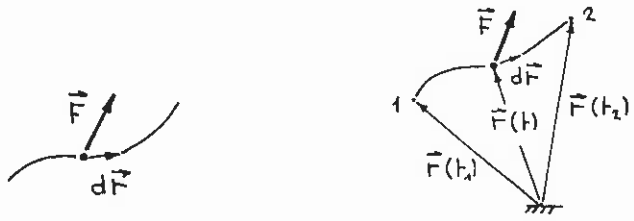


Figure 4.12 **(a)** Differential displacement. **(b)** Finite displacement.

When the particle moves during a finite time interval Δt from time $t = t_1$ to $t = t_2$ from point 1 to point 2, the *work* (työ) ($[W] = \text{Nm} = \text{J}$) done by the force

F is defined as sum of the differential works along the path of the particle (Figure 4.12 (b)):

$$W = \int_1^2 \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{t_1}^{t_2} \boldsymbol{F} \cdot \boldsymbol{v}\, \mathrm{d}t = \int_{t_1}^{t_2} P\, \mathrm{d}t. \tag{7}$$

The first form is a line integral along the particle path and the last two forms are ordinary integrals along the time axis.

The differential work done on a particle system is defined as the sum of the differential works done on the particles and similarly for the finite work. Thus

$$\mathrm{d}'W = \sum \mathrm{d}'W_i = \sum \boldsymbol{R}_i \cdot \mathrm{d}\boldsymbol{r}_i = \sum \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_i + \sum \boldsymbol{f}_i \cdot \mathrm{d}\boldsymbol{r}_i \tag{8}$$

and

$$W = \sum W_i = \int_1^2 \boldsymbol{R}_i \cdot \mathrm{d}\boldsymbol{r}_i = \int_1^2 \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_i + \int_1^2 \boldsymbol{f}_i \cdot \mathrm{d}\boldsymbol{r}_i \tag{9}$$

where in the latter the points 1 and 2 refer for each particle to the endpoints of its path. The differential work $\mathrm{d}'W$ consists thus of the differential work of the external forces $\mathrm{d}'W_{\mathrm{ext}}$ and of the differential work of the internal forces $\mathrm{d}'W_{\mathrm{int}}$:

$$\boxed{\mathrm{d}'W = \mathrm{d}'W_{\mathrm{ext}} + \mathrm{d}'W_{\mathrm{int}}} \tag{10}$$

where

$$\begin{aligned} \mathrm{d}'W_{\mathrm{ext}} &= \sum \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_i\,, \\ \mathrm{d}'W_{\mathrm{int}} &= \sum \boldsymbol{f}_i \cdot \mathrm{d}\boldsymbol{r}_i = -\sum_{ij} S\mathrm{d}s. \end{aligned} \tag{11}$$

The last form of the last equation is explained and derived in Example 4.3.

Equation (9) can be presented similarly in the form

$$\boxed{W = W_{\mathrm{ext}} + W_{\mathrm{int}}} \tag{12}$$

where the meaning of the notations is obvious.

Finally, we can write for the power

$$\boxed{P = P_{\mathrm{ext}} + P_{\mathrm{int}}} \tag{13}$$

where

$$\begin{aligned} P_{\mathrm{ext}} &= \sum \boldsymbol{F}_i \cdot \boldsymbol{v}_i\,, \\ P_{\mathrm{int}} &= \sum \boldsymbol{f}_i \cdot \boldsymbol{v}_i = -\sum_{ij} S\dot{s}. \end{aligned} \tag{14}$$

Again, the last form of the last equation is explained in Example 4.3.

Example 4.3. We consider the work done by two pairwise forces f_{ij} and f_{ji} acting on particles i and j in Figure (a).

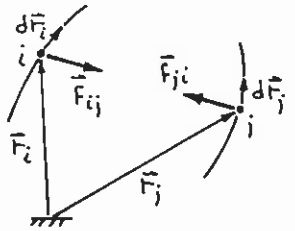


Figure (a)

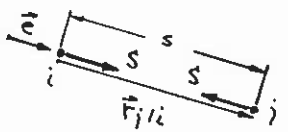


Figure (b)

We obtain from Figures (a) and (b) the kinematical result

$$r_{j/i} = r_j - r_i \tag{a}$$

and by differentiation

$$\mathrm{d}r_{j/i} = \mathrm{d}r_j - \mathrm{d}r_i. \tag{b}$$

The differential work done by the forces f_{ij} and f_{ji} is

$$\begin{aligned} \mathrm{d}'W_{ij}^{\mathrm{int}} &= f_{ij} \cdot \mathrm{d}r_i + f_{ji} \cdot \mathrm{d}r_j = -f_{ji} \cdot \mathrm{d}r_i + f_{ji} \cdot \mathrm{d}r_j \\ &= f_{ji} \cdot (\mathrm{d}r_j - \mathrm{d}r_i) = f_{ji} \cdot \mathrm{d}r_{j/i}. \end{aligned} \tag{c}$$

To further develop this expression, we denote (Figure (b))

$$r_{j/i} = se, \qquad f_{ji} = -Se \tag{d}$$

where e is a unit vector directed from point i towards point j and and where S is defined similarly as in Section 4.1.3. Expression (c) becomes

$$\mathrm{d}'W_{ij}^{\mathrm{int}} = -Se \cdot \mathrm{d}(se) = -Se \cdot (\mathrm{d}se + s\mathrm{d}e) = -S\mathrm{d}se \cdot e - Sse \cdot \mathrm{d}e \tag{e}$$

or

$$\mathrm{d}'W_{ij}^{\mathrm{int}} = -S\mathrm{d}s. \tag{f}$$

Use have been made of the well known fact that the differential ($\mathrm{d}e$) of a vector of constant magnitude (e) is perpendicular to the original vector or zero.

When equation (f) is divided by the time differential $\mathrm{d}t$, we obtain for the power of forces f_{ij} and f_{ji}

$$P_{ij}^{\mathrm{int}} = -S\dot{s}. \tag{g}$$

The total differential work $d'W^{int}$ done by the internal forces and similarly the total power P^{int} of internal forces are obviously obtained by summing all the contributions from the particle pairs considered above. Thus

$$d'W^{int} = -\sum_{ij} S ds \tag{h}$$

and

$$P^{int} = -\sum_{ij} S \dot{s}. \tag{i}$$

The summation notation with indices ij means here the sum over all the particle pairs (with each pair only once). To simplify formulas, S and s have not been equipped with indices but this should not cause confusion.

Formulas (h) and (i) are considerable more illuminating than the corresponding middle forms in (11) and (14).

First, it is realized that *the work and power of the internal forces do not depend on the choice of the frame of reference* because the extensions ds and the extension rates $\dot{s}$ are naturally the same (they depend only on the change of shape of the system) in different coordinate systems. In contrast, the work and power of the external forces clearly generally depend on the choice of the frame of reference if the frames are moving with respect of each other. In coordinate systems, in rest with respect to each other, the work and power of external forces, however, are the same.

Second, *in a rigid body the work and power of internal forces are zero*, because in the rigid body model the distances between particle pairs are assumed to remain constants. This result is an important advantage in connection of applications of rigid body mechanics. On the other hand, it is obvious that in some cases the rigid body model is in this respect unrealistic. A notable case for this are impact problems where local deformations near the areas of contact are not negligible and the work of internal forces must somehow be taken into account even if the rigid body model would be quite accurate say in the evaluation of momentum and moment of momentum.

4.5.2 Kinetic energy

The *kinetic energy* (liike-energia, kineettinen energia) K ($[K] = \mathrm{J}$) of a particle is defined as

$$\boxed{K = \frac{1}{2} m \boldsymbol{v} \cdot \boldsymbol{v}} = \frac{1}{2} m v^2. \tag{15}$$

Here v is the *speed* (vauhti) of the particle, that is, it is the magnitude $|\boldsymbol{v}|$ of the velocity vector $\boldsymbol{v}$.

The kinetic energy K of a *particle system* is defined as the sum of the kinetic energies of its particles:

$$\boxed{K = \frac{1}{2} \sum m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i} = \frac{1}{2} \sum m_i v_i^2. \tag{16}$$

The kinetic energy is thus a non-negative quantity. It disappears only if all the velocities of the particles of the system are zero. The value of the kinetic energy depends on the choice of the frame of reference if the frames are moving with respect to each other.

4.5.3 State function

Using the terminology of thermodynamics, e.g. Ziegler (1983), work is not a *state function* or dependent state variable (tilafunktio, tilanfunktio). For a particle system the *independent state variables* (tilamuuttuja) would be for instance the position vectors and velocities of the particles of the system. They determine the *state* (tila) of the system and they are then considered as independent variables from the point of view of the state. (What kind of quantites is included in a definition of a state depends on the application and on the specific models used in the thermodynamic theory at hand.) The state functions are thus functions of the state variables. For example, the kinetic energy K of the system depends only on the velocities of the particles of the system (the masses of the system are fixed through the selection of the system) so K is a state function.

The work done on a system (calculated starting from a given reference state) depends in general on the paths of the particles during the evaluation and *work is thus not a state function*. A state function is a *property* of a system. Correspondingly, for example, we may say that a system has such and such amount of kinetic energy but we cannot say that it has such and such amount of work.

The above is connected to the use of notation $d'W$ (or often $\overline{dW}$ or $đW$ or δW) and to the term *differential work* (and not the differential of the work) which emphasizes that we are not dealing with the differential of a quantity called work but we have only an *infinitesimal expression* (infinitesimaalinen lauseke). Similar notational conventions have been discussed already in Remark 3.1 in connection of the infinitesimal angular rotation $d'\boldsymbol{\theta}$ where a quantity $\boldsymbol{\theta}$ did not exist at all. Here, however, is a certain difference because now we can still define a quantity $W(t)$ connected to a specific process.

As a further explanation, let us consider Figure 4.13. It represents the work W done on a system starting from a certain moment of time as a function of time. It should be emphasized that *in particle mechanics the only truly independent variable is the time*. Any quantity in particle mechanics can be represented in the manner shown in Figure 4.13 as a function of time in a certain event called in the wordings of thermodynamics as a *change of state* or *process* (tilan muutos, prosessi). When discussing above the state variables as independent variables, this was meant from the point of view of the state of the system. In Figure 4.13 the change $d'W$ would be the total differential of function $W(t)$ but

as the relationship $W = W(t)$ depends on the path taken by the process, the relationship is unknown in advance and this interpretation has no practical use.

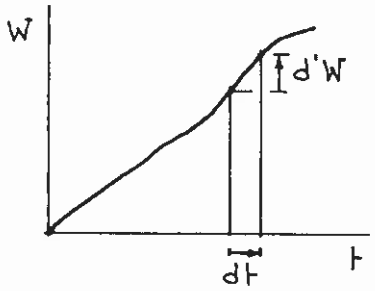


Figure 4.13 Work done on a system as a function of time.

4.5.4 Principle of balance of mechanical energy

In the following manipulation, the equation of motion

$$F_i + f_i = m_i \dot{v}_i \tag{17}$$

of each particle i of the system is multiplied (scalar product) by the velocity v_i of the particle and all the resulting equations are added together to give one scalar equation:

$$\begin{aligned} &\textstyle\sum F_i \cdot v_i + \sum f_i \cdot v_i = \sum m_i \dot{v}_i \cdot v_i , \\ &P_{\text{ext}} + P_{\text{int}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\sum m_i v_i \cdot v_i , \\ &\boxed{P = \dot{K}} \end{aligned} \tag{18}$$

or the total power of the external and internal forces acting on a system is equal to the rate of change of the kinetic energy of the system.

By integrating equation (18) between the time interval from $t = t_1$ to $t = t_2$, we obtain the equation

$$\int_{t_1}^{t_2} P\,\mathrm{d}t = \int_{t_1}^{t_2} \dot{K}\,\mathrm{d}t = \Big|_{t_1}^{t_2} K = K(t_2) - K(t_1)$$

$$\boxed{W = \Delta K} \tag{19}$$

or the total work done by the external and internal forces acting on the system is equal to change of the kinetic energy of the system.

Equations (18) and (19) have many varying names in the literature: principle of energy, principle of work and kinetic energy, work principle, energy principle, law of kinetic energy, etc. We will call both of them as the *principle of balance of mechanical energy* (mekaanisen energian taseen periaate). The context shows if the "power" form or the"work" form of the principle is used.

Remark 4.3. In continuum mechanics, one of the axioms is called the *principle of the balance of energy* (energian taseen periaate), Chapter 9. It should be noted that the principle of the balance of mechanical energy is obtained just by manipulating the equations of motion and contains no thermal terms and is not directly equivalent to the continuum mechanics axiom. □

Remark 4.4. Instead of starting the manipulation of the equations of motion by the velocity vectors of the particles, we can start with

$$\begin{aligned} &\textstyle\sum F_i \cdot \mathrm{d}r_i + \sum f_i \cdot \mathrm{d}r_i = \sum m_i \dot{v}_i \cdot \mathrm{d}r_i , \\ &\mathrm{d}'W_{\text{ext}} + \mathrm{d}'W_{\text{int}} = \textstyle\sum m_i \dot{v}_i \cdot v_i \mathrm{d}t , \\ &\mathrm{d}'W_{\text{ext}} + \mathrm{d}'W_{\text{int}} = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}\textstyle\sum m_i v_i \cdot v_i)\mathrm{d}t , \\ &\mathrm{d}'W = \mathrm{d}K \end{aligned} \tag{20}$$

or the total differential work done by the external and internal forces acting on the system is equal to the differential change of the kinetic energy of the system.

Dividing (20) by $\mathrm{d}t$ gives (18) and integrating (20) with respect to time gives (19). □

4.5.5 Rigid body

Let us consider Figure 4.14 (a). The differential displacement of a point i of a rigid body is according to formula (3.2.3)

$$\mathrm{d}r_i = \mathrm{d}r_A + \mathrm{d}'\boldsymbol{\theta} \times r_{i/A} , \tag{21}$$

where A is an arbitrary particle of the body and $\mathrm{d}'\boldsymbol{\theta}$ the differential angular displacement of the body. The total differential work $\mathrm{d}'W_{\text{ext}} = \mathrm{d}'W$ of the external point forces acting on the body is thus

$$\begin{aligned} \mathrm{d}'W = \textstyle\sum F_i \cdot \mathrm{d}r_i &= \sum F_i \cdot (\mathrm{d}r_A + \mathrm{d}'\boldsymbol{\theta} \times r_{i/A}) = \sum F_i \cdot \mathrm{d}r_A + \sum F_i \cdot (\mathrm{d}'\boldsymbol{\theta} \times r_{i/A}) \\ &= \textstyle\sum F_i \cdot \mathrm{d}r_A + \sum r_{i/A} \times F_i \cdot \mathrm{d}'\boldsymbol{\theta} = (\sum F_i) \cdot \mathrm{d}r_A + (\sum r_{i/A} \times F_i) \cdot \mathrm{d}'\boldsymbol{\theta} \end{aligned} \tag{22}$$

or

$$\boxed{d'W = \boldsymbol{F} \cdot d\boldsymbol{r}_A + \boldsymbol{M}_A \cdot d'\boldsymbol{\theta}.} \tag{22}$$

The summations above refer to the external point forces. Use have been made of some of the calculation rules of the scalar triple product. According to this formula, each force system having the same resultant and moment produces the same differential work on a rigid body. The formula is naturally valid also for distributed forces.Due to the kinematical constraints of the rigid body model, the expressions of work and kinetic energy obtain specific detailed forms.

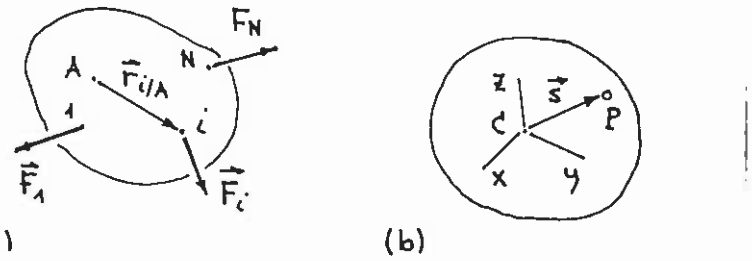


Figure 4.14 (**a**) Forces acting on a rigid body. (**b**) Coordinate system for kinetic energy evaluation.

Dividing (22) by dt gives a corresponding form

$$\boxed{P = \boldsymbol{F} \cdot \boldsymbol{v}_A + \boldsymbol{M}_A \cdot \boldsymbol{\omega}} \tag{23}$$

concerning the power. Finite work is obtained by integration of (22) or (23).

For the evaluation of the kinetic energy of a rigid body it is convenient to use a coordinate system with its origin at the center of mass similarly as in Section 4.4.2. Again, from formula (3.2.6), the velocity of a generic point is

$$\boldsymbol{v} = \bar{\boldsymbol{v}} + \boldsymbol{\omega} \times \boldsymbol{s}. \tag{24}$$

When this is substituted in the kinetic energy expression for a continuum

$$K = \frac{1}{2}\int \boldsymbol{v} \cdot \boldsymbol{v}\, dm \tag{25}$$

there is finally obtained the formula

$$\boxed{K = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\{\omega\}^T[\bar{J}]\{\omega\}.} \tag{26}$$

Here $\bar{v}$ is the speed of the center of mass and $\{\omega\}$ and $[\bar{J}]$ have the meanings explained in Section 4.4.2. If a rigid body is performing rotation about a fixed point, an alternative kinetic energy expression contains only the latter part of (26) and $[J]$ is then evaluated in a coordinate system with its origin at the fixed point.

4.5.6 Conservative systems

The work done by conservative forces is of quite different nature from the general case.

The differential work done by an external conservative force $\boldsymbol{F}_i$ acting on particle i is

$$\begin{aligned} d'W_i^{ext} &= \boldsymbol{F}_i \cdot d\boldsymbol{r}_i = -\nabla V_i^{ext} \cdot d\boldsymbol{r}_i \\ &= -\left(\frac{\partial V_i^{ext}}{\partial x_i}\boldsymbol{i} + \frac{\partial V_i^{ext}}{\partial y_i}\boldsymbol{j} + \frac{\partial V_i^{ext}}{\partial z_i}\boldsymbol{k}\right) \cdot (dx_i\boldsymbol{i} + dy_i\boldsymbol{j} + dz_i\boldsymbol{k}) \\ &= -\left(\frac{\partial V_i^{ext}}{\partial x_i}dx_i + \frac{\partial V_i^{ext}}{\partial y_i}dy_i + \frac{\partial V_i^{ext}}{\partial z_i}dz_i\right) \end{aligned} \tag{27}$$

or

$$\boxed{d'W_i^{ext} = -dV_i^{ext}} \tag{28}$$

or the differential work done by a conservative force is equal to minus of the total differential of the potential energy of the force due to the change $d\boldsymbol{r}_i$ of the argument $\boldsymbol{r}_i$ and not just an infinitesimal expression.

Similarly, finite work done by an external conservative force $\boldsymbol{F}_i$ is

$$W_i^{ext} = \int_1^2 d'W_i^{ext} = -\int_1^2 dV_i^{ext} = -[V_i^{ext}(r_{i2}) - V_i^{ext}(r_{i1})] \tag{29}$$

or

$$\boxed{W_i^{ext} = -\Delta V_i^{ext}} \tag{30}$$

or the work done by a conservative force is equal to minus of the change of the potential energy of the force due to the change $\Delta\boldsymbol{r}_i$ of the argument $\boldsymbol{r}_i$.

If the pairwise forces f_{ij} and f_{ji} considered in Example 4.3 are conservative, the differential work done by them is

$$\mathrm{d}'W_{ij}^{\mathrm{int}} = -S\mathrm{d}s = -\frac{\mathrm{d}V_{ij}^{\mathrm{int}}}{\mathrm{d}s}\mathrm{d}s \tag{31}$$

or

$$\boxed{\mathrm{d}'W_{ij}^{\mathrm{int}} = -\mathrm{d}V_{ij}^{\mathrm{int}}.} \tag{32}$$

Similarly,

$$W_{ij}^{\mathrm{int}} = \int_1^2 \mathrm{d}'W_{ij}^{\mathrm{int}} = -\int_1^2 \mathrm{d}V_{ij}^{\mathrm{int}} = -[V_{ij}^{\mathrm{int}}(s_2) - V_{ij}^{\mathrm{int}}(s_1)] \tag{33}$$

or

$$\boxed{W_{ij}^{\mathrm{int}} = -\Delta V_{ij}^{\mathrm{int}}.} \tag{34}$$

The verbal interpretation of formulas (32) and (34) is similar as in the case of the external conservative force. Let us remark that we have considered only those conservative forces which can be expressed via a force field. However, as the work done by the other kind of conservative forces such as constraint forces is zero, the formulas are valid in general.

If all the forces acting on a system are conservative, the system is said to be a *conservative system* (konservatiivinen systeemi). The total work done by conservative forces is

$$\begin{aligned} W &= W_{\mathrm{ext}} + W_{\mathrm{int}} = -\sum \Delta V_i^{\mathrm{ext}} - \sum_{ij} \Delta V_{ij}^{\mathrm{int}} \\ &= -\Delta \sum V_i^{\mathrm{ext}} - \Delta \sum_{ij} V_{ij}^{\mathrm{int}} = -\Delta(\sum V_i^{\mathrm{ext}} + \sum_{ij} V_{ij}^{\mathrm{int}}) \end{aligned} \tag{35}$$

or

$$\boxed{W = -\Delta V} \tag{36}$$

where

$$\begin{aligned} V &= V_{\mathrm{ext}} + V_{\mathrm{int}}, \\ V_{\mathrm{ext}} &= \sum V_i^{\mathrm{ext}}, \\ V_{\mathrm{int}} &= \sum_{ij} \Delta V_{ij}^{\mathrm{int}}. \end{aligned} \tag{37}$$

Here V is called the (total) *potential energy of the system* (systeemin (kokonais)-potentiaalienergia). It consists of the *potential energy of the external forces* V_{ext} (systeemin ulkoisten voimien potentiaalienergia) and of the *potential energy of the internal forces of the system* V_{int} (systeemin sisäisten voimien potentiaalienergia). V_{int} is often called the *strain energy* (muodonmuutosenergia, kimmoenergia) of the system. The datum for each potential energy can be selected arbitrarily.

For infinitesimal changes in a conservative system, the equivalent of (36) is

$$\boxed{\mathrm{d}'W = -\mathrm{d}V.} \tag{38}$$

For a conservative system, the principle of balance of mechanical energy (19) obtains the form

$$\boxed{\Delta(K+V) = 0} \tag{39}$$

or in a conservative system, the *mechanical energy* (mekaaninen energia) $K+V$ is conserved.

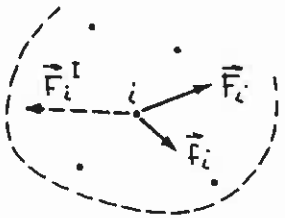
This is called the *principle of conservation of mechanical energy* (mekaanisen energian säilymisen periaate). This feature of certain quantity preserving (conserving) its value is in fact behind the term "conservative" force.

4.6 INERTIA FORCE METHOD

In the following manipulation, the equations of motion for a particle system,

$$\boldsymbol{F}_i + \boldsymbol{f}_i = m_i \boldsymbol{a}_i, \tag{1}$$

are written in the form



$$\boxed{\boldsymbol{F}_i + \boldsymbol{f}_i + \boldsymbol{F}_i^{\mathrm{I}} = \boldsymbol{0},} \tag{2}$$

Figure 4.15 Forces acting on a particle.

where the apparent force

$$\boxed{F_i^{\mathrm{I}} = -m_i a_i} \tag{3}$$

is called *inertia force* (hitausvoima). The content of these equations is called often the *inertia force principle* or perhaps more properly the *inertia force method* (hitausvoimaperiaate tai paremminkin hitausvoimamenettely):

> A particle system is in equilibrium under the force system consisting of the real external and internal forces and of the apparent inertia forces. (4)

The inertia force method transforms a problem of dynamics superficially to a problem of statics. However, as the resulting equations are still the original differential equations (1) in disguise, the inertia force method can only be of some help in forming the governing equations and not in the actual solution of them.

If the solution of a dynamics problem is obtained, the inertia force method can be applied sometimes conveniently in a post-processing manner for instance in the determination of internal stresses in a link of a mechanism at a certain moment of time by employing familiar procedures of statics.

Remark 4.5. Quite often in the literature concerning solid mechanics the starting point is statics and only later the more complicated case of dynamics is treated. In this way one gets familiar with new concepts in a relative simple context, which are then elaborated on. In this approach we step from a statics problem into a dynamics problem by just introducing the inertia forces into a statics formulation through the *substitution* (sijoitus)

$$\boxed{F_i := F_i + F_i^{\mathrm{I}}\,.} \tag{5}$$

This practise will be used also in this text at some places. □

4.7 REFERENCES

Gantmacher, F. (1975): *Lectures in Analytical Mechanics*, Mir Publishers, Moscow.

Goldstein, H. (1971): *Classical mechanics*, Addison-Wesley, Reading, Massachusetts.

Lanczos, C. (1974): *The Variational Principles of Mechanics*, University of Toronto Press, Toronto.

Rosenberg, R. M. (1977): *Analytical Dynamics of Discrete Systems*, Plenum Press, New York.

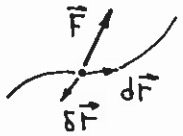
Ziegler, H. (1983): *An Introduction to Thermomechanics*, 2nd. rev. ed., North-Holland, Amsterdam.

CHAPTER 5

VIRTUAL WORK

5.1 VIRTUAL WORK DONE BY A FORCE

The *virtual work* (virtuaalinen työ) $\delta' W$ ($[\delta' W] = \mathrm{Nm} = \mathrm{J}$) done by a force $\boldsymbol{F}$ acting on a particle is defined as



$$\boxed{\delta' W = \boldsymbol{F} \cdot \delta \boldsymbol{r},} \tag{1}$$

Figure 5.1 Real differential displacement d$\boldsymbol{r}$ and virtual displacement $\delta \boldsymbol{r}$.

where $\delta \boldsymbol{r}$ is called *virtual displacement* (virtual displacement) of the particle. It is an infinitesimal quantity where the δ-notation is employed instead of the d-notation to indicate that it is a virtual, assumed, imaginary displacement, a product of a thought experiment. Virtual displacement is assumed to take place *without any time increment*. When calculating virtual work, $\boldsymbol{F}$ *is the actual force acting on the particle preserving its magnitude and direction "during" the virtual displacement.*

Most of the expressions of Section 4.5 valid for real differential work obtain here analogous forms with d′ or d replaced by δ' or δ. For instance, in rectangular cartesian coordinates,

$$\begin{aligned} \boldsymbol{F} &= F_x \boldsymbol{i} + F_y \boldsymbol{j} + F_z \boldsymbol{k}, \\ \delta \boldsymbol{r} &= \delta x \boldsymbol{i} + \delta y \boldsymbol{j} + \delta z \boldsymbol{k} \end{aligned} \tag{2}$$

and the virtual work obtains the expression

$$\delta' W = F_x \delta x + F_y \delta y + F_z \delta z \tag{3}$$

Remark 5.1. As explained in Section D.2.2, the δ-notation refers to the first variation or shortly just variation of a quantity. A virtual displacement $\delta \boldsymbol{r}$ is thus the variation of the position vector $\boldsymbol{r}$ of the particle. The notation $\delta' W$ is used similarly as in Section 4.5.3 for d′W because it is not a variation of a quantity called work. It is just an infinitesimal expression. □

For a system of particles the virtual work done by the forces acting on the particles is defined as the sum

$$\delta' W = \sum \delta' W_i = \sum \boldsymbol{R}_i \cdot \delta \boldsymbol{r}_i = \sum \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i + \sum \boldsymbol{f}_i \cdot \delta \boldsymbol{r}_i . \tag{4}$$

The virtual work $\delta' W$ consists thus of the virtual work $\delta' W_{\mathrm{ext}}$ of the external forces and of the virtual work $\delta' W_{\mathrm{int}}$ of the internal forces or

$$\boxed{\delta' W = \delta' W_{\mathrm{ext}} + \delta' W_{\mathrm{int}},} \tag{5}$$

where

$$\begin{aligned} \delta' W_{\mathrm{ext}} &= \sum \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i , \\ \delta' W_{\mathrm{int}} &= \sum \boldsymbol{f}_i \cdot \delta \boldsymbol{r}_i = -\sum_{ij} S \delta s. \end{aligned} \tag{6}$$

The last form of the last formula can be derived similarly as in the case of differential work in Section 4.5.1. The quantity δs is the *virtual extension* (virtuaalinen pitenemä), that is, the differential change of the distance s due to the virtual displacements $\delta \boldsymbol{r}_i$ and $\delta \boldsymbol{r}_j$ (see example 5.1).

Example 5.1. We consider the virtual work done by two pairwise forces $\boldsymbol{f}_{ij}$ and $\boldsymbol{f}_{ji}$ acting on particles i and j in Figure (a).

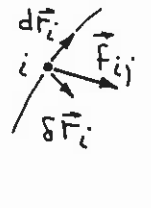


Figure (a)

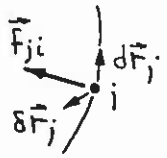
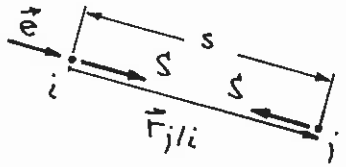


Figure (b)



The treatment is completely similar to the one in Example 4.3. Thus we obtain immediately the expressions

$$\boldsymbol{r}_{j/i} = \boldsymbol{r}_j - \boldsymbol{r}_i , \tag{a}$$

$$\delta \boldsymbol{r}_{j/i} = \delta \boldsymbol{r}_j - \delta \boldsymbol{r}_i , \tag{b}$$

$$\begin{aligned} \delta' W_{ij}^{\mathrm{int}} &= \boldsymbol{f}_{ij} \cdot \delta \boldsymbol{r}_i + \boldsymbol{f}_{ji} \cdot \delta \boldsymbol{r}_j = -\boldsymbol{f}_{ji} \cdot \delta \boldsymbol{r}_i + \boldsymbol{f}_{ji} \cdot \delta \boldsymbol{r}_j \\ &= \boldsymbol{f}_{ji} \cdot (\delta \boldsymbol{r}_j - \delta \boldsymbol{r}_i) = \boldsymbol{f}_{ji} \cdot \delta \boldsymbol{r}_{j/i} , \end{aligned} \tag{c}$$

$$\boldsymbol{r}_{j/i} = s\boldsymbol{e}, \qquad \boldsymbol{f}_{ji} = -S\boldsymbol{e}, \tag{d}$$

$$\delta' W_{ij}^{\mathrm{int}} = -S\boldsymbol{e} \cdot \delta(s\boldsymbol{e}) = -S\boldsymbol{e} \cdot (\delta s \boldsymbol{e} + s \delta \boldsymbol{e}) = -S \delta s \boldsymbol{e} \cdot \boldsymbol{e} - S s \boldsymbol{e} \cdot \delta \boldsymbol{e}, \tag{e}$$

$$\delta' W_{ij}^{\text{int}} = -S\delta s. \tag{f}$$

Expression (6) is obtained thus from the summation of contributions (f).

5.2 PRINCIPLE OF VIRTUAL WORK

The equations of motion of a particle system are

$$\boldsymbol{F}_i + \boldsymbol{f}_i = m_i \boldsymbol{a}_i, \quad i = 1,2,\cdots,N \tag{1}$$

or in component form form

$$\begin{aligned} (F_i)_x + (f_i)_x &= m_i (a_i)_x, \\ (F_i)_y + (f_i)_y &= m_i (a_i)_y, \\ (F_i)_z + (f_i)_z &= m_i (a_i)_z. \end{aligned} \tag{1'}$$

We again manipulate these to obtain new forms. Equation (1) is multiplied scalarly by an *arbitrary* vector

$$\boldsymbol{w}_i = (w_i)_x \boldsymbol{i} + (w_i)_y \boldsymbol{j} + (w_i)_z \boldsymbol{k} \tag{2}$$

and all the resulting equations are added to give an equation

$$\sum (\boldsymbol{F}_i + \boldsymbol{f}_i) \cdot \boldsymbol{w}_i = \sum m_i \boldsymbol{a}_i \cdot \boldsymbol{w}_i. \tag{3}$$

An equivalent equation is arrived at by multiplying equations (1') by arbitrary scalars $(w_i)_x$, $(w_i)_y$, $(w_i)_z$, respectively, and by adding the resulting equations to give

$$\begin{aligned} &[(F_1)_x + (f_1)_x](w_1)_x + [(F_1)_y + (f_1)_y](w_1)_y + [(F_1)_z + (f_1)_z](w_1)_z + \\ &[(F_2)_x + (f_2)_x](w_2)_x + \cdots + [(F_N)_z + (f_N)_z](w_N)_z = \\ &m_1 (a_1)_x (w_1)_x + m_1 (a_1)_y (w_1)_y + m_1 (a_1)_z (w_1)_z + \\ &m_2 (a_2)_x (w_2)_x + \cdots + m_N (a_N)_z (w_N)_z. \end{aligned} \tag{3'}$$

Because the vectors $\boldsymbol{w}_i$ or the components $(w_i)_x$, $(w_i)_y$, $(w_i)_z$ are completely arbitrary, *equation* (3) *is equivalent to equations* (1). We can namely make first say the selection $(w_1)_x \neq 0$, all the other components equal to zero, to give an equation

$$[(F_1)_x + (f_1)_x](w_1)_x = m_1 (a_1)_x (w_1)_x \tag{4}$$

and thus as $(w_1)_x$ is arbitrary, the first equation (1'), etc., is arrived at.

Equation (3) can be considered to represent some kind of weighted average equation of motion for the whole system with the multipliers $\boldsymbol{w}_i$ acting as weights. The usefulness of the equation increases if we take the interpretation

$$\boldsymbol{w}_i = \delta \boldsymbol{r}_i \tag{5}$$

or

$$\begin{aligned} (w_i)_x &= \delta x_i, \\ (w_i)_y &= \delta y_i, \\ (w_i)_z &= \delta z_i. \end{aligned} \tag{5'}$$

Now equation (3) can be written as (see (5.1.4))

$$\boxed{\delta' W = \sum m_i \boldsymbol{a}_i \cdot \delta \boldsymbol{r}_i} \tag{6}$$

or equation (6) is valid with respect to any virtual displacements of the system.

This is called the *principle of virtual work* (dynamics version of it) (virtuaalisen työn periaate) or also the *principle of virtual displacements* (virtuaalisten siirtymien periaate). We also shall call this as the *virtual work equation* (virtuaalisen työn yhtälö).

Remark 5.2. The derivation above was performed on purpose by manipulating the equations of motion instead by the virtual displacements $\delta \boldsymbol{r}_i$ first by arbitrary vectors $\boldsymbol{w}_i$. This was done in an effort to avoid some of the mystique often connected to the principle of the virtual work. In mathematics, the term "virtual" does not usually appear. So what is the mathematical content of the principle of virtual work?. The answer is simple. Equation (6) or (3) or (3') is a *linear combination (lineaarikombinaatio) of the equations of motion of the system.* One could think that a large amount of information disappears as from $3N$ equations only one scalar equation (6) is produced by the manipulation. However, the virtual work equation is only apparently just one equation, because the virtual displacements can be selected in it in many ways to produce several independent equations. In contrast, in the derivation of the energy principle in Remark 4.4 much information really vanishes as strictly only one scalar equation is finally produced as the infinitesimal displacements $d\boldsymbol{r}_i$ used in the derivation are those corresponding to the actual motion of the system and not to be selected at the free will of the applier. The explanation for taking the "weights " $\boldsymbol{w}_i$ as quantities having the dimension of displacement, which are infinitesimally small and which can be imagined as displacements of the particles of the system would demand involved justifications. Here we may just be satisfied to say that this interpretation has proved to be extremely useful in practice. Still one further fundamental difference between the concepts of real and virtual work is that in the latter we have no use for a finite, integrated work quantity. □

Remark 5.3. From the way of derivation of the principle of virtual work it is now obvious that time must be considered "freezed" when the principle is applied, because no time increment can be associated in the mathematical manipulation of forming the linear combination of the equations of motion. It is also understood that to say — as is sometimes seen — that the virtual displacements take place infinitely fast is not a particularly succesfull way to indicate that time is freezed as it may raise the question about the corresponding inertia forces connected to the displacement. Similarly, no changes of the magnitudes or directions of the forces depending on the virtual displacements should occur as due to the derivation the forces must be those acting at time t corresponding to which the linear combination is formed. As an example of this, let us consider Figure 5.2.

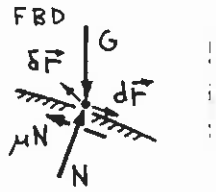


Figure 5.2 Particle moving on a rough surface.

A particle — a system — moves on a rough surface under gravity. Even if we take the virtual displacement so that the particle will lift from the surface or it will go inside the surface, the normal force N and the friction force μN keep their original values when the virtual work is evaluated. In evaluating the real work for such a displacement, completely different expressions would result. For instance, if the displacement would lift the particle from the surface, N and μN would immediately vanish and only the gravity would do work on the particle. □

Remark 5.4. Sometimes the principle of virtual work or the principle of virtual displacements is termed the *principle of virtual power* (virtuaalisen tehon periaate) or the *principle of virtual velocities* (virtuaalisten nopeuksien periaate). Then the interpretation (5) is replaced by the interpretation

$$w_i = v_i^{\text{virt}} \tag{7}$$

where v_i^{virt} is an arbitrary quantity having the dimension of velocity. By taking $v_i^{\text{virt}} = \delta r_i / \delta t$ where δt is an infinitesimal time, we in fact operate with the principle of virtual work with all the terms divided by δt. This form of the principle has the advantage that in using quantitities v_i^{virt} emphasizes that the principle concerns the system at a certain moment of time and we have no need to speak about virtual displacement "happening". □

Remark 5.5. In some cases we will write the virtual work equation (6) using the inertia force concept discussed in Section 4.6 to have a nice standard form "something on the left-hand side equals zero":

$$\boxed{\delta' W^* \equiv \delta' W_{\text{ext}} + \delta' W_{\text{int}} + \delta' W^{\text{I}} = 0,} \tag{8}$$

where

$$\delta' W^{\text{I}} = \sum F_i^{\text{I}} \cdot \delta r_i = -\sum m_i a_i \cdot \delta r_i \tag{9}$$

is the virtual work of the inertia forces. The star superscipt in formula (8) is used to remind that the inertia force method is employed. □

In *statics* the virtual work equation (6) simplifies to the form

$$\boxed{\delta' W = 0} \tag{10}$$

or

$$\delta' W_{\text{ext}} + \delta' W_{\text{int}} = 0. \tag{10'}$$

We realize that the principle of the virtual work in the dynamic case could have been derived with shorter notations by starting from the equilibrium equations

$$F_i + f_i = 0, \quad i = 1, 2, \cdots, N \tag{11}$$

to obtain first (10) and then the substitution

$$F_i := F_i + F_i^{\text{I}} \tag{12}$$

or

$$\delta' W_{\text{ext}} := \delta' W_{\text{ext}} + \delta' W^{\text{I}} \tag{13}$$

discussed in Remark 4.5 would have given (6).

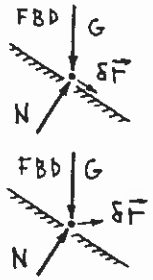
It may be mentioned that in some presentations the virtual work of internal forces is defined with an opposite sign to that used here. Then for instance the statics virtual work equation (10') would look

$$\delta' W_{\text{int}} = \delta' W_{\text{ext}}. \tag{14}$$

This may convenient in some applications but it is not theoretically well grounded.

5.3 CONSTRAINTS AND VIRTUAL WORK

If the motion of the particles of a system are constrained in some way, as is normally the case, the virtual displacements can be classified with respect to the constraints as follows:



Kinematically admissible virtual displacement or virtual displacement satisfying the constraints (kinemaattisesti luvallinen virtuaalinen siirtymä),

Kinematically inadmissible virtual displacement or virtual displacement violating the constraints (kinemaattisesti luvaton virtuaalinen siirtymä).

Figure 5.3 Classification of virtual displacements.

Kinematically admissible virtual displacements are infinitesimal displacements, which satisfy the constraints valid for real differential displacements. Further, time must be considered freezed for possible *given* motions, see also Remark 5.6. Kinematically inadmissible virtual displacements are infinitesimal displacements, which violate the constraints valid for real differential displacements. Figure 5.3 shows schematically one admissible and one inadmissible virtual displacement. The constraint is that the particle must move or rest on the inclined plane.

The previous classification is associated with the following important result:

> The virtual work done by constraint forces is equal to zero in a kinematically admissible virtual displacement of the system. (1)

This is taken as an axiom in some presentations, for instance Lanczos (1974). Here we can see this to be valid just from case to case basis. For instance in the case shown in Figure 5.3, where a frictionless contact is assumed, it is seen that the constraint force N is perpendicular to the kinematically admissible virtual displacement, which must be tangential to the inclined plane, and thus the virtual work of the constraint force disappears.

It is often advantageous to try to eliminate the constraint forces from the formulation of a problem. On the basis of statement (1) it is seen that this can be accomplished by selecting the virtual displacements as kinematically admissible. (There are however exceptions to this. The notable exception is Coulomb friction. For instance in the case shown in Figure 5.2 the friction force μN, which is a constitutive force would do a nonzero virtual work in a

kinematically admissible virtual displacement and would thus bring the constraint force N indirectly into the formulation.)

Remark 5.6. The elimination of the constraint forces from the resulting virtual work equation when using kinematically admissible virtual displacements can be explained also as follows. The constraint forces — as all forces — appear linearly in the equations of motion or rest; see for instance Example 5.2, equations (c). The elimination of the constraint forces would take place the way elimination is done in mathematics in general when dealing with a system of linear equations: suitable linear combinations of the equations are formed. But as discussed in Remark 5.2, the virtual work equation is in fact a linear combination of the equations of motion or equilibrium. In a kinematically admissible virtual displacement the principle of virtual work "wisely selects just the right combinations". □

Remark 5.7. Usually in the literature the term virtual displacement refers to a kinematically admissible virtual displacement. But if we want to find out constraint forces using the principle of virtual work, we have to employ kinematically inadmissible virtual displacements. □

5.4 GENERALIZED FORCES

5.4.1 General case

We now turn to the use of generalized coordinates in connection with the principle of virtual work. Only the holonomic case is considered here. Thus the positions of the particles of a system can be given in the form

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \cdots, q_n, t), \qquad i = 1,2,\cdots,N, \tag{1}$$

where the generalized coordinates can be varied arbitrarily without violating the constraints. Variation of (1) using the chain rule and remembering that time as the independent variable is freezed gives

$$\boxed{\delta \mathbf{r}_i = \sum_{j=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j}, \qquad i = 1,2,\cdots,N, \tag{2}$$

or using component forms

$$\delta x_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \delta q_j ,$$

$$\delta y_i = \sum_{j=1}^{n} \frac{\partial y_i}{\partial q_j} \delta q_j , \qquad (2')$$

$$\delta z_i = \sum_{j=1}^{n} \frac{\partial z_i}{\partial q_j} \delta q_j .$$

It is immediately realized that by using these expressions we are due to the starting point (1) in fact applying kinematically admissible virtual displacements.

The virtual work done on the system obtains the form

$$\begin{aligned} \delta' W^* &= \delta' W_{\text{ext}} + \delta' W_{\text{int}} + \delta' W^{\text{I}} \\ &= \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j + \sum_{i=1}^{N} \boldsymbol{f}_i \cdot \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j + \sum_{i=1}^{N} \boldsymbol{F}_i^{\text{I}} \cdot \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j \\ &= \sum_{j=1}^{n} \Big(\sum_{i=1}^{N} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \Big) \delta q_j + \sum_{j=1}^{n} \Big(\sum_{i=1}^{N} \boldsymbol{f}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \Big) \delta q_j + \sum_{j=1}^{n} \Big(\sum_{i=1}^{N} \boldsymbol{F}_i^{\text{I}} \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \Big) \delta q_j \end{aligned} \qquad (3)$$

or

$$\boxed{\delta' W^* = \sum_{j=1}^{n} Q_j^* \delta q_j ,} \qquad (4)$$

where

$$Q_j^* = Q_j^{\text{ext}} + Q_j^{\text{int}} + Q_j^{\text{I}} \qquad (5)$$

with

$$\begin{aligned} Q_j^{\text{ext}} &= \sum \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} , \\ Q_j^{\text{int}} &= \sum \boldsymbol{f}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} = -\sum_{kl} S \frac{\partial s}{\partial q_j} , \\ Q_j^{\text{I}} &= \sum \boldsymbol{F}_i^{\text{I}} \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} = -\sum m_i \boldsymbol{a}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} . \end{aligned} \qquad (6)$$

The quantities Q_j are called *generalized forces* (yleistetty voima). The important formula (4) thus represents the final expression for the virtual work

when generalized coordinates are employed. The formula can also be considered as the definition for the generalized forces. A generalized force needs not have the dimension of a force, its dimension is determined by the fact that each product $Q_j \delta q_j$ must have the dimension of work. Thus, for example, if the generalized coordinate is an angle or a volume, the corresponding generalized force has the dimension of a moment or of a pressure.

The last form of the second formula (6) follows from the last form of the last formula (5.1.6). We can write $s = s(q_1, q_2, \cdots, q_n, t)$ so that

$$\delta s = \sum_{j=1}^{n} \frac{\partial s}{\partial q_j} \delta q_j . \qquad (7)$$

Continuing similarly as above gives the last form. The summation over the particle pairs is indicated by the indices kl as j is now reserved for numbering of the generalized coordinates.

If the inertia force concept is not directly used, we write instead of (4)

$$\boxed{\delta' W = \sum_{j=1}^{n} Q_j \delta q_j ,} \qquad (8)$$

where

$$Q_j = Q_j^{\text{ext}} + Q_j^{\text{int}} . \qquad (9)$$

Remark 5.8. The generalized forces Q^{ext} can often be determined conveniently in an alternative way by drawing the virtual displacement pattern consecutively due to each variation δq_j, by determining from the figure the corresponding virtual work $\delta' W^{\text{ext}}$, whereafter the generalized force is obtained due to definition (4) from

$$Q_j^{\text{ext}} = \frac{\delta' W^{\text{ext}}}{\delta q_j} . \qquad (10) \ \square$$

The next step is based on the principle of virtual work and on the fact that the virtual displacements can be selected arbitrarily. When expression (4) is substituted into the virtual work equation $\delta' W^* = 0$, there follows

$$\sum_{j=1}^{n} Q_j^* \delta q_j = 0 . \qquad (11)$$

The generalized coordinates are independent and thus their variations can be selected in an arbitrary way. We thus obtain from the one scalar equation (11) in general several equations (Take first say $\delta q_1 \neq 0,\ \delta q_2 = 0, \delta q_3 = 0, \cdots, \delta q_n = 0$ to give $Q_1^* \delta q_1 = 0$ and thus finally $Q_1^* = 0$, etc.)

$$\boxed{Q_j^* \equiv Q_j^{\text{ext}} + Q_j^{\text{int}} + Q_j^{\text{I}} = 0,} \qquad j = 1,2,\cdots,n. \tag{12}$$

These are the *equations of motion* of the system expressed in the generalized coordinates. Constraint forces do not (usually) appear in them.

In *statics* the inertia forces disappear and the *equations of equilibrium* are arrived at:

$$\boxed{Q_j \equiv Q_j^{\text{ext}} + Q_j^{\text{int}} = 0,} \qquad j = 1,2,\cdots,n. \tag{13}$$

Remark 5.9. Virtual displacements have been considered above as the variations $\delta \boldsymbol{r}_i(t)$ of the position vectors $\boldsymbol{r}_i(t)$ of the particles of the system. The same result is obtained by starting from the variation of the displacement vector $\boldsymbol{u}_i(t)$ as because of the representation $\boldsymbol{r}_i(t) = {}^0\boldsymbol{r}_i(t) + \boldsymbol{u}_i(t)$,

$$\delta \boldsymbol{r}_i(t) = \delta \boldsymbol{u}_i(t). \tag{14}$$

or

$$\begin{aligned} \delta x_i(t) &= \delta u_i(t), \\ \delta y_i(t) &= \delta v_i(t), \\ \delta z_i(t) &= \delta w_i(t). \end{aligned} \tag{14}$$

The equivalents of formulas (2) and (2') are

$$\delta \boldsymbol{u}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{u}_i}{\partial q_j} \delta q_j, \qquad i = 1,2,\cdots,N, \tag{15}$$

and

$$\begin{aligned} \delta u_i &= \sum_{j=1}^{n} \frac{\partial u_i}{\partial q_j} \delta q_j, \\ \delta v_i &= \sum_{j=1}^{n} \frac{\partial v_i}{\partial q_j} \delta q_j, \\ \delta w_i &= \sum_{j=1}^{n} \frac{\partial w_i}{\partial q_j} \delta q_j \end{aligned} \tag{15'}$$

and expressions (6) change accordingly with respect to the notations. If one operates instead of position vectors with displacements, the generalized coordinates can be then called *generalized displacements* (yleistetty siirtymä); see also Remark 4.1. □

Remark 5.10. A precise mathematical definition of a kinematically admissible virtual displacement is that it is the *difference* between two possible differential displacements satisfying the constraints of the system. In the case of representation (1) we have

$$\mathrm{d}\boldsymbol{r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \mathrm{d}q_j + \frac{\partial \boldsymbol{r}_i}{\partial t} \mathrm{d}t, \tag{16}$$

and using the following notations for some other possible differential displacement:

$$\bar{\mathrm{d}}\boldsymbol{r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \bar{\mathrm{d}}q_j + \frac{\partial \boldsymbol{r}_i}{\partial t} \mathrm{d}t, \tag{17}$$

we have the difference

$$\delta \boldsymbol{r}_i \equiv \bar{\mathrm{d}}\boldsymbol{r}_i - \mathrm{d}\boldsymbol{r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j \tag{18}$$

with $\delta q_j \equiv \bar{\mathrm{d}}q_j - \mathrm{d}q_j$. It is seen that this definition automatically removes the contribution of the time derivative term in a rheonomic case. □

5.4.2 Rigid body

A kinematically admissible virtual displacement of a rigid body is of the form

$$\delta \boldsymbol{r}_i = \delta \boldsymbol{r}_{\mathrm{A}} + \delta' \boldsymbol{\theta} \times \boldsymbol{r}_{i/\mathrm{A}}. \tag{19}$$

This is analogous to formula (3.2.3). Quantity $\delta' \boldsymbol{\theta}$ is a virtual angular displacement of the rigid body. A kinematically admissible virtual displacement of a rigid body is such that the distance between any two particles remain constant; no gaps are allowed to appear. As then $\delta s = 0$, the *virtual work of internal forces of a rigid body is zero in a kinematically admissible virtual displacement*. This result is in accordance of statement (5.3.1) as in a rigid body the internal forces are constraint forces.

Example 5.2. We consider the same simple case as in the kinematical study of Example 4.1: a particle moving in plane motion in the xy-plane on an inclined plane of a wedge in given motion. The system is the particle and the plane is assumed frictionless and the particle is also under the influence of gravity.

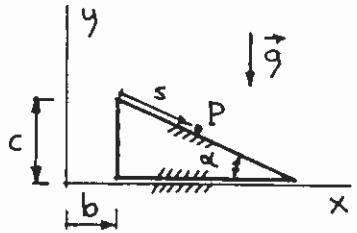


Figure (a)

The position of the wedge is given by the measure b in the form

$$b = \hat{b}\sin\omega t, \tag{a}$$

where $\hat{b}$ and ω are constants. We have a one degree of freedom holonomic rheonomic system. The measure s is taken as the generalized coordinate.

The equivalent of formula (1) is

$$r = (s\cos\alpha + \hat{b}\sin\omega t)i + (c - s\sin\alpha)j, \tag{b}$$

or

$$\begin{aligned} x &= s\cos\alpha + \hat{b}\sin\omega t, \\ y &= c - s\sin\alpha. \end{aligned} \tag{b'}$$

We first write the equations of motion using the free body diagram of Figure (b). There is obtained

$$\begin{aligned} &\rightarrow N\sin\alpha = m\ddot{x}, \\ &\uparrow N\cos\alpha - mg = m\ddot{y}. \end{aligned} \tag{c}$$

Figure (b)

The kinematical relations (b') give by differentiations

$$\begin{aligned} \ddot{x} &= \ddot{s}\cos\alpha - \hat{b}\omega^2\sin\omega t, \\ \ddot{y} &= -\ddot{s}\sin\alpha. \end{aligned} \tag{d}$$

The four equations (c) and (d), which describe the problem with appropriate initial conditions, contain four unknown functions: $x(t)$, $y(t)$, $s(t)$, $N(t)$. An alternative formulation is arrived at by eliminating the constraint force N. A linear combination is formed by multiplying the first of (c) by $\cos\alpha$ and the second by $-\sin\alpha$. This produces an equation

$$mg\sin\alpha = m\ddot{x}\cos\alpha - m\ddot{y}\sin\alpha, \tag{e}$$

which does not any more contain the constraint force. Further substitution of expressions (d) into (e) and some arrangement gives finally a differential equation

$$\ddot{s} = g\sin\alpha + \hat{b}\omega^2\cos\alpha\sin\omega t. \tag{f}$$

Its solution is found to be

$$s = A + Bt + \frac{1}{2}g\sin\alpha\cdot t^2 - \hat{b}\cos\alpha\sin\omega t. \tag{g}$$

The integration constants A and B can be determined from the initial conditions, which are not considered in detail in this study. The other unknowns x, y, and N are then found easily from equations (b'), (d), and (c) by substitutions.

It may be mentioned, that according to the terminology of structural mechanics, the set presented by equations (c) and (d) is called a mixed formulation (as both forces and displacements apppear as unknowns) and the problem presented by equation (f) is called a displacement formulation (only displacements appear as unknowns).

We now consider the problem using the principle of virtual work. The equivalent of formula (2) is

$$\delta r = \frac{\partial r}{\partial s}\delta s = (\cos\alpha\, i - \sin\alpha\, j)\delta s, \tag{h}$$

whereas the real differential displacement is (formula (16))

$$\mathrm{d}r = \frac{\partial r}{\partial s}\mathrm{d}s + \frac{\partial r}{\partial t}\mathrm{d}t = (\cos\alpha\, i - \sin\alpha\, j)\mathrm{d}s + \hat{b}\omega\cos\omega t\, i\,\mathrm{d}t. \tag{i}$$

The difference of the expressions is seen in Figure (c). In the real motion the wedge moves during the the time increment $\mathrm{d}t$ and the displacement $\mathrm{d}r$ given by (i) is not in general parallel to the inclined plane. The constraint force N does non-zero work in the real motion and it is thus also non-conservative (compare with Figure 4.6). The displacement δr given by (h) is, however, seen to be parallel to the inclined plane and the constraint force N does no virtual work.

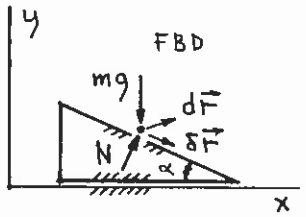


Figure (c)

From (d), the acceleration of the particle is

$$a = (\underset{s}{\ddot{s}}\cos\alpha - \hat{b}\omega^2\sin\omega t)i - \underset{s}{\ddot{s}}\sin\alpha\, j. \tag{j}$$

We now generate the equation of motion according to formula (12). The generalized forces are (formula (6); a one particle system has no internal forces)

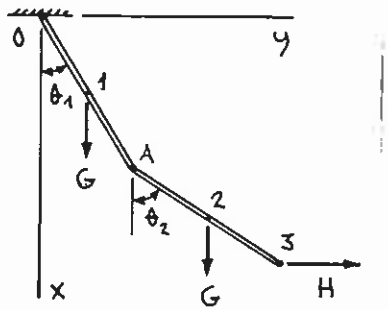
$$
\begin{aligned}
Q^{\text{ext}} &= \boldsymbol{F}\cdot\frac{\partial \boldsymbol{r}}{\partial s} = [N\sin\alpha\,\boldsymbol{i} + (N\cos\alpha - mg)\boldsymbol{j}]\cdot(\cos\alpha\,\boldsymbol{i} - \sin\alpha\,\boldsymbol{j}) \\
&= N\sin\alpha\cos\alpha - N\cos\alpha\sin\alpha + mg\sin\alpha = mg\sin\alpha, \\
Q^{\text{int}} &= 0, \\
Q^{\text{I}} &= \boldsymbol{F}^{\text{I}}\cdot\frac{\partial \boldsymbol{r}}{\partial s} = -m\boldsymbol{a}\cdot\frac{\partial \boldsymbol{r}}{\partial s} \\
&= -m[(\ddot{s}\cos\alpha - \hat{b}\omega^2\sin\omega t)\boldsymbol{i} - \ddot{s}\sin\alpha\,\boldsymbol{j}]\cdot(\cos\alpha\,\boldsymbol{i} - \sin\alpha\,\boldsymbol{j}) \\
&= -m(\ddot{s}\cos^2\alpha - \hat{b}\omega^2\cos\alpha\sin\omega t + \ddot{s}\sin^2\alpha) \\
&= -m(\ddot{s} - \hat{b}\omega^2\cos\alpha\sin\omega t).
\end{aligned} \tag{k}
$$

When calculating the generalized force Q^{ext}, we could have written directly $Q^{\text{ext}} = -mg\boldsymbol{j}\cdot\partial\boldsymbol{r}/\partial s$ as we know from the theory that the contribution from the constraint force N will vanish. The expressions above, however, show clearly how the application of the principle of the virtual work automatically eliminates the constraint force N from the formulation (compare with the manipulation to arrive at formula (f)). The equation of motion is thus

$$
Q^* \equiv mg\sin\alpha - m(\ddot{s} - \hat{b}\omega^2\cos\alpha\sin\omega t) = 0, \tag{l}
$$

which is after some arrangement equal to (f).

Example 5.3. We consider the system of Figure (a), whose kinematics was studied already in example 4.2.



The pins at O and A are frictionless. Forces G act at the midpoints of the bars, whose lengths are l. The directions of the forces remain constant irrespective of the position of the system. The equilibrium position of the system is determined by the principle of virtual work.

The only forces doing non-zero virtual work in a kinematically admissible virtual displacement are the constitutive external forces

$$
\begin{aligned}
\boldsymbol{F}_1 &= G\boldsymbol{i}, \\
\boldsymbol{F}_2 &= G\boldsymbol{i}, \\
\boldsymbol{F}_3 &= H\boldsymbol{j}.
\end{aligned} \tag{a}
$$

Figure (a)

The displacement of points 1, 2, 3, where the forces act, are according to formulas (c) and (d) of Example 4.2

$$
\begin{aligned}
\boldsymbol{u}_1 &= (\frac{l}{2}\cos\theta_1 - \frac{l}{2})\boldsymbol{i} + \frac{l}{2}\sin\theta_1\boldsymbol{j}, \\
\boldsymbol{u}_2 &= (l\cos\theta_1 + \frac{l}{2}\cos\theta_2 - \frac{3l}{2})\boldsymbol{i} + (l\sin\theta_1 + \frac{l}{2}\sin\theta_2)\boldsymbol{j}, \\
\boldsymbol{u}_3 &= (l\cos\theta_1 + l\cos\theta_2 - 2l)\boldsymbol{i} + (l\sin\theta_1 + l\sin\theta_2)\boldsymbol{j}.
\end{aligned} \tag{b}
$$

The generalized forces are according to formulas (6) (see Remark 5.9)

$$
\begin{aligned}
Q_1 = Q_1^{\text{ext}} &= \sum_{i=1}^{3} \boldsymbol{F}_i\cdot\frac{\partial \boldsymbol{u}_i}{\partial\theta_1} \\
&= G\boldsymbol{i}\cdot(-\frac{l}{2}\sin\theta_1\,\boldsymbol{i} + \frac{l}{2}\cos\theta_1\boldsymbol{j}) + \\
&+ G\boldsymbol{i}\cdot(-l\sin\theta_1\,\boldsymbol{i} + l\cos\theta_1\boldsymbol{j}) + \\
&+ H\boldsymbol{j}\cdot(-l\sin\theta_1\,\boldsymbol{i} + l\cos\theta_1\boldsymbol{j}) \\
&= -G\frac{l}{2}\sin\theta_1 - Gl\sin\theta_1 + Hl\cos\theta_1 = -\frac{3Gl}{2}\sin\theta_1 + Hl\cos\theta_1, \\
Q_2 = Q_2^{\text{ext}} &= \sum_{i=1}^{3} \boldsymbol{F}_i\cdot\frac{\partial \boldsymbol{u}_i}{\partial\theta_2} \\
&= G\boldsymbol{i}\cdot\boldsymbol{0} + \\
&+ G\boldsymbol{i}\cdot(-\frac{l}{2}\sin\theta_2\,\boldsymbol{i} + \frac{l}{2}\cos\theta_2\boldsymbol{j}) + \\
&+ H\boldsymbol{j}\cdot(-l\sin\theta_2\,\boldsymbol{i} + l\cos\theta_2\boldsymbol{j}) \\
&= -G\frac{l}{2}\sin\theta_2 + Hl\cos\theta_2 = -\frac{Gl}{2}\sin\theta_2 + Hl\cos\theta_2.
\end{aligned} \tag{c}
$$

The equations of equilibrium (13) are

$$
\begin{aligned}
Q_1 &\equiv -\frac{3Gl}{2}\sin\theta_1 + Hl\cos\theta_1 = 0, \\
Q_2 &\equiv -\frac{Gl}{2}\sin\theta_2 + Hl\cos\theta_2 = 0,
\end{aligned} \tag{d}
$$

and their solution is

$$
\theta_1 = \arctan\frac{2H}{3G}, \qquad \theta_2 = \arctan\frac{2H}{G}. \tag{e}
$$

The alternative approach based on free body diagrams of the two bars would consist of six equations of equilibrium having as unknowns four constraint force components corresponding to pins O and A and the two angles θ_1 and θ_2.

5.5 VIRTUAL WORK DONE BY CONSERVATIVE AND MONOGENIC FORCES

If an external force $\boldsymbol{F}_i$ is conservative (see Section 4.1.3)), the virtual work done by it is

$$
\begin{aligned}
\delta' W_i^{\text{ext}} &= \boldsymbol{F}_i\cdot\delta\boldsymbol{r}_i = -\nabla V_i^{\text{ext}}\cdot\delta\boldsymbol{r}_i = \\
&= -(\frac{\partial V_i^{\text{ext}}}{\partial x_i}\boldsymbol{i} + \frac{\partial V_i^{\text{ext}}}{\partial y_i}\boldsymbol{j} + \frac{\partial V_i^{\text{ext}}}{\partial z_i}\boldsymbol{k})\cdot(\delta x_i\,\boldsymbol{i} + \delta y_i\,\boldsymbol{j} + \delta z_i\,\boldsymbol{k}) \\
&= -(\frac{\partial V_i^{\text{ext}}}{\partial x_i}\delta x_i + \frac{\partial V_i^{\text{ext}}}{\partial y_i}\delta y_i + \frac{\partial V_i^{\text{ext}}}{\partial z_i}\delta z_i)
\end{aligned} \tag{1}
$$

or

$$\boxed{\delta' W_i^{\text{ext}} = -\delta V_i^{\text{ext}}} \tag{2}$$

or the virtual work done by a conservative force is equal to minus the variation of the potential energy of the force due to the change δr_i of the argument r_i and not just an infinitesimal expression. (3)

The derivation is completely analogous to that used in obtaining the result (4.5.28).

For conservative pairwise forces f_{ij} and f_{ji}, we obtain

$$\delta' W_{ij}^{\text{int}} = -S\delta s = -\frac{\partial V_{ij}^{\text{int}}}{\partial s}\delta s \tag{4}$$

or

$$\boxed{\delta' W_{ij}^{\text{int}} = -\delta V_{ij}^{\text{int}}.} \tag{5}$$

The verbal interpretation of formula (5) is similar to that for formula (2).

For a conservative system we obtain similarly

$$\boxed{\delta' W = -\delta V,} \tag{6}$$

where δV is the variation of the total potential energy $V(r_1, r_2, \cdots, r_N)$ due to the variations of $r_1, r_2, \cdots, r_N$ and not just an infinitesimal expression.

If generalized coordinates are employed, V becomes through the relations (5.4.1) a function of the generalized coordinates (see Remark 5.11):

$$V = V(q_1, q_2, \cdots, q_n). \tag{7}$$

Taking the variation, formula (6) can now be expressed in the form

$$\delta' W = -\sum_{j=1}^{n} \frac{\partial V}{\partial q_j}\delta q_j . \tag{8}$$

Introducing formula (5.4.8) defining generalized coordinates and remembering that the variations δq_j are arbitrary, gives the expressions

$$\boxed{Q_j = -\frac{\partial V}{\partial q_j},} \qquad j = 1, 2, \cdots, n, \tag{9}$$

which can be used conveniently to evaluate the generalized forces corresponding to conservative forces.

In all the formulas above we can finally replace the symbol V for potential energy of conservative forces with the symbol V' for the time dependent potential energy of monogenic forces to obtain a more general theory. As mentioned in Section 4.1.3, this is due to the fact that time is considered freezed in the principle of virtual work. As an example, let us consider the time dependent potential energy

$$V' = V'(q_1, q_2, \cdots, q_n, t) \tag{10}$$

of a system. Its differential change in a real motion is

$$\mathrm{d}V' = -\sum_{j=1}^{n} \frac{\partial V'}{\partial q_j} dq_j + \frac{\partial V'}{\partial t}\mathrm{d}t \tag{11}$$

but its variation is simply

$$\delta V' = -\sum_{j=1}^{n} \frac{\partial V'}{\partial q_j}\delta q_j . \tag{12}$$

It is thus realized that the derivations presented above proceed quite similarly even if time is explicitly present in the expressions for potential energy.

Remark 5.11. Time can emerge into the expression of potential energy even for a conservative force when the constraints are rheonomic. For example, in Example 4.1 the potential energy of the gravity force acting on the particle is $V = mgy$. If x is taken as the generalized coordinate, the final expression for the potential energy is $V = mg[c - (x - \hat{b}\sin\omega t)\tan\alpha]$. This looks like a time dependent potential energy of a monogenic force can can be dealt with without problems in an analogous way. □

Remark 5.12. It should be finally emphasized that Lagrange's equations, Hamilton's principle and the principle of stationary potential energy to be introduced in the following are all just special forms of the general principle of virtual work. □

5.6 LAGRANGE'S EQUATIONS

In the dynamic case the equations of motion (5.4.12) derived by the principle of virtual work and written here in the form $Q_j^{\text{ext}} + Q_j^{\text{int}} = -Q_j^{\text{I}}$ or

$$Q_j = \sum m_i \ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}, \qquad j = 1,2,\cdots,n \tag{1}$$

demand the calculation of the accelerations $\ddot{\boldsymbol{r}}_i$ from formulas (5.4.1): $\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, \cdots, q_n, t)$. This may be a rather heavy task, if the original expressions are complicated. The right hand sides of (1) can be manipulated into another form which does not any more contain accelerations. The manipulation is as follows:

$$\begin{aligned}
\sum m_i \ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} &= \sum [\frac{\mathrm{d}}{\mathrm{d}t}(m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}) - m_i \dot{\boldsymbol{r}}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \boldsymbol{r}_i}{\partial q_j}] \\
&= \sum [\frac{\mathrm{d}}{\mathrm{d}t}(m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_j}) - m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j}] \\
&= \sum [\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_j}(\frac{1}{2} m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i) - \frac{\partial}{\partial q_j}(\frac{1}{2} m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i)] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_j}(\sum \frac{1}{2} m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i) - \frac{\partial}{\partial q_j}(\sum \frac{1}{2} m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j}.
\end{aligned} \tag{2}$$

Here between the first and second lines the following formulas

$$\frac{\partial \boldsymbol{r}_i}{\partial q_j} = \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_j}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \boldsymbol{r}_i}{\partial q_j} = \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j}, \tag{3}$$

which can be shown to be valid, have been used.

Introducing (2) into (1) gives the *Lagrange's equations of motion* or shortly *Lagrange's equations* (Lagrangen yhtälöt)

$$\boxed{Q_j = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j},} \qquad j = 1,2,\cdots,n. \tag{4}$$

For the kinetic energy $K = \sum 1/2 \cdot m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i$ of the system, only the expressions for the velocities

$$\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t}, \qquad i = 1,2,\cdots,N \tag{5}$$

of the particles are needed. On the other hand, the additional operations indicated in (4) must be performed. The kinetic energy expression is of the form

$$K = K(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t). \tag{6}$$

The quantities $\dot{q}_j$ are called *generalized velocities* (yleistetty nopeus). When forming the partial derivatives in (4), the arguments shown in (6) are to be considered independent. Only when applying the operator $\mathrm{d}/\mathrm{d}t$, it should be realized that both the q_j and the $\dot{q}_j$ are in fact functions of t. The Lagrange's equations are thus finally a set of ordinary differential equations of second order with time as the independent variable. As a special case we have the conventional equations of motion $(F_i)_x + (f_i)_x = m_i \ddot{x}_i$, etc.

If the system contains conservative or monogenic forces, the corresponding generalized forces can be evaluated also via the potential energy V or V' using formulas (5.5.9) or its counterpart with V replaced by V'. In that case Lagrange's equations (4) obtain the specific form

$$\boxed{Q_j^{\mathrm{n}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j},} \qquad j = 1,2,\cdots,n, \tag{7}$$

where

$$L = K - V \quad (= K - V') \tag{8}$$

is called the *Lagrangian function* (Lagrangen funktio) and Q_j^{n} is the generalized force due to nonconservative (polygenic) forces. If all the forces acting on a system are conservative (monogenic), the left-hand sides in (7) disappear.

Example 5.4. We consider the simple problem of a particle moving on a wedge in given motion considered in Example 5.2 (Figure (a)). Now the Lagrange's equation is used to derive the equation of motion.

From Example 5.2,

$$\begin{aligned} x &= s\cos\alpha + \hat{b}\sin\omega t, \\ y &= c - s\sin\alpha. \end{aligned} \tag{a}$$

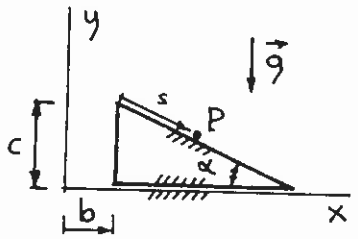


Figure (a)

Application of differentiation similarly as in (5) gives

$$\dot{x} = \dot{s}\cos\alpha + \hat{b}\omega\cos\omega t, \qquad \dot{y} = -\dot{s}\sin\alpha \tag{b}$$

and the kinetic energy of the particle

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{s}^2\cos^2\alpha + 2\dot{s}\cos\alpha\,\hat{b}\omega\cos\omega t + \hat{b}^2\omega^2\cos^2\omega t + \dot{s}^2\sin^2\alpha)$$
$$= \frac{1}{2}m(\dot{s}^2 + 2\dot{s}\cos\alpha\,\hat{b}\omega\cos\omega t + \hat{b}^2\omega^2\cos^2\omega t). \tag{c}$$

Here the kinetic energy happens not to depend on the generalized coordinate. The potential energy of the gravity force is

$$V = mgy = mg(c - s\sin\alpha), \tag{d}$$

when the plane $y = 0$ is taken as the datum. Gravity force is conservative and the constraint force N does no virtual work so we can use form (7) with Q^n zero:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0 \tag{e}$$

with

$$L = \frac{1}{2}m(\dot{s}^2 + 2\dot{s}\cos\alpha\,\hat{b}\omega\cos\omega t + \hat{b}^2\omega^2\cos^2\omega t) - mg(c - s\sin\alpha). \tag{f}$$

We have

$$\frac{\partial L}{\partial \dot{s}} = m\dot{s} + m\cos\alpha\,\hat{b}\omega\cos\omega t,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{s}} = m\ddot{s} - m\cos\alpha\,\hat{b}\omega^2\sin\omega t, \tag{g}$$
$$\frac{\partial L}{\partial s} = mg\sin\alpha,$$

and Lagrange's equation (e) becomes

$$m\ddot{s} - m\cos\alpha\,\hat{b}\omega^2\sin\omega t - mg\sin\alpha = 0 \tag{h}$$

or again

$$\ddot{s} = g\sin\alpha + \hat{b}\omega^2\cos\alpha\sin\omega t. \tag{i}$$

5.7 HAMILTON'S PRINCIPLE

The starting point in the derivation of the principle of virtual work was equation (5.2.3):

$$\sum(\boldsymbol{F}_i + \boldsymbol{f}_i)\cdot\boldsymbol{w}_i - \sum m_i\boldsymbol{a}_i\cdot\boldsymbol{w}_i = 0, \tag{1}$$

where the interpretation $\boldsymbol{w}_i = \delta\boldsymbol{r}_i$ was taken for the weights $\boldsymbol{w}_i$. The derivation was applied at a fixed moment of time. Nothing prevents us to take the weights as functions of time: $\boldsymbol{w}_i = \boldsymbol{w}_i(t)$. When equation (1) is integrated over the time interval $[t_1, t_2]$, we arrive at an equation (we denote for shortness $\boldsymbol{R}_i = \boldsymbol{F}_i + \boldsymbol{f}_i$)

$$\int_{t_1}^{t_2}[\sum\boldsymbol{R}_i\cdot\boldsymbol{w}_i - \sum m_i\boldsymbol{a}_i\cdot\boldsymbol{w}_i]\mathrm{d}t = 0, \tag{2}$$

where the quantities $\boldsymbol{w}_i(t)$ are completely arbitrary. When again the interpretation $\boldsymbol{w}_i(t) = \delta\boldsymbol{r}_i(t)$ is taken, we obtain in more detail

$$\int_{t_1}^{t_2}[\sum\boldsymbol{R}_i\cdot\delta\boldsymbol{r}_i - \sum m_i\ddot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i]\mathrm{d}t = 0. \tag{3}$$

In addition to the real path $\boldsymbol{r}_i(t)$ of a particle i, we thus imagine that the particle could have taken a varied path $\boldsymbol{r}_i(t) + \delta\boldsymbol{r}_i(t)$, which is considered in Hamilton's principle to such that *it coincides with the real path at $t = t_1$ and $t = t_2$* (Figure 5.4), that is, we take

$$\delta\boldsymbol{r}_i(t_1) = 0, \qquad \delta\boldsymbol{r}_i(t_2) = 0. \tag{4}$$

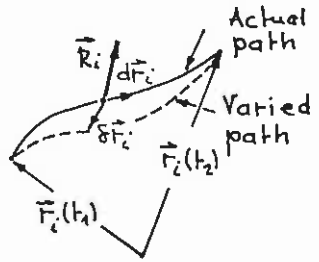


Figure 5.4. Actual and varied path of a particle.

The integrals $\int_{t_1}^{t_2} m_i\ddot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i\,\mathrm{d}t$ are transformed by integration by parts:

$$-\int_{t_1}^{t_2} m_i\ddot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i\,\mathrm{d}t = \int_{t_1}^{t_2} m_i\dot{\boldsymbol{r}}_i\cdot\delta\dot{\boldsymbol{r}}_i\,\mathrm{d}t - \Big|_{t_1}^{t_2} m_i\dot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i. \tag{5}$$

Essentially formula (B.1.1a) has been used with $x \to t$. It is not difficult to see that the formula is valid in an analogous form even when the two functions g and h are vectors and when the ordinary product is replaced by the scalar product. This can be proved by first writing $\ddot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i = \ddot{x}_i\delta x_i + \cdots$ and then applying the conventional integration by parts formula to the ordinary products. It should be further noticed that m_i is constant in time and from the rules of variational calculus $\mathrm{d}(\delta\boldsymbol{r}_i)/\mathrm{d}t = \delta\dot{\boldsymbol{r}}_i$. Further, using again rules of variational calculus:

$$m_i \dot{r}_i \cdot \delta\dot{r}_i = \delta(\frac{1}{2} m_i \dot{r}_i \cdot \dot{r}_i) = \delta K_i . \tag{6}$$

When we take into account that the last term in (5) disappears because of (4), equation (3) obtains thus the form

$$\int_{t_1}^{t_2} (\delta K + \sum \boldsymbol{R}_i \cdot \delta \boldsymbol{r}_i)\, \mathrm{d}t = 0. \tag{7}$$

If the forces acting on the system are conservative (monogenic)

$$\delta' W \equiv \sum \boldsymbol{R}_i \cdot \delta \boldsymbol{r}_i = -\delta V \quad (= -\delta V) \tag{8}$$

and equation (7) becomes *Hamilton's principle* (Hamiltonin periaate)

$$\boxed{\delta \int_{t_1}^{t_2} L\, \mathrm{d}t = 0,} \tag{9}$$

or when the forces acting on the system are conservative (monogenic), the systems moves so that the functional $\int L\,\mathrm{d}t$, where $L = K - V \quad (= K - V')$, obtains a stationary value. The admissible functions must satisfy the kinematical constraints present, and they have to coincide with the values corresponding to the actual paths at the moments of time $t = t_1$ and $t = t_2$.

When we employ generalized coordinates, the stationary condition gives Lagrange's equations as the Euler equations.

If all the forces acting on the system are not conservative (monogenic), Hamilton's principle cannot any more be given as a pure variational principle but must be presented in the form

$$\delta \int_{t_1}^{t_2} L\, \mathrm{d}t + \int_{t_1}^{t_2} \delta' W^{\mathrm{n}}\, \mathrm{d}t = 0, \tag{10}$$

where $\delta' W^{\mathrm{n}}$ is the virtual work done by the nonconservative (polygenic) forces.

Because problems of dynamics are initial value problems with respect to time, the condition at the future value $t = t_2$ of time is physically unnatural. However, this condition does not prevent the application of the principle, see Example D.19. In numerical solution in dynamics, it seems that the Hamilton's principle has little to offer compared with the direct application of the principle of virtual work.

5.8 VIRTUAL WORK IN STATICS

5.8.1 General

The principle of virtual work is presented in statics in many varied forms depending on the reference.

The virtual work equation (5.2.10) or

$$\boxed{\delta' W_{\mathrm{ext}} + \delta' W_{\mathrm{int}} = 0} \tag{1}$$

can be stated first as a special case of principle (5.2.6) valid in the dynamic case: If a system (a body) is in equilibrium, the virtual work done by the forces acting on it is zero with respect to any virtual displacement of the system. But second, reversing the argument, we can start by assuming that the virtual work equation is valid with respect to any virtual displacement of the system to derive the equilibrium equations similarly as was explained in the dynamic case on page 5-3 to arrive at the equations of motion.

We can combine the statements above in one theorem:

A system is in equilibrium, if and only if the virtual work done by the forces acting on it is zero with respect to any virtual displacement of the system. (2)

A system is said here to be in *equilibrium* (tasapaino) or in a *state of equilibrium* (tasapainotila), if the resultants of the forces acting on every particle of the system are all zero. It should be noticed that this definition does not necessarily imply state of rest of a system. For instance, according to this definition, a particle moving with constant velocity on a horizontal plane is a system in equilibrium.

The most usual applications deal with kinematically admissible virtual displacements. In this case the virtual work of constraint forces is zero, and there remains only the virtual work of the constitutive forces. We have thus the following theorem:

When a system is in equilibrium, the virtual work done by the constitutive forces acting on the system is zero with respect to any kinematically admissible virtual displacement of the system. (3)

This is the most usual form of the principle of virtual work in statics (although the term "constitutive force" is not in general use). It should be noticed that the "if and only if"-form of the principle (2) is no more here, as we cannot test for

any concievable virtual displacement because of the satisfaction of the the constraints. This fact does not limit the applicability of the principle.

In statics the principle of virtual work is applied in some cases so that the internal forces (in a continuum the stress field) are selected on purpose to differ from the real internal forces. For a statically indeterminate structure several (= number of statical indeterminacy, for a continuum an infinite number) independent distributions of internal forces can be taken which satisfy the equations of equilibrium with the given external forces. This kind of internal force system is called *statically admissible* (staattisesti luvallinen) and the whole force system can be called an *equilibrium system* (tasapainosysteemi). Because of the constitutive relations present in each specific case, only a certain statically admissible internal force system corresponds to the real one. Remembering how the virtual work equation was derived, we see that the *principle of virtual work is valid in statics for an arbitrary equilibrium force system.* This is the starting point for the *principle of virtual forces* (virtuaalisten voimien periaate) and the *principle of stationary complementary potential energy* (komplementaarisen energian stationaarisuuden periaate), which are, however, not considered in this text, see e.g. Washizu (1982).

The so-called *unit dummy load method* (yksikkövoimamenettely) is one application of the principle of virtual work in the form discussed in the preceding paragraph. The idea is to determine the displacements of the points of a structure — provided the small displacement linear kinematical theory is valid — when the deformations (for a continuum the strains) corresponding to the displacement distribution are known. The virtual displacements and deformations (strains) are taken to be the real displacements and strains. To determine a certain displacement component of a certain point, a unit external force is applied in the direction in question at that point. Any statically admissible internal force system giving an equilibrium system with the point load is determined. This can be selected for instance by making the system in some way statically determinate by "cuts" and by finding the corresponding internal forces. We can now evaluate the term $\delta' W_{\text{int}}$ in (1) and the term $\delta' W_{\text{ext}}$ becomes simply one times the unknown displacement, so the equation gives immediately the value of the displacement.

Remark 5.13. Some comments are perhaps needed to make the unit dummy load method more clear. First, it may seem contradictory that finite, even if small, real displacements can be determined using the principle of virtual work as it was emphasized earlier that the virtual displacements are infinitely small. This is easiest to explain using notation, which is explained only later. In Section 12.1.2 we find that for a continuum in the small displacement theory the virtual work of the internal forces is given in the form

$$\delta' W_{\text{int}} = -\int_V \sigma_{ij} \delta \varepsilon_{ij} \mathrm{d}V, \tag{4}$$

where

$$\delta \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_j}{\partial a_i}\right) \tag{5}$$

is the virtual strain and δu_i is the virtual displacement and the rest of the notation is explained in Section 12.1.2. When the derivation of the principle of virtual work is studied in detail, it is realized that it could have been performed equally well using the notation

$$\delta' W_{\text{int}} = -\int_V \sigma_{ij} \varepsilon_{ij} \mathrm{d}V, \tag{6}$$

where

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right) \tag{7}$$

and it is nowhere necessary to consider the weighting function w to be infinitesimal. In the large displacement case, the situation is more involved.

Second, there is strictly speaking no such thing as a unit force. Actually a force having the general symbol F can be used and finally we divide by F to obtain the displacement. This all is avoided by shortly operating with a "unit force" if so wanted. If the supports obtain certain displacements, the corresponding external virtual work must naturally be included. As the support forces corresponding to the equilibrium force system are known, these contributions can however be evaluated. □

Example 5.5. We consider the statically determinate plane truss shown in Figure (a). Bar 5 obtains an elongation Δs say due to a temperature rise in the bar. We determine the vertical displacement of joint 4 upwards due to the elongation.

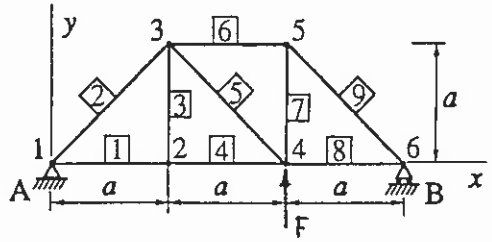


Figure (a)

For a truss, the principle of virtual work is

$$\sum_i [(F_i)_x \delta u_i + (F_i)_y \delta v_i] - \sum_{ij} N \delta s = 0, \tag{a}$$

where the meaning of the notation is rather obvious and is explained in more detail in Section 6.4.1. When we apply this to real displacements and extensions, it reads

$$\sum_i [(F_i)_x u_i + (F_i)_y v_i] - \sum_{ij} N \Delta s = 0. \tag{b}$$

In the unit dummy load method here only bar 5 has a non-zero extension and the only external force doing virtual work is the force F shown in Figure (a). Thus (b) is in detail

$$F v_4 - N_5 \Delta s = 0. \tag{c}$$

For a statically determinate structure, there is only one internal force distribution corresponding to the external force F. Because of the form of (c) we need only bar force N_5. It is easily found for instance by the section method that

$$N_5 = -\frac{\sqrt{2}}{3} F. \tag{d}$$

Substitution in (b) gives

$$F v_4 - \frac{\sqrt{2}}{3} F \Delta s = 0, \tag{e}$$

from which the vertical displacement is found to be

$$v_4 = \frac{\sqrt{2}}{3} \Delta s. \tag{f}$$

5.8.2 Principle of stationary potential energy

For a conservative system, the principle of virtual work (3): $\delta' W = 0$ for kinematically admissible virtual displacements obtains due to the result (5.5.6): $\delta' W = -\delta V$ the form

$$\boxed{\delta V = 0} \tag{8}$$

or when a conservative system is in equilibrium, the potential energy of the system has a stationary value.

This is called the *principle of stationary potential energy* (potentiaalienergian stationaarisuuden periaate). Due to the way of derivation of this principle, the potential energy V is to be considered as a function of the position (or displacements) of the system and the variations of the position (or displacements) must be kinematically admissible.

If generalized coordinates are used, $V = V(q_1, q_2, \cdots, q_n)$,

$$\delta V = \sum_{j=1}^{n} \frac{\partial V}{\partial q_j} \delta q_j \tag{9}$$

and the stationary condition can be expressed also in the form

$$\boxed{\frac{\partial V}{\partial q_j} = 0,} \quad j = 1, 2, \cdots, n. \tag{10}$$

These equations are the equilibrium equations of a conservative system. Their solution gives the *equilibrium position* (tasapainoasema) of the system.

5.8.3 Principle of minimum potential energy

The equilibrium position of a system is said to be *stable* (stabiili) if small disturbancies do not cause the system to move far from its equilibrium position. If this is not so, the equilibrium position is called *unstable* (epästabiili).

To study the stability of a system in a general case would mean a consideration of the nature of the solution of a dynamics problem. For conservative system, however, an alternative possibility exists. It can be shown that

The equilibrium position of a conservative system is stable, if and only if the potential energy of the system has at it an essential minimum. (11)

This is called the *principle of minimum potential energy* (potentiaalienergian minimin periaate).

We do not prove this theorem. The proof is based on the facts that the mechanical energy $K + V$ of a conservative system is constant (formula (4.5.39)) and that the kinetic energy K is a non-negative quantity.

5.9 REFERENCES

Lanczos, C. (1974): *The Variational Principles of Mechanics*, University of Toronto Press, Toronto.

Washizu, K. (1982): *Variational Methods in Elasticity & Plasticity*, 3rd ed., Pergamon Press, Oxford.

CHAPTER 6

APPLICATIONS IN STATICS

We consider here the use of the principle of virtual work the analysis of elastic plane trusses including large displacements and deformations. This should be considered as an introduction in a relative simple setting to some of the concepts needed in the more complicated continuum case.

6.1 KINEMATICS

As mentioned already in Section 4.2, a truss can be modelled roughly as a particle system with all its mass concentrated at the joints, which are then considered as the particles of the system and the bars of the truss just indicate the kind of interaction between the particles. In the following, we will call the particles joints.

We consider the kinematics of a typical bar connecting two joints i and j in Figure 6.1.

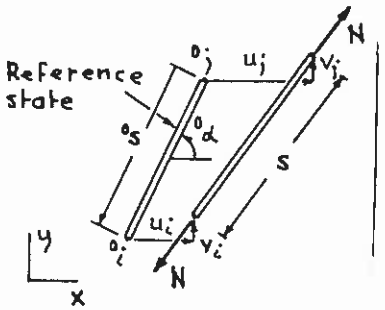


Figure 6.1 Initial and current position of a bar.

As before, the reference or initial state is indicated with the left superscript 0. The general expression for the square of the length of the bar is

$$s^2 = (x_j - x_i)^2 + (y_j - y_i)^2. \tag{1}$$

The coordinates of the current state are connected with the coordinates of the reference state and with the displacement components by

$$\begin{aligned} x_i &= {}^0x_i + u_i, \quad x_j = {}^0x_j + u_j, \\ y_i &= {}^0y_i + v_i, \quad y_j = {}^0y_j + v_j. \end{aligned} \tag{2}$$

Substitution of these into (1) gives

$$s^2 = ({}^0x_j - {}^0x_i + u_j - u_i)^2 + ({}^0y_j - {}^0y_i + v_j - v_i)^2. \tag{3}$$

In the initial state the displacements are zero by definition and thus

$${}^0s^2 = ({}^0x_j - {}^0x_i)^2 + ({}^0y_j - {}^0y_i)^2. \tag{4}$$

To shorten the formulas we denote

$${}^0s_x = {}^0x_j - {}^0x_i, \quad {}^0s_y = {}^0y_j - {}^0y_i \tag{5}$$

and obtain

$$\begin{aligned} s^2 &= {}^0s_x^2 + 2\,{}^0s_x \cdot (u_j - u_i) + (u_j - u_i)^2 + \\ &\quad + {}^0s_y^2 + 2\,{}^0s_y \cdot (v_j - v_i) + (v_j - v_i)^2, \\ {}^0s^2 &= {}^0s_x^2 + {}^0s_y^2. \end{aligned} \tag{6}$$

The axial *strain* (venymä, suhteellinen venymä) E ($[E] = 1$) is defined by

$$s^2 - {}^0s^2 = 2E\,{}^0s^2 \tag{7}$$

or

$$\boxed{E = \frac{1}{2}\frac{s^2 - {}^0s^2}{{}^0s^2} = \frac{1}{2}\left[\left(\frac{s}{{}^0s}\right)^2 - 1\right].} \tag{8}$$

The definition (7) is a special case of the general form of the definition

$$(\mathrm{d}s)^2 - (\mathrm{d}\,{}^0s)^2 = 2E_{ij}\,\mathrm{d}\,{}^0x_i\,\mathrm{d}\,{}^0x_j \tag{9}$$

of the *Green-Lagrange strain tensor* or shortly the *Green strain tensor* (Green-Lagrangen venymätensori tai lyhyemmin Greenin venymätensori) E_{ij}, which is discussed in more detail in Section 10.2.4.

Expression (8) can be put in the form

$$E = \frac{1}{2}\frac{(s + {}^0s)(s - {}^0s)}{{}^0s^2} = \frac{1}{2}\frac{(s + {}^0s)\Delta s}{{}^0s^2} \approx \frac{1}{2}\frac{2 \cdot {}^0s \cdot \Delta s}{{}^0s^2} = \frac{\Delta s}{{}^0s}. \tag{10}$$

The approximate expression is valid in the cases where the extension $\Delta s = s - {}^0s$ is small compared to the original bar length ${}^0s$ in which case $s \approx {}^0s$. This approximation produces the formula

$$\varepsilon = \frac{\Delta s}{{}^0s} = \frac{s - {}^0s}{{}^0s} = \frac{s}{{}^0s} - 1 \tag{11}$$

for the so-called *infinitesimal strain* or *engineering strain* or *small strain* (infinitesimaalinen venymä, insinöörivenymä, pieni venymä) ε.

We need the strain expressions as functions of the displacement components. Substitution of (6) into (8) gives

$$E = \frac{1}{2}\frac{2\,{}^0s_x\cdot(u_j-u_i)+(u_j-u_i)^2+2\,{}^0s_y\cdot(v_j-v_i)+(v_j-v_i)^2}{{}^0s^2}$$
$$= \frac{{}^0s_x}{{}^0s^2}(u_j-u_i)+\frac{{}^0s_y}{{}^0s^2}(v_j-v_i)+\frac{1}{2\,{}^0s^2}[(u_j-u_i)^2+(v_j-v_i)^2]. \tag{12}$$

Similarly as in the continuum case, the Green strain is seen to consist exactly of linear and quadratic terms in displacement components. The small strain expression is obtained from (12) by dropping the quadratic terms:

$$\varepsilon = \frac{{}^0s_x}{{}^0s^2}(u_j-u_i)+\frac{{}^0s_y}{{}^0s^2}(v_j-v_i) = \frac{1}{{}^0s}\left[\frac{{}^0s_x}{{}^0s}(u_j-u_i)+\frac{{}^0s_y}{{}^0s}(v_j-v_i)\right] \tag{13}$$

or

$$\varepsilon = \frac{1}{{}^0s}[\cos{}^0\alpha\cdot(u_j-u_i)+\sin{}^0\alpha\cdot(v_j-v_i)]. \tag{14}$$

In applying this formula, the angle ${}^0\alpha$ in the reference configuration must be measured so that it is the angle between the directed line starting from point i and towards point j and the positive x-axis and positive in the counterclockwise direction from the x-axis.

We record still one expression for the bar length for further use:

$$s = [({}^0s_x+u_j-u_i)^2+({}^0s_y+v_j-v_i)^2]^{1/2}. \tag{15}$$

6.2 STRESS

The rather complicated concepts concerning stresses in large deformation problems are considered in more detail in Chapter 11. Figure 6.2 shows a cross-section of a bar in the reference configuration (Figure (a)) and in the current configuration (Figures (b) and (c)). The complications emerge because in the so-called Lagrangian representation used in solid mechanics, "the mathematics happens in the reference configuration but the physics happen in the current

configuration". A differential material cross-sectional area element $\mathrm{d}{}^0A$ in the reference state transforms to a material area element $\mathrm{d}A$ in the current state. The conventional engineering stress concept is the so-called *Euler stress* or *Cauchy stress* or *engineering stress* (Eulerin jännitys, Cauchyn jännitys, insinöörijännitys) σ giving the differential force $\mathrm{d}\boldsymbol{F}$ acting on the area element $\mathrm{d}A$ by

$$\mathrm{d}\boldsymbol{F} = \sigma\,\mathrm{d}A\boldsymbol{n}, \tag{1}$$

where $\boldsymbol{n}$ is the unit normal vector to the current area element. Thus, the normal force vector $\boldsymbol{N}$ in the bar (for notational clarity, we use here instead of the symbol S the symbol N to avoid confusion with the second Piola-Kirchhoff stress tensor symbol S_{ij}) is

$$\boldsymbol{N} = \sigma A\boldsymbol{n} \qquad (N\vec{n} = \sigma A\vec{n}) \tag{2}$$

assuming constant stress distribution over the current cross-sectional area.

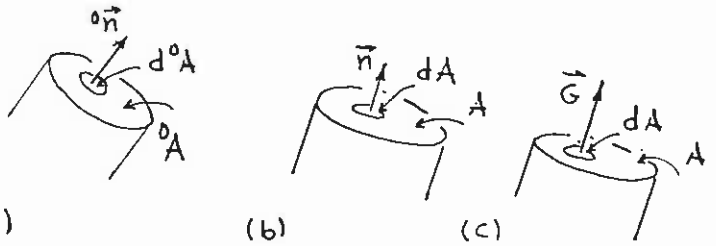


Figure 6.2 (a) Cross-section in the reference configuration. **(b)** and **(c)** Cross-sections in the current configuration.

In large deformation theory, the *second Piola-Kirchhoff stress tensor* (toisen lajin Piola-Kirchhoffin jännitystensori) or here shortly the *Kirchhoff stress tensor* (Kirchhoffin jännitystensori) is the most widely used stress concept. The differential force vector $\mathrm{d}\boldsymbol{F}$ acting on the area element $\mathrm{d}A$ in the current configuration is given by

$$\mathrm{d}\boldsymbol{F} = S\,\mathrm{d}{}^0A\boldsymbol{G}. \tag{3}$$

$\boldsymbol{G}$ is a basis vector in the same direction as $\boldsymbol{n}$ but not of unit length:

$$\boldsymbol{G} = \frac{\mathrm{d}s}{\mathrm{d}{}^0s}\boldsymbol{n} = \frac{s}{{}^0s}\boldsymbol{n}. \tag{4}$$

The Kirchhoff stress S is thus referred to the original area and as $\boldsymbol{G}$ is not a unit vector, it is a rather complicated quantity. The normal force vector $\boldsymbol{N}$ in the bar is obtained — assuming again constant stress distribution — from

$$N = S\,{}^0A G. \qquad (N\vec{n} = S\,{}^0A\,\frac{s}{{}^0s}\,\vec{n}) \tag{5}$$

The scalar bar forces are thus

$$N = \sigma A \tag{6}$$

using the Euler stress and

$$N = \frac{s}{{}^0s} S\,{}^0A \tag{7}$$

using the Kirchhoff stress.

6.3 CONSTITUTIVE RELATION

We assume here elastic material with a linear constitutive relation between the Kirchhoff stress and the Green strain:

$$S = CE. \tag{1}$$

C is a material constant (Young's modulus).

6.4 LARGE DISPLACEMENT THEORY

6.4.1 Virtual work equation

The principle of virtual work must be applied for the structure in the current equilibrium position which is unknown in advance. Figure 6.3 shows a part of a structure in the reference configuration and in the current equilibrium configuration with virtual displacements superposed on it.

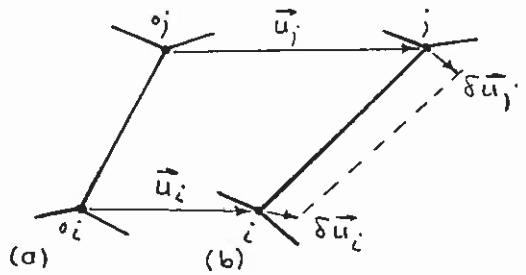


Figure 6.3 (a) Reference configuration. (b) Current configuration.

The virtual work equation is

$$-\delta' W \equiv -\delta' W_{\text{int}} - \delta' W_{\text{ext}} = 0 \tag{1}$$

where here

$$-\delta' W_{\text{int}} = \sum_{ij} N\delta s, \tag{2}$$

and

$$-\delta' W_{\text{ext}} = -\sum_i [(F_i)_x \delta u_i + (F_i)_y \delta v_i]. \tag{3}$$

Signs have been changed to obtain a conventional formulation. The summation in the internal work is over the bars (or particle pairs connected by bars) and in the external work over the nodes not constrained by supports.

Using formulas (6.2.7), (6.3.1) and (6.1.12) we arrive at

$$\begin{aligned} N &= \frac{s}{{}^0s} S\,{}^0A = \frac{s}{{}^0s} CE\,{}^0A = \frac{C\,{}^0A}{{}^0s} sE \\ &= \frac{C\,{}^0A}{{}^0s} s \frac{1}{2} \frac{2\,{}^0s_x \cdot (u_j - u_i) + (u_j - u_i)^2 + 2\,{}^0s_y \cdot (v_j - v_i) + (v_j - v_i)^2}{{}^0s^2}. \end{aligned} \tag{4}$$

The variation of (6.1.15):

$$s = [({}^0s_x + u_j - u_i)^2 + ({}^0s_y + v_j - v_i)^2]^{1/2} \tag{5}$$

gives

$$\delta s = \frac{1}{s}[({}^0s_x + u_j - u_i)(\delta u_j - \delta u_i) + ({}^0s_y + v_j - v_i)(\delta v_j - \delta v_i)]. \tag{6}$$

Thus finally,

$$\begin{aligned} N\delta s = \frac{C\,{}^0A}{{}^0s} \frac{1}{2} &\frac{2\,{}^0s_x \cdot (u_j - u_i) + (u_j - u_i)^2 + 2\,{}^0s_y \cdot (v_j - v_i) + (v_j - v_i)^2}{{}^0s^2} \cdot \\ &\cdot [({}^0s_x + u_j - u_i)(\delta u_j - \delta u_i) + ({}^0s_y + v_j - v_i)(\delta v_j - \delta v_i)]. \end{aligned} \tag{7}$$

This is to be substituted in (2). The system equations are obtained by taking the consecutive variations $\delta u_1 \neq 0$, $\delta v_1 \neq 0$, $\delta u_2 \neq 0$, etc. for those nodes not constrained by supports. Clearly very complicated equations with respect to the unknown nodal displacements u_1, v_1, u_2, etc. are arrived at. After the displacements have been determined, the strains, stresses and bar forces are easily evaluated in a post-processing manner.

6.4.2 System equations assembly

We describe here some steps towards obtaining a systematic way to generate in more detail the system equations in the spirit of the finite element method. Some of the terminology appearing here is discussed also in Section D.3.4. In our application we can consider the truss to consist of two kind of elements: line (bar) elements and point (joint) elements. The line elements produce here the internal virtual work contributions and the point elements the external virtual work contributions. The line elements of the system are identified by numbering them starting from one: 1, 2, ⋯ and similarly for the point elements. A line element has two nodes numbered locally as 1 and 2 and four locally numberd generalized displacements (g.d.) $q_1 = u_1$, $q_2 = v_1$, $q_3 = u_2$, $q_4 = v_2$. A point element has only one node which is numbered locally simply by 1 and two locally numbered generalized displacements $q_1 = u_1$, $q_2 = v_1$. We can now derive the generalized force expression corresponding to the generalized displacements present in the elements. The system equilibrium equations

$$-Q_j = 0, \qquad j = 1,2,\cdots,n \tag{8}$$

are then obtained by summation from the element generalized force contributions. (We use the minus sign as a consequence from writing the virtual work equation in the form $-\delta' W = 0$.) Only the right places to put the contributions must be known. The generalized displacements for the system are numbered globally starting from one. The information for the assembly comes from the correspondence between the local and global generalized displacement numbers. Figure 6.4 gives an example.

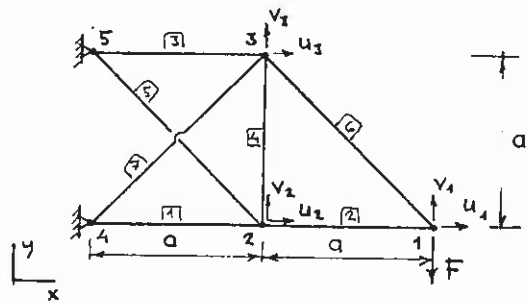


Figure 6.4 Numbering of a truss.

The joints=nodes have been numbered as shown in the figure and based on that (each node has two generalized displacements) the global generalized displacements have been listed as

$$\{q\}_{10\times 1}^{\mathrm{T}} = [q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}]$$

$$= [u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4, u_5, v_5]. \tag{9}$$

The displacements of joints 4 and 5 have been included in the list although these displacements are finally constrained.

The information of the line elements shown in the figure are collected in Table 6.1:

Table 6.1 Line element data

Line element number	Global node numbers	Global g.d. numbers
1	4 2	7 8 3 4
2	2 1	3 4 1 2
3	5 3	9 10 5 6
4	2 3	3 4 5 6
5	5 2	9 10 3 4
6	3 1	5 6 1 2
7	4 3	7 8 5 6

For instance, the information concerning line element 1 tells that the local node 1 is at the global node 4 and the local node 2 is at the global node 2. From this we can infer the following correspondences for line element 1:

$$\begin{aligned} &\text{local } q_1 = u_1 \mathrel{\hat{=}} \text{ global } q_7 = u_4, \\ &\text{local } q_2 = v_1 \mathrel{\hat{=}} \text{ global } q_8 = v_4, \\ &\text{local } q_3 = u_2 \mathrel{\hat{=}} \text{ global } q_3 = u_2, \\ &\text{local } q_4 = v_2 \mathrel{\hat{=}} \text{ global } q_4 = v_2. \end{aligned} \tag{10}$$

This kind of information is collected in the column titled "global g.d. numbers". A column titled "local g.d. numbers" consisting of rows all having the numbers 1 2 3 4 could be put in the table but this information would be superluous.

The point elements are numbered here simply according to the joint numbering and we obtain a trivial Table 6.2:

Table 6.2 Point element data

Point element number	Global node number	Global g.d. numbers
1	1	1 2
2	2	3 4
3	3	5 6
4	4	7 8
5	5	9 10

For instance, we have for point element 3 the correspondence

$$\text{local } q_1 = u_1 \triangleq \text{ global } q_5 = u_3,$$
$$\text{local } q_2 = v_1 \triangleq \text{ global } q_6 = v_3. \tag{11}$$

We now determine the element contributions to the generalized forces from the line and and point elements. No detailed indexing is used for simplicity as the meaning should be clear from the context.

For a *line element*

$$\begin{aligned} -\delta' W &= -Q_1\delta q_1 - Q_2\delta q_2 - Q_3\delta q_3 - Q_4\delta q_4 \\ &= -\underset{4\times1}{\{\delta q\}^{\mathrm{T}}}\ \underset{4\times1}{\{Q\}} = -\underset{4\times1}{\{Q\}^{\mathrm{T}}}\ \underset{4\times1}{\{\delta q\}}. \end{aligned} \tag{12}$$

Comparison with expression (7) and making the associations $i=1$ and $j=2$ gives

$$\begin{aligned} -Q_1 &= -B\cdot({}^0s_x + q_3 - q_1), \\ -Q_2 &= -B\cdot({}^0s_y + q_4 - q_2), \\ -Q_3 &= B\cdot({}^0s_x + q_3 - q_1), \\ -Q_4 &= B\cdot({}^0s_y + q_4 - q_2) \end{aligned} \tag{13}$$

with

$$B = \frac{C\,{}^0A}{{}^0s}\,\frac{1}{2}\,\frac{2\,{}^0s_x\cdot(q_3-q_1)+(q_3-q_1)^2+2\,{}^0s_y\cdot(q_4-q_2)+(q_4-q_2)^2}{{}^0s^2}. \tag{14}$$

For a *point element*

$$\begin{aligned} -\delta' W &= -Q_1\delta q_1 - Q_2\delta q_2 \\ &= -\underset{2\times1}{\{\delta q\}^{\mathrm{T}}}\ \underset{2\times1}{\{Q\}} = -\underset{2\times1}{\{Q\}^{\mathrm{T}}}\ \underset{2\times1}{\{\delta q\}}. \end{aligned} \tag{15}$$

Comparison with expression (3) and making the association $i=1$ gives

$$\begin{aligned} -Q_1 &= -(F_1)_x, \\ -Q_2 &= -(F_1)_y. \end{aligned} \tag{16}$$

When these expressions are applied in the assembly process, the local indices must be replaced by the global ones in the manner described in (10) and (11).

The system equations in the finite element method are normally numbered in the order determined by the nodal parameter numbering, that is, the first equation is associated to the first nodal parameter, etc. Here the nodal parameters are the nodal displacements or generalized displacements and the first system equation

$-Q_1 = 0$ is associated with the first global generalized displacement q_1, etc. The assembly of the system equations can now be stated as follows. If the system equations are written as

$$\boxed{-Q_i = 0,} \qquad i = 1,2,\cdots,n \tag{17}$$

the left hand sides are obtained by the summation formula

$$\boxed{-Q_i = \sum_e -Q_r^{(e)},} \tag{18}$$

where the summation is over the elements and element e gives a contribution (or can at most give a non-zero contribution) to the generalized force Q_i if the element local g.d. number r corresponds to the global g.d. number i.

As an example, the third system equation is obtained from

$$-Q_3 = -Q_3^{(1)} - Q_1^{(2)} - Q_1^{(4)} - Q_3^{(5)} - Q_1^{(\bar{2})} = 0. \tag{19}$$

The superscripts refer to the element numbers and the point element number is equipped here with an overbar. Line elements 3, 6, 7 and point elements 1, 3, 4, 5 give no contributions.

The support conditions (assuming given displacements here equal to zero)

$$q_7 = u_4 = 0, \quad q_8 = v_4 = 0, \quad q_9 = u_5 = 0, \quad q_{10} = v_5 = 0, \tag{20}$$

can be introduced at the final stage to the system equations. In fact, if kinematically admissible virtual displacements are considered, it is not necessary to generate the four last equations.

6.4.3 Incremental formulation

In practice the system equations must be solved in an incremental and linearised manner. For instance references Washizu (1982) and Bathe (1996) contain detailed formulations for incremental solution procedures for large deformation solid mechanics problems.

The loading path is divided in a number of equilibrium states. It is assumed that we have obtained the solution at a state described by the left superscript t and we have to find the the next neighbouring equilibrium state denoted by the left superscript $t+\Delta t$. In the so-called *total Lagrangian formulation* (kokonais-Lagrangen formulaatio) we have the relevant quantities to deal with collected in

Table 6.3. It should be noted that Δs has here a different meaning than in Section 6.1.

All the quantities in the table associated with the state ${}^{t}\Omega$ are assumed to be known from the analysis performed this far. The aim is to generate a set of system equations for the basic unknown nodal displacement increments Δu_i and Δv_i which are further linear in them.

Table 6.3 Variables in the incremental total Lagrangian formulation

	State ${}^{t}\Omega$	State ${}^{t+\Delta t}\Omega$
Bar force	N	$N+\Delta N$
Kirchhoff stress	S	$S+\Delta S$
Bar length	s	$s+\Delta s$
Green strain	E	$E+\Delta E$
Nodal displacements	$\boldsymbol{u}$	$\boldsymbol{u}+\Delta\boldsymbol{u}$
External nodal forces	$\boldsymbol{F}$	$\boldsymbol{F}+\Delta\boldsymbol{F}$
Given displacements	$\bar{\boldsymbol{u}}$	$\bar{\boldsymbol{u}}+\Delta\bar{\boldsymbol{u}}$

The virtual work equation for the state ${}^{t}\Omega$ is

$$\sum_{ij} N\delta s - \sum_{i}[(F_i)_x\delta u_i + (F_i)_y\delta v_i] = 0. \tag{21}$$

The same for the new unknown state ${}^{t+\Delta t}\Omega$ is

$$\sum_{ij}(N+\Delta N)\delta(s+\Delta s) + \\ -\sum_{i}\{[(F_i)_x + (\Delta F_i)_x]\delta(u_i+\Delta u_i) + [(F_i)_y + (\Delta F_i)_y]\delta(v_i+\Delta v_i)]\} = 0. \tag{22}$$

It should be emphasized that here the virtual displacements are to be superposed onto the unknown state ${}^{t+\Delta t}\Omega$. However, in taking the variation we can consider the known values s and $\boldsymbol{u}$ at state ${}^{t}\Omega$ given and fixed so that

$$\delta(s+\Delta s) = \delta\Delta s, \quad \delta(\boldsymbol{u}+\Delta\boldsymbol{u}) = \delta\Delta\boldsymbol{u} \tag{23}$$

(or simply we are free to select the virtual displacemen the way we please) and (22) obtains the form

$$\sum_{ij}(N+\Delta N)\delta\Delta s - \sum_{i}\{[(F_i)_x + (\Delta F_i)_x]\delta\Delta u_i + [(F_i)_y + (\Delta F_i)_y]\delta\Delta v_i]\} = 0. \tag{24}$$

The bar length increment from (5) is

$$\Delta s = \{[({}^{0}s_x + u_j - u_i) + (\Delta u_j - \Delta u_i)]^2 + \\ + [({}^{0}s_y + v_j - v_i) + (\Delta v_j - \Delta v_i)]^2\}^{1/2} +$$

$$- [({}^{0}s_x + u_j - u_i)^2 + ({}^{0}s_y + v_j - v_i)^2]^{1/2}. \tag{25}$$

A truncated representation with respect to the displacement increments Δu and Δv using binomial series gives

$$\Delta s \approx \Delta s^{(1)} + \Delta s^{(2)}, \tag{26}$$

where

$$\Delta s^{(1)} = \frac{1}{s}[({}^{0}s_x + u_j - u_i)(\Delta u_j - \Delta u_i) + ({}^{0}s_y + v_j - v_i)(\Delta v_j - \Delta v_i)]. \tag{27}$$

is linear and

$$\Delta s^{(2)} = \frac{1}{s}\frac{1}{2}[(\Delta u_j - \Delta u_i)^2 + (\Delta v_j - \Delta v_i)^2] \tag{28}$$

quadratic in the displacement increments. The variation is

$$\delta\Delta s \approx \delta\Delta s^{(1)} + \delta\Delta s^{(2)} \tag{29}$$

with

$$\delta\Delta s^{(1)} = \frac{1}{s}[({}^{0}s_x + u_j - u_i)(\delta\Delta u_j - \delta\Delta u_i) + ({}^{0}s_y + v_j - v_i)(\delta\Delta v_j - \delta\Delta v_i)] \tag{31}$$

and

$$\delta\Delta s^{(2)} = \frac{1}{s}[(\Delta u_j - \Delta u_i)(\delta\Delta u_j - \delta\Delta u_i) + (\Delta v_j - \Delta v_i)(\delta\Delta v_j - \delta\Delta v_i)]. \tag{32}$$

Substituting (29) into the virtual work equation (24) gives first

$$\sum_{ij} N\delta\Delta s^{(1)} + \sum_{ij}\Delta N\delta\Delta s^{(1)} + \sum_{ij} N\delta\Delta s^{(2)} + \sum_{ij}\Delta N\delta\Delta s^{(2)} + \\ -\sum_{i}[[(F_i)_x + (\Delta F_i)_x]\delta\Delta u_i + [(F_i)_y + (\Delta F_i)_y]\delta\Delta v_i]\} = 0. \tag{33}$$

Neglecting one higher order term and arranging gives finally

$$\sum_{ij}\Delta N\delta\Delta s^{(1)} + \sum_{ij} N\delta\Delta s^{(2)} + \sum_{ij} N\delta\Delta s^{(1)} + \\ -\sum_{i}[[(F_i)_x + (\Delta F_i)_x]\delta\Delta u_i + [(F_i)_y + (\Delta F_i)_y]\delta\Delta v_i]\} = 0. \tag{34}$$

This is a valid form for a truss irrespective the material law. Expressions (31) and (32) are to be used in it. The system equations for the unknown

displacement increments are obtained by applying the consecutive virtual displacements $\delta\Delta u_1 \neq 0$, $\delta\Delta v_1 \neq 0$, $\delta\Delta u_2 \neq 0$, etc.

For an elastic truss, considered here,

$$N = \frac{C\,{}^0A}{{}^0s} sE \tag{35}$$

and thus

$$N + \Delta N = \frac{C\,{}^0A}{{}^0s}(s+\Delta s)(E+\Delta E) = \frac{C\,{}^0A}{{}^0s} sE + \frac{C\,{}^0A}{{}^0s} s\Delta E + \frac{C\,{}^0A}{{}^0s}\Delta sE \tag{36}$$

so neglecting one higher order term and taking (30) into account gives

$$\Delta N \approx \frac{C\,{}^0A}{{}^0s} s\Delta E + \frac{C\,{}^0A}{{}^0s}\Delta sE. \tag{37}$$

We may apply here the linearized form $\Delta s^{(1)}$ of Δs. For ΔE, the linearized form from (6.1.12) is found to be

$$\Delta E^{(1)} = \frac{1}{{}^0s^2}[{}^0s_x \cdot (\Delta u_j - \Delta u_i) + (u_j - u_i)(\Delta u_j - \Delta u_i) + \\ + {}^0s_y \cdot (\Delta v_j - \Delta v_i) + (v_j - v_i)(\Delta v_j - \Delta v_i)]. \tag{38}$$

So finally, the bar force increment is evaluted from

$$\Delta N \approx \frac{C\,{}^0A}{{}^0s} s\Delta E^{(1)} + \frac{C\,{}^0A}{{}^0s}\Delta s^{(1)} E. \tag{39}$$

The system equation are a linear set with respect to the displacement increments. The assembly of the system equations is considered in more detail in the next section.

After the new total displacements have been determined, the corresponding srains, stresses, bar forces, etc should be evaluated directly from the corresponding basic formulas.

6.4.4 System equations assembly

We continue to present more detailed element contributions similarly as in Section 6.4.2. Some of the expressions become rather complicated.

(1) Term $\Delta N\delta\Delta s^{(1)}$ in (34) for a line element. To shorten the formulas we denote

$$s_x = {}^0s_x + u_j - u_i, \qquad s_y = {}^0s_y + v_j - v_i, \tag{40}$$

that is, s_x and s_y are the current x- and y-components of the directed line segment ij and for instance

$$s^2 = s_x^2 + s_y^2. \tag{41}$$

Using these notations, we have

$$\Delta s^{(1)} = \frac{1}{s}[s_x(\Delta u_j - \Delta u_i) + s_y(\Delta v_j - \Delta v_i)], \\ \delta\Delta s^{(1)} = \frac{1}{s}[s_x(\delta\Delta u_j - \delta\Delta u_i) + s_y(\delta\Delta v_j - \delta\Delta v_i)], \\ \Delta E^{(1)} = \frac{1}{{}^0s^2}[s_x(\Delta u_j - \Delta u_i) + s_y(\Delta v_j - \Delta v_i)]. \tag{42}$$

Further,

$$\Delta N = \frac{C\,{}^0A}{{}^0s} s\Delta E^{(1)} + \frac{C\,{}^0A}{{}^0s}\Delta s^{(1)} E \\ = \frac{C\,{}^0A}{{}^0s}\left(\frac{s}{{}^0s^2} + \frac{E}{s}\right)[s_x(\Delta u_j - \Delta u_i) + s_y(\Delta v_j - \Delta v_i)] \tag{43}$$

and

$$\Delta N\delta\Delta s^{(1)} = \frac{C\,{}^0A}{{}^0s}\left(\frac{1}{{}^0s^2} + \frac{E}{s^2}\right)[s_x(\Delta u_j - \Delta u_i) + s_y(\Delta v_j - \Delta v_i)] \cdot \\ \cdot [s_x(\delta\Delta u_j - \delta\Delta u_i) + s_y(\delta\Delta v_j - \delta\Delta v_i)]. \tag{44}$$

Making again the associations $i = 1$ and $j = 2$ similarly as in Section 6.4.2 gives

$$-Q_1 = -s_x B, \\ -Q_2 = -s_y B, \\ -Q_3 = +s_x B, \\ -Q_4 = +s_y B, \tag{45}$$

with

$$B = \frac{C\,{}^0A}{{}^0s}\left(\frac{1}{{}^0s^2} + \frac{E}{s^2}\right)[-s_x\Delta q_1 - s_y\Delta q_2 + s_x\Delta q_3 + s_y\Delta q_4]. \tag{46}$$

It is seen that the generalized forces consist of linear terms with respect to the displacement increments Δq. Thus we may write the result (46) in a more compact form

$$-\underset{4\times1}{\{Q\}} = \underset{4\times4}{[K]}\underset{4\times1}{[\Delta q]} \tag{47}$$

where the meaning of the notations is obvious. The coefficient matrix is

$$[K] = \frac{C\,{}^0A}{{}^0s}\left(\frac{1}{{}^0s^2} + \frac{E}{s^2}\right)\begin{bmatrix} s_x^2 & s_x s_y & -s_x^2 & -s_x s_y \\ s_x s_y & s_y^2 & -s_x s_y & -s_y^2 \\ -s_x^2 & -s_x s_y & s_x^2 & s_x s_y \\ -s_x s_y & -s_y^2 & s_x s_y & s_y^2 \end{bmatrix}. \tag{48}$$

In continuum problems this kind of matrix is often separated in two parts, the first one consisting of terms that stay constant in each increment and to an updated part. Here we simply update the whole coefficient matrix for each increment.

(2) Term $N\delta\Delta s^{(2)}$ in (34) for a line element. We obtain

$$N\delta\Delta s^{(2)} = \frac{N}{s}[(\Delta u_j - \Delta u_i)(\delta\Delta u_j - \delta\Delta u_i) + (\Delta v_j - \Delta v_i)(\delta\Delta v_j - \delta\Delta v_i)]. \tag{49}$$

Thus

$$\begin{aligned} -Q_1 &= -\frac{N}{s}(\Delta q_3 - \Delta q_1), \\ -Q_2 &= -\frac{N}{s}(\Delta q_4 - \Delta q_2), \\ -Q_3 &= +\frac{N}{s}(\Delta q_3 - \Delta q_1), \\ -Q_4 &= +\frac{N}{s}(\Delta q_4 - \Delta q_2), \end{aligned} \tag{50}$$

or

$$-\underset{4\times1}{\{Q\}} = \underset{4\times4}{[K]}\underset{4\times1}{[\Delta q]} \tag{51}$$

with

$$[K] = \frac{N}{s}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}. \tag{52}$$

This matrix is called the *initial stress stiffness matrix* (alkujännitysjäykkyys-matriisi), Washizu (1982).

(3) Term $N\delta\Delta s^{(1)}$ in (34) for a line element. We obtain

$$N\delta\Delta s^{(1)} = \frac{N}{s}[s_x(\delta\Delta u_j - \delta\Delta u_i) + s_y(\delta\Delta v_j - \delta\Delta v_i)]. \tag{53}$$

Thus

$$\begin{aligned} -Q_1 &= -\frac{s_x}{s}N, \\ -Q_2 &= -\frac{s_y}{s}N, \\ -Q_3 &= +\frac{s_x}{s}N, \\ -Q_4 &= +\frac{s_y}{s}N. \end{aligned} \tag{54}$$

These are constants with respect to the displacement increments and in matrix notation

$$-\underset{4\times1}{\{Q\}} = -\underset{4\times1}{\{b\}} = -N\begin{Bmatrix} s_x/s \\ s_y/s \\ -s_x/s \\ -s_y/s \end{Bmatrix}. \tag{55}$$

(4) Term

$$-[(F_i)_x + (\Delta F_i)_x]\delta\Delta u_i + [(F_i)_y + (\Delta F_i)_y]\delta\Delta v_i \tag{56}$$

in (34) for a point element. We obtain

$$\begin{aligned} -Q_1 &= -(F_1)_x - (\Delta F_1)_x, \\ -Q_2 &= -(F_1)_y - (\Delta F_1)_y \end{aligned} \tag{57}$$

or in matrix notation

$$-\underset{2\times1}{\{Q\}} = -\underset{2\times1}{\{b\}} = -\begin{Bmatrix}(F_1)_x + (\Delta F_1)_x \\ (F_1)_y + (\Delta F_1)_y\end{Bmatrix}. \tag{58}$$

The system equations are here of the form

$$-Q_i \equiv \sum_{j=1}^{n} K_{ij}\Delta q_j - b_i = 0, \quad i = 1,2,\cdots,n \tag{59}$$

or

$$\sum_{j=1}^{n} K_{ij}\Delta q_j = b_i, \quad i = 1,2,\cdots,n \tag{60}$$

or in matrix notation

$$\boxed{\underset{n\times n}{[K]}\underset{n\times 1}{\{\Delta q\}} = \underset{n\times 1}{\{b\}}.} \tag{61}$$

The assembly process described in Section 6.4.2 can be here more specific. Now the elements of the *system matrices* (systeemimatriisi) $[K]$ and $\{b\}$ can be assembled from elements of the *element matrices* (elementtimatriisi) $[K]^{(e)}$ and $\{b\}^{(e)}$. After some study we arrive at the following assembly rules

$$\boxed{\begin{aligned} K_{ij} &= \sum_e K_{rs}^{(e)}, \\ b_i &= \sum_e b_r^{(e)}. \end{aligned}} \tag{62}$$

Element e gives a contribution to term K_{ij} at most, if the element has the local g.d. numbers corresponding to global g.d. numbers i and j. The local numbers r and s must then be given the numbers corresponding to i and j. Term b_i obtains a contribution at most, if the element has a local g.d number corresponding to the global g.d. number i. The local number r must then be given the value corresponding to i.

6.5 SMALL DISPLACEMENT THEORY

In the small displacement theory there is no need to follow the updated geometry of the structure and we can for simplicity drop without confusion the left superscript 0 from all formulas. The kinematical expressions are linear in the displacements. We deal with the engineering strain

$$\varepsilon = \frac{\Delta s}{s} = \frac{1}{s}\left[\frac{s_x}{s}(u_j - u_i) + \frac{s_y}{s}(v_j - v_i)\right] \tag{1}$$

and stress σ. The axial force in a bar is

$$N = \sigma A \tag{2}$$

and the constitutive relation for an elastic material is

$$\sigma = C\varepsilon. \tag{3}$$

The virtual work equation is simply

$$\sum_{ij} N\delta s - \sum_i [(F_i)_x \delta u_i + (F_i)_y \delta v_i] = 0 \tag{4}$$

with respect to the reference configuration and the final system equations are of the form

$$\boxed{\underset{n\times n}{[K]}\underset{n\times 1}{\{q\}} = \underset{n\times 1}{\{b\}}.} \tag{5}$$

The element contributions can be picked from the expressions in the previous section by dropping the terms depending on the updated geometry.

A line element produces a coefficient matrix (see formula (48))

$$[K] = \frac{CA}{s}\frac{1}{s^2}\begin{bmatrix} s_x^2 & s_x s_y & -s_x^2 & -s_x s_y \\ s_x s_y & s_y^2 & -s_x s_y & -s_y^2 \\ -s_x^2 & -s_x s_y & s_x^2 & s_x s_y \\ -s_x s_y & -s_y^2 & s_x s_y & s_y^2 \end{bmatrix}. \tag{6}$$

and a point element a column matrix (see formula (58))

$$\{b\} = \begin{Bmatrix}(F_x)_1 \\ (F_y)_1\end{Bmatrix}. \tag{7}$$

The assembly of system equations proceeds as explained in the previous section.

6.6 REFERENCES

Bathe, K.-J. (1996): *Finite Element Procedures*, Prentice Hall, Englewoods Ciffs.

Washizu, K. (1982): *Variational Methods in Elasticity & Plasticity*, 3rd ed., Pergamon Press, Oxford.

PART II

VIRTUAL WORK IN SOLID MECHANICS

CHAPTER 8

INTRODUCTION

As mentioned in Chapter 1, when using the *continuum model*, we assume the material of a body to be distributed continuously in space. The mechanics dealing with the continuum model is called *continuum mechanics.*

Continuum mechanics is classified according to the state of the matter of the body under study broadly as described in Figure 8.1 into two categories: *solid mechanics* (kiinteän aineen mekaniikka, solidimekaniikka) and *fluid mechanics* (nestemekaniikka, fluidimekaniikka).

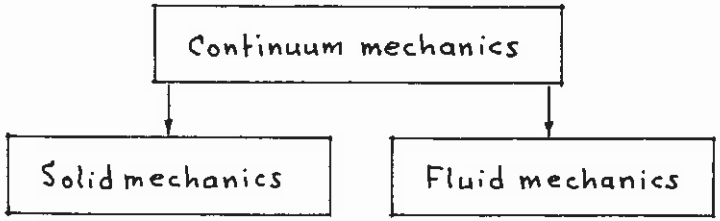
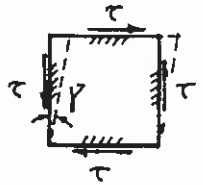


Figure 8.1 Main division of continuum mechanics.

From the point of view of mechanics the division between a solid and a fluid takes place according to the definition given in Figure 8.2.



No shear stresses τ can exist in a fluid without non-zero shearing strain rates $\dot{\gamma}$, in a solid this is not the case.

Figure 8.2 Difference between a solid and a fluid.

Thus shearing stresses vanish in a fluid at rest. In some limiting cases this classification may depend on the time interval available and on the accuracy of

observation. The mechanics dealing with materials having both the properties of solids and fluids is called *rheology* (reologia).

Fluids are further classified according to their compressibility into *liquids* (neste) and to *gases* (kaasu). In this text we only consider solids.

Although the axioms of continuum mechanics are the same for solids and fluids, the final governing equations can be quite different. This is partly due to the difference in the nature of constitutive relations but particularly because of the way of describing the kinematics in solid mechanics and in fluid mechanics. In solid mechanics, the conventional representation is the *Lagrangian description* (Lagrangen esitystapa) and in fluid mechanics, the *Eulerian description* (Eulerin esitystapa).

As some basic concepts have been considered in some detail in Part I of this text in connection of particle mechanics, we often in this second part can find the appropriate expressions simply using the steps described in formulas (4.4.7):

$$m_i \to \mathrm{d}m, \qquad \Sigma \to \int. \tag{1}$$

CHAPTER 9

AXIOMS OF CONTINUUM MECHANICS

The basic axioms of continuum mechanics are the following four:

(1) *Principle of conservation of mass* (massan säilymisen periaate): The mass m of a body is constant or

$$\boxed{\dot{m} = 0.} \tag{1}$$

(2) *Principle of balance of momentum* (liikemäärän taseen periaate): The resultant $\boldsymbol{F}$ of the external forces acting on a body is equal to the rate of change of the momentum $\boldsymbol{p}$ of the body:

$$\boxed{\boldsymbol{F} = \dot{\boldsymbol{p}}.} \tag{2}$$

(3) *Principle of balance of moment of momentum* (liikemäärämomentin taseen periaate): The moment $\boldsymbol{M}$ of the external forces acting on a body about a fixed point is equal to the rate of change of the moment of momentum $\boldsymbol{L}$ of the body about the fixed point:

$$\boxed{\boldsymbol{M} = \dot{\boldsymbol{L}}.} \tag{3}$$

(4) *Principle of balance of energy* (energian taseen periaate): The work W_{ext} done by the external forces acting on a body plus the heat W_Q received by the body is equal to the change of the kinetic energy ΔK plus the change of the internal energy ΔE of the body or

$$\boxed{P_{\text{ext}} + P_Q = \dot{K} + \dot{E}.} \tag{4}$$

In addition to these there exists the *principle of entropy growth* (entropian kasvun periaate) which we do not need in this text.

Principles (2) (3) and (4) are valid in an inertial frame. If they are applied in a non-inertial frame, the terms arising from apparent forces given by (4.1.42) must be included. Principles (2) and (3) were derived in Sections 4.3 and 4.4 as consequencies of particle mechanics axioms. Here they are taken conversely as axioms. Principle (4) is often called also the *first law of thermodynamics* (termodynamiikan ensimmäinen päälause). Some of the notations appearing here are elaborated later on.

The principles above are formulated to apply for a finite body. To each principle there corresponds an equation which can be called the *integral form* or *finite form* or *global form* of the principle (integraalimuoto, äärellinen muoto, globaali muoto). By certain mathematical manipulations one can produce from each global form a *differential form* or *local form* (paikallinen muoto, lokaali muoto) valid for an arbitrary differential continuum element.

CHAPTER 10

KINEMATICS

10.1 GENERAL

The kinematics of the continuum model differs from the kinematics of the particle system model in the respect that instead of the motion of a finite set of separate particles the motion of a set of infinitely dense set of particles is considered. This means that the number of independent variables grows from one (time) to the number four (three space coordinates and time) in three dimensions.

It should be emphasized that in the concept "particle" has in continuum mechanics a different meaning than in particle mechanics. In particle mechanics each particle has a finite definite mass. In a continuum, we cannot associate a finite mass to a particle, as the mass density would become infinite. In a continuum, a particle means a *material point* (ainepiste) or a continuum element or physical point in contrast to a *spatial point* (spatiaalipiste). Each continuum element or particle contains in fact — to be definable in practice — an enormous number of other kind of "particles": molecules. A body in mechanics is a *closed system* consisting all the time of the same continuum particles. When a body moves, the values of the coordinates of its particles change.

10.2 LAGRANGIAN DESCRIPTION

10.2.1 Motion

In the *Lagrangian description* or material description the independent variables are the *material coordinates* or *Lagrangian coordinates* (ainekoordinaatti, Lagrangen koordinaatti) a, b, c and time t. Quantities a, b, c are the coordinates of a generic particle of the continuum in the initial or *reference configuration* (referenssitila) selected for the body (Figure 10.1). The reference configuration is taken normally to be the configuration of the body at the moment of time $t={}^0t$ when an event under consideration starts. The quantities in the reference configuration are often equipped with a left superscript 0; see Remark 10.1.

The relation

$$\boxed{\boldsymbol{r} = \boldsymbol{r}({}^0\boldsymbol{r}, t)} \tag{1}$$

or

$$\begin{aligned} x &= x(a,b,c,t), \\ y &= y(a,b,c,t), \\ z &= z(a,b,c,t) \end{aligned} \tag{1'}$$

gives the configuration of the body at an arbitrary moment of time t. Representation (1) corresponds to the representation $\boldsymbol{r}_i = \boldsymbol{r}_i(t)$ of particle mechanics. As the continuum contains an unbounded number of particles, we cannot, however, identify them in practice by numbering them. The values a, b, c serve for this purpose. The formulas above thus tell that a particle which had in the reference configuration coordinate values a, b, c, has at time t coordinate values x, y, z.

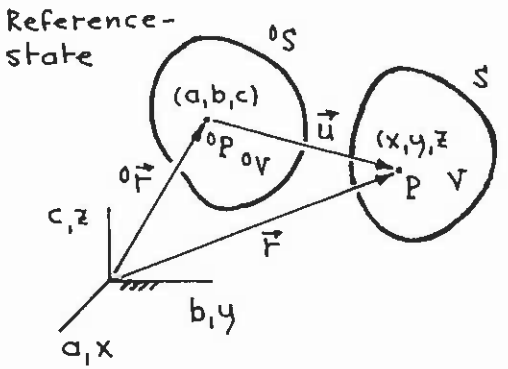


Figure 10.1 Motion of a body.

Remark 10.1. The symbols used in the literature for the coordinates especially when large displacements are considered vary quite a lot. Table 10.1 gives some kind of summary of the notations to be used in this text.

Table 10.1 Some notations for coordinates

$a\ b\ c$	$a_1\ a_2\ a_3$	${}^0x\ {}^0y\ {}^0z$	${}^0x_1\ {}^0x_2\ {}^0x_3$	
$x\ y\ z$	$x_1\ x_2\ x_3$	${}^tx\ {}^ty\ {}^tz$	${}^tx_1\ {}^tx_2\ {}^tx_3$	${}^{t+\Delta t}x_1\ {}^{t+\Delta t}x_2\ {}^{t+\Delta t}x_3$

The first and second row contain the symbols used in the reference and in the current configuration, respectively. It depends on the case at hand which symbols seem to be the most appropriate. Left superscripts are employed extensively in Bathe (1996) and we follow somewhat that convention. In this section the notations of the first column are used. □

10.2.2 Material time derivative

Let us consider an arbitrary function of position and time $f({}^0\boldsymbol{r},t)$ in the Lagrangian description. An example could be say the density $\rho({}^0\boldsymbol{r},t)$ of a continuum. The differential change of the value of f (we employ here rectangular cartesian coordinates so that $f = f(a,b,c,t)$) due to the changes $\mathrm{d}a$, $\mathrm{d}b$, $\mathrm{d}c$, $\mathrm{d}t$ of the independent variables a, b, c, t is

$$\mathrm{d}f = \frac{\partial f}{\partial a}\mathrm{d}a + \frac{\partial f}{\partial b}\mathrm{d}b + \frac{\partial f}{\partial c}\mathrm{d}c + \frac{\partial f}{\partial t}\mathrm{d}t. \tag{2}$$

In mechanics we are usually interested in the changes experienced by continuum material elements. For instance, given the change of density of a certain continuum material element (particle), we can determine from the appropriate constitutive law the corresponding pressure change for that element. Thus when calculating the change we must consider a certain particle and keep the values of a, b, c fixed in (2) so that

$$\mathrm{d}a = 0, \quad \mathrm{d}b = 0, \quad \mathrm{d}c = 0. \tag{3}$$

Dividing equation (2) further by the time increment $\mathrm{d}t$ and denoting $\mathrm{d} \to \mathrm{D}$ we obtain the following expression for the rate of change of f:

$$\boxed{\dot{f} \equiv \frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t}.} \tag{4}$$

The term $\mathrm{D}f/\mathrm{D}t \equiv \dot{f}$ is called *material time derivative* or briefly *material derivative* (aineellinen aikaderivaatta, ainederivaatta) of f. Other names used are total derivative, derivative following the particle, substantial derivative. As the names imply, *this derivative measures the rate of change experienced by a certain fixed material element.*

In words, *the material time derivative of a function in Lagrangian representation is obtained as the partial derivative of this function with respect to time.* This result is valid also for vector or tensor functions. In operator form we have

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t}. \tag{5}$$

Using (4), it is easy to show that the material time derivative obeys the conventional calculation rules for derivatives such as, $\mathrm{D}(f_1 + f_2)/\mathrm{D}t =$, $\mathrm{D}f_1/\mathrm{D}t + \mathrm{D}f_2/\mathrm{D}t$, $\mathrm{D}(f_1 f_2)/\mathrm{D}t = \mathrm{D}f_1/\mathrm{D}t \cdot f_2 + f_1 \cdot \mathrm{D}f_2/\mathrm{D}t$, etc.

Remark 10.2. The notation $\mathrm{D}f/\mathrm{D}t$ is more usual in fluid mechanics than in solid mechanics. In fluid mechanics, when the Eulerian representation is used, the material time derivative expression is essentially more complicated than in

solid mechanics. In fact, it makes most of the the fluid mechanics equations right from the start nonlinear. Let us also remark that the notation $()^{\cdot}$ where the dot is to the right of the symbol is not very common for the material time derivative. (It appears for instance in the reference Prager (1961).) Often just the notation $\dot{()}$ is used. However, we reserve the notation $\dot{()}$ for quantities which depend only on one independent variable (time t). □

Remark 10.3. We also need to measure the rate of change experienced by material bodies. These derivatives may be again called material time derivatives. However, the *quantities associated with a finite body are functions of only of time* (integration over the mass removes the dependence on space coordinates) and here the material time derivatives are just ordinary time derivatives and

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\mathrm{d}f}{\mathrm{d}t} = \dot{f}. \tag{6}$$

This for instance explains the dot notations used in expressing the axioms in Chapter 9. □

10.2.3 Displacement, velocity and acceleration

The path of a certain particle is obtained from relationship (1) or (1') by keeping ${}^0\boldsymbol{r}$ or a, b, c fixed and letting time t change. The *velocity* $\boldsymbol{v}$ is obtained thus as the material time derivative of function $\boldsymbol{r}$ or

$$\boxed{\boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{\mathrm{D}\boldsymbol{r}}{\mathrm{D}t} = \frac{\partial \boldsymbol{r}({}^0\boldsymbol{r},t)}{\partial t}.} \tag{7}$$

Let us remark that the symbols a, b, c and x, y, z in (1') could mean any coordinates defining the positions of the particles. In the following they, however, mean simply rectangular cartesian coordinates so that the a- and x-axes coincide, etc., as is shown in Figure 10.1. When the corresponding unit basis vectors are $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, we thus obtain the expressions

$${}^0\boldsymbol{r} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \tag{8}$$

and

$$\boldsymbol{r} = x(a,b,c,t)\boldsymbol{i} + y(a,b,c,t)\boldsymbol{j} + z(a,b,c,t)\boldsymbol{k}. \tag{9}$$

As the basis vectors do not depend on time, a detailed expression is

$$\begin{aligned}\boldsymbol{v} &= \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} = \frac{\partial x}{\partial t}\boldsymbol{i} + \frac{\partial y}{\partial t}\boldsymbol{j} + \frac{\partial z}{\partial t}\boldsymbol{k} \\ &= v_a\boldsymbol{i} + v_b\boldsymbol{j} + v_c\boldsymbol{k}.\end{aligned} \tag{7'}$$

The *acceleration* $\boldsymbol{a}$ is obtained as the material time derivative of velocity:

$$\boxed{\boldsymbol{a} = \boldsymbol{v}^{\cdot} = \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \frac{\partial \boldsymbol{v}({}^{0}\boldsymbol{r},t)}{\partial t} = \boldsymbol{r}^{\cdot\cdot} = \frac{\mathrm{D}^2\boldsymbol{r}}{\mathrm{D}t^2} = \frac{\partial^2 \boldsymbol{r}({}^{0}\boldsymbol{r},t)}{\partial t^2}} \tag{10}$$

or in rectangular cartesian coordinates

$$\begin{aligned} \boldsymbol{a} &= x^{\cdot\cdot}\boldsymbol{i} + y^{\cdot\cdot}\boldsymbol{j} + z^{\cdot\cdot}\boldsymbol{k} = \frac{\partial^2 x}{\partial t^2}\boldsymbol{i} + \frac{\partial^2 y}{\partial t^2}\boldsymbol{j} + \frac{\partial^2 x}{\partial t^2}\boldsymbol{k} \\ &= a_a\boldsymbol{i} + a_b\boldsymbol{j} + a_c\boldsymbol{k}. \end{aligned} \tag{10'}$$

In practice the relationships (1) and (1') are written in the forms

$$\boxed{\boldsymbol{r} = {}^{0}\boldsymbol{r} + \boldsymbol{u}({}^{0}\boldsymbol{r},t)} \tag{11}$$

or

$$\begin{aligned} x &= a + u(a,b,c,t), \\ y &= b + v(a,b,c,t), \\ z &= c + w(a,b,c,t), \end{aligned} \tag{11'}$$

where

$$\boldsymbol{u} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k} \tag{12}$$

is the *displacement*. Because ${}^{0}\boldsymbol{r}$ is a constant for a continuum particle, we obtain for the velocity and acceleration the following alternative formulas:

$$\boxed{\boldsymbol{v} = \boldsymbol{u}^{\cdot} = \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}({}^{0}\boldsymbol{r},t)}{\partial t}} \tag{13}$$

or

$$\begin{aligned} \boldsymbol{v} &= u^{\cdot}\boldsymbol{i} + v^{\cdot}\boldsymbol{j} + w^{\cdot}\boldsymbol{k} = \frac{\partial u}{\partial t}\boldsymbol{i} + \frac{\partial v}{\partial t}\boldsymbol{j} + \frac{\partial w}{\partial t}\boldsymbol{k} \\ &= v_a\boldsymbol{i} + v_b\boldsymbol{j} + v_c\boldsymbol{k} \end{aligned} \tag{13'}$$

and

$$\boxed{\boldsymbol{a} = \boldsymbol{u}^{\cdot\cdot} = \frac{\mathrm{D}^2\boldsymbol{u}}{\mathrm{D}t^2} = \frac{\partial^2 \boldsymbol{u}({}^{0}\boldsymbol{r},t)}{\partial t^2}} \tag{14}$$

or

$$\begin{aligned} \boldsymbol{a} &= u^{\cdot\cdot}\boldsymbol{i} + v^{\cdot\cdot}\boldsymbol{j} + w^{\cdot\cdot}\boldsymbol{k} = \frac{\partial^2 u}{\partial t^2}\boldsymbol{i} + \frac{\partial^2 v}{\partial t^2}\boldsymbol{j} + \frac{\partial^2 w}{\partial t^2}\boldsymbol{k} \\ &= a_a\boldsymbol{i} + a_b\boldsymbol{j} + a_c\boldsymbol{k}. \end{aligned} \tag{14'}$$

10.2.4 Deformation

In this section we follow mainly the presentation in Washizu (1982). We consider the change of shape or *deformation* (muodonmuutos, deformaatio) of a differential continuum element due to the motion. We first note that as in the reference configuration

$${}^{0}\boldsymbol{r} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}, \tag{15}$$

we have the formulas

$$\boldsymbol{i} = \frac{\partial {}^{0}\boldsymbol{r}}{\partial a}, \quad \boldsymbol{j} = \frac{\partial {}^{0}\boldsymbol{r}}{\partial b}, \quad \boldsymbol{k} = \frac{\partial {}^{0}\boldsymbol{r}}{\partial c} \tag{16}$$

for the unit basis vectors.

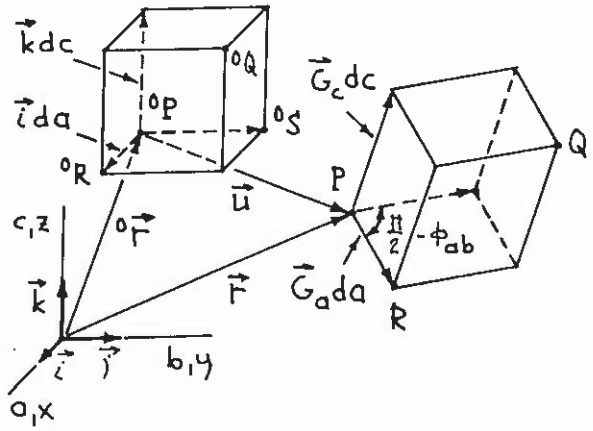


Figure 10.2 Deformation of a continuum element.

The representation (1) or (11) or using rectangular cartesian coordinates:

$$\boldsymbol{r} \equiv x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (a+u)\boldsymbol{i} + (b+v)\boldsymbol{j} + (c+w)\boldsymbol{k} \tag{17}$$

can be considered as a mapping which maps the domain ${}^0V$ in the reference configuration into a domain V in the current configuration. Two arbitrary infinitely close neighbouring points ${}^0P$ and ${}^0Q$ in the reference configuration are mapped into points P and Q in the current configuration (Figure 10.2).

In the reference configuration the position vector of point ${}^0Q$ with respect to point ${}^0P$ is the total differential

$$\mathrm{d}^0\boldsymbol{r} = \frac{\partial {}^0\boldsymbol{r}}{\partial a}\mathrm{d}a + \frac{\partial {}^0\boldsymbol{r}}{\partial b}\mathrm{d}b + \frac{\partial {}^0\boldsymbol{r}}{\partial c}\mathrm{d}c = \mathrm{d}a\boldsymbol{i} + \mathrm{d}b\boldsymbol{j} + \mathrm{d}c\boldsymbol{k}. \tag{18}$$

Formulas (16) have been made use of. Similarly, in the current configuration the position vector of point Q with respect to point P is the total differential

$$\mathrm{d}\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial a}\mathrm{d}a + \frac{\partial \boldsymbol{r}}{\partial b}\mathrm{d}b + \frac{\partial \boldsymbol{r}}{\partial c}\mathrm{d}c \tag{19}$$

or (see (17))

$$\begin{aligned}
\mathrm{d}x &= \frac{\partial x}{\partial a}\mathrm{d}a + \frac{\partial x}{\partial b}\mathrm{d}b + \frac{\partial x}{\partial c}\mathrm{d}c = (1+\frac{\partial u}{\partial a})\mathrm{d}a + \frac{\partial u}{\partial b}\mathrm{d}b + \frac{\partial u}{\partial c}\mathrm{d}c,\\
\mathrm{d}y &= \frac{\partial y}{\partial a}\mathrm{d}a + \frac{\partial y}{\partial b}\mathrm{d}b + \frac{\partial y}{\partial c}\mathrm{d}c = \frac{\partial v}{\partial a}\mathrm{d}a + (1+\frac{\partial v}{\partial b})\mathrm{d}b + \frac{\partial v}{\partial c}\mathrm{d}c,\\
\mathrm{d}z &= \frac{\partial z}{\partial a}\mathrm{d}a + \frac{\partial z}{\partial b}\mathrm{d}b + \frac{\partial z}{\partial c}\mathrm{d}c = \frac{\partial w}{\partial a}\mathrm{d}a + \frac{\partial w}{\partial b}\mathrm{d}b + (1+\frac{\partial w}{\partial c})\mathrm{d}c.
\end{aligned} \tag{20}$$

It should be realized that we here consider the dependence of function $\boldsymbol{r}$ on the position in the reference state at a certain moment of time and in calculating the total differential we thus put $\mathrm{d}t = 0$.

As a special case we consider the motion of the vectorial material fibers

$$\frac{\partial {}^0\boldsymbol{r}}{\partial a}\mathrm{d}a = \boldsymbol{i}\,\mathrm{d}a, \quad \frac{\partial {}^0\boldsymbol{r}}{\partial b}\mathrm{d}b = \boldsymbol{j}\,\mathrm{d}b, \quad \frac{\partial {}^0\boldsymbol{r}}{\partial c}\mathrm{d}c = \boldsymbol{k}\,\mathrm{d}c \tag{21}$$

of a rectangular parallelepiped in the reference configuration (Figure 10.2). They are mapped into the vectors

$$\frac{\partial \boldsymbol{r}}{\partial a}\mathrm{d}a = \boldsymbol{G}_a\,\mathrm{d}a, \quad \frac{\partial \boldsymbol{r}}{\partial b}\mathrm{d}b = \boldsymbol{G}_b\,\mathrm{d}b, \quad \frac{\partial \boldsymbol{r}}{\partial c}\mathrm{d}c = \boldsymbol{G}_c\,\mathrm{d}c, \tag{22}$$

where

$$\begin{aligned}
\boldsymbol{G}_a &\equiv \frac{\partial \boldsymbol{r}}{\partial a} = (1+\frac{\partial u}{\partial a})\boldsymbol{i} + \frac{\partial v}{\partial a}\boldsymbol{j} + \frac{\partial w}{\partial a}\boldsymbol{k},\\
\boldsymbol{G}_b &\equiv \frac{\partial \boldsymbol{r}}{\partial b} = \frac{\partial u}{\partial b}\boldsymbol{i} + (1+\frac{\partial v}{\partial b})\boldsymbol{j} + \frac{\partial w}{\partial b}\boldsymbol{k},\\
\boldsymbol{G}_c &\equiv \frac{\partial \boldsymbol{r}}{\partial c} = \frac{\partial u}{\partial c}\boldsymbol{i} + \frac{\partial v}{\partial c}\boldsymbol{j} + (1+\frac{\partial w}{\partial b})\boldsymbol{k}.
\end{aligned} \tag{23}$$

$\boldsymbol{G}_a$, $\boldsymbol{G}_b$, $\boldsymbol{G}_c$ are usually not perpendicular to each other and not unit vectors except in a rigid body motion. The rectangular parallelepiped material element in the reference configuration is thus transformed into a parallelopided usually no more of a rectangular shape.

We now try to charaterize the change of shape in more detail. The square of the length of the material line element ${}^0P\,{}^0Q$ in the reference configuration is

$$(\mathrm{d}^0s)^2 = \mathrm{d}^0\boldsymbol{r}\cdot\mathrm{d}^0\boldsymbol{r} = \mathrm{d}a\mathrm{d}a + \mathrm{d}b\mathrm{d}b + \mathrm{d}c\mathrm{d}c. \tag{24}$$

The square of the length of the deformed line element PQ or the the vector

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{G}_a\mathrm{d}a + \boldsymbol{G}_b\mathrm{d}b + \boldsymbol{G}_c\mathrm{d}c \tag{25}$$

is

$$\begin{aligned}
(\mathrm{d}s)^2 = \mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r} = {} & G_{aa}\mathrm{d}a\mathrm{d}a + G_{ab}\mathrm{d}a\mathrm{d}b + G_{ac}\mathrm{d}a\mathrm{d}c +\\
& + G_{ba}\mathrm{d}b\mathrm{d}a + G_{bb}\mathrm{d}b\mathrm{d}b + G_{bc}\mathrm{d}b\mathrm{d}c +\\
& + G_{ca}\mathrm{d}c\mathrm{d}a + G_{cb}\mathrm{d}c\mathrm{d}b + G_{cc}\mathrm{d}c\mathrm{d}c,
\end{aligned} \tag{26}$$

where the notations

$$\begin{aligned}
&G_{aa} = \boldsymbol{G}_a\cdot\boldsymbol{G}_a, \quad G_{ab} = G_{ba} = \boldsymbol{G}_a\cdot\boldsymbol{G}_b,\\
&\cdots \qquad\qquad \cdots
\end{aligned} \tag{27}$$

have been used. In index notation

$$(\mathrm{d}^0s)^2 = \delta_{ij}\mathrm{d}a_i\mathrm{d}a_j = \mathrm{d}a_i\mathrm{d}a_i \tag{24'}$$

and

$$(\mathrm{d}s)^2 = G_{ij}\mathrm{d}a_i\mathrm{d}a_j. \tag{26'}$$

The general form of (26') is

$$\boxed{(\mathrm{d}s)^2 = \mathrm{d}^0\boldsymbol{r}\cdot\boldsymbol{G}\cdot\mathrm{d}^0\boldsymbol{r}.} \tag{27}$$

G is called *Green's deformation tensor* (Greenin deformaatiotensori), Malvern (1969). G_{ij} is the index representation of it in rectangular cartesian coordinates.

Let us consider the geometrical meaning of the term G_{aa}. The squares of the lengths of the line element ${}^0P\,{}^0R$ in the reference and in the current configuration are (Figure 10.2), respectively,

$$(\mathrm{d}\,{}^0s)^2 = (\mathrm{d}a)^2, \qquad (\mathrm{d}s)^2 = G_{aa}(\mathrm{d}a)^2. \tag{28}$$

Thus the relative change in length is

$$\frac{\mathrm{d}s - \mathrm{d}\,{}^0s}{\mathrm{d}\,{}^0s} = \sqrt{G_{aa}} - 1. \tag{29}$$

The interpretation of the terms G_{bb} and G_{cc} is analogous.

Next we consider the line elements ${}^0P\,{}^0R$ and ${}^0P\,{}^0S$ (Figure 10.2), which are before the deformation perpendicular to each other. When the angle between the line elements PR and PS is denoted by $\pi/2 - \phi_{ab}$ (ϕ_{ab} is thus the angle indicating the decrease of the right angle), we obtain from the definition of the scalar product

$$\boldsymbol{G}_a\,\mathrm{d}a \cdot \boldsymbol{G}_b\,\mathrm{d}b = |\boldsymbol{G}_a||\boldsymbol{G}_b|\mathrm{d}a\mathrm{d}b\cos(\frac{\pi}{2} - \phi_{ab}) \tag{30}$$

or

$$G_{ab}\mathrm{d}a\mathrm{d}b = \sqrt{G_{aa}}\sqrt{G_{bb}}\mathrm{d}a\mathrm{d}b\sin\phi_{ab} \tag{31}$$

so that

$$G_{ab} = G_{ba} = \sqrt{G_{aa}}\sqrt{G_{bb}}\sin\phi_{ab}\,. \tag{32}$$

This gives some kind of geometric interpretation for the term G_{ab}. The interpretation of the terms G_{bc} and G_{ca} is analogous.

The six independent components G_{ij} of the deformation tensor determine thus completely the geometry of the deformed parallelepiped. The strain tensor $\boldsymbol{E}$ is defined in general form by the following equation:

$$\boxed{(\mathrm{d}s)^2 - (\mathrm{d}\,{}^0s)^2 = 2\,\mathrm{d}\,{}^0\boldsymbol{r} \cdot \boldsymbol{E} \cdot \mathrm{d}\,{}^0\boldsymbol{r}.} \tag{33}$$

Using index notation and rectangular cartesian coordinates, we have the counterpart

$$\boxed{(\mathrm{d}s)^2 - (\mathrm{d}\,{}^0s)^2 = 2\,E_{ij}\mathrm{d}a_i\mathrm{d}a_j\,.} \tag{33'}$$

Comparison with formulas (24') and (26') gives the expression

$$E_{ij} = \frac{1}{2}(G_{ij} - \delta_{ij}). \tag{34}$$

Continuing for brevity in index notation, we have

$$\boldsymbol{r} = (a_k + u_k)\boldsymbol{i}_k\,, \tag{8'}$$

$$\boldsymbol{G}_i = \frac{\partial \boldsymbol{r}}{\partial a_i} = \frac{\partial a_k}{\partial a_i}\boldsymbol{i}_k + \frac{\partial u_k}{\partial a_i}\boldsymbol{i}_k = \delta_{ki}\boldsymbol{i}_k + \frac{\partial u_k}{\partial a_i}\boldsymbol{i}_k = \boldsymbol{i}_i + \frac{\partial u_k}{\partial a_i}\boldsymbol{i}_k\,, \tag{23'}$$

$$\begin{aligned} G_{ij} &= \boldsymbol{G}_i \cdot \boldsymbol{G}_j = (\boldsymbol{i}_i + \frac{\partial u_k}{\partial a_i}\boldsymbol{i}_k) \cdot (\boldsymbol{i}_j + \frac{\partial u_l}{\partial a_j}\boldsymbol{i}_l) \\ &= \boldsymbol{i}_i \cdot \boldsymbol{i}_j + \frac{\partial u_l}{\partial a_j}\boldsymbol{i}_i \cdot \boldsymbol{i}_l + \frac{\partial u_k}{\partial a_i}\boldsymbol{i}_k \cdot \boldsymbol{i}_j + \frac{\partial u_k}{\partial a_i}\frac{\partial u_l}{\partial a_j}\boldsymbol{i}_k \cdot \boldsymbol{i}_l \\ &= \delta_{ij} + \frac{\partial u_l}{\partial a_j}\delta_{il} + \frac{\partial u_k}{\partial a_i}\delta_{kj} + \frac{\partial u_k}{\partial a_i}\frac{\partial u_l}{\partial a_j}\delta_{kl} = \delta_{ij} + \frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}. \end{aligned} \tag{35}$$

Some of the manipulation rules presented in Section A.2 have been made use of. Thus the components of the strain tensor (34) are expressed in displacements

$$\boxed{E_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}).} \tag{36}$$

Tensor $\boldsymbol{E}$ is called usually the *Green-Lagrange (finite) strain tensor* or here briefly the *Green strain tensor* (Green-Lagrangen venymätensori, Greenin venymätensori). Its components (36) are in detail, if the summation convention is not used,

$$\begin{aligned} &E_{aa} = \frac{\partial u}{\partial a} + \frac{1}{2}[(\frac{\partial u}{\partial a})^2 + (\frac{\partial v}{\partial a})^2 + (\frac{\partial w}{\partial a})^2], \\ &\cdots, \\ &E_{bc} = E_{cb} = \frac{1}{2}(\frac{\partial v}{\partial c} + \frac{\partial w}{\partial b} + \frac{\partial u}{\partial b}\frac{\partial u}{\partial c} + \frac{\partial v}{\partial b}\frac{\partial v}{\partial c} + \frac{\partial w}{\partial b}\frac{\partial w}{\partial c}), \\ &\cdots. \end{aligned} \tag{37}$$

We obtain from formula (34)

$$G_{aa} = 1 + 2E_{aa}, \quad G_{ab} = 2E_{ab}, \quad \ldots \quad \ldots \tag{38}$$

If the strain components are small ($E_{ij} << 1$), formulas (29) and (32) give the approximate expressions

$$\frac{ds - d^0 s}{d^0 s} = \sqrt{1 + 2E_{aa}} - 1 \approx 1 + \frac{1}{2} 2E_{aa} - 1 = E_{aa},$$
$$\phi_{ab} = \arcsin \frac{2E_{ab}}{\sqrt{1 + 2E_{aa}}\sqrt{1 + 2E_{bb}}} \approx \arcsin(2E_{ab}) = 2E_{ab}. \tag{39}$$

Thus in the *small deformation theory* (pienten muodonmuutosten teoria), the terms E_{aa}, E_{bb}, E_{cc} represent unit extensions or *normal strains* (suhteellinen pituudenmuutos, venymä) of material line elements originally in the a-, b-, c-directions in the reference configuration. The terms $2E_{ab}$, $2E_{bc}$, $2E_{ca}$ represent the changes in the angles between line element originally perpendicula to each other or so called *shearing strains* (liukuma, leikkausmuodonmuutos).

It should be realized that the (local) deformations can remain small even if the body as whole can obtain large displacements (consider for instance the displacements of a spring in a watch). In engineering structures the strains can be often considered as small. For instance for steel the strain components $\leq 1/1000$ for the behaviour to remain elastic.

In the *infinitesimal displacement* or *small displacement theory* (pienten siirtymien teoria) in the strain expressions (36) only the linear terms in the displacements are left:

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right).} \tag{40}$$

These strain components are called *infinitesimal strain* components or *small strain* components or sometimes *engineering strain* components (infinitesimaalinen venymä, pieni venymä, insinöörivenymä). These expressions are valid if all derivatives $\partial u_i / \partial a_j$ are small, that is $\partial u_i / \partial a_j << 1$, in which case the products of the derivatives can be neglected in comparison to the derivatives.

Remark 10.4. In engineerin literature, the following notations are often used for the engineering strains. First $\varepsilon_{aa} \to \varepsilon_a$, etc. and $2\varepsilon_{ab} \to \gamma_{ab}$. The resulting expressions are thus

$$\varepsilon_a = \frac{\partial u}{\partial a}, \quad \gamma_{bc} = \gamma_{cb} = \frac{\partial v}{\partial c} + \frac{\partial w}{\partial b},$$
$$\varepsilon_b = \frac{\partial v}{\partial b}, \quad \gamma_{ca} = \gamma_{ac} = \frac{\partial w}{\partial a} + \frac{\partial u}{\partial c}, \tag{41}$$
$$\varepsilon_c = \frac{\partial w}{\partial c}, \quad \gamma_{ab} = \gamma_{ba} = \frac{\partial u}{\partial b} + \frac{\partial v}{\partial a}.$$

Further, as in the small displacement theory there is usually no need to make a difference between the reference and the current configuration so that we can proceed with only one coordinate system, we often let the Lagrange coordinates be denoted by the symbols x, y, z so to achieve perhaps the most common notation found in the engineering literature. □

10.2.5 Material time derivative of a volume integral

In mehanics, we need to evaluate in addition to the rate of change of functions of position and time also the rate of change of certain volume integrals. This has been commented on already in Remark 10.3. In fact the axioms of Chapter 9 concern finite bodies and the quantities appearing in them such as mass, momentum, kinetic energy, etc. are expressed as volume integrals.

In the Lagrangian representation the mathematical domain under study is the domain of space filled by the material body in the reference configuration. The body may move in any way but "*the mathematics takes place all the time in the fixed reference domain*".

Thus in the Lagrangian representation, a typical volume integral dealing with a body is of the form

$$I(t) = \int_{{}^0V} f({}^0\mathbf{r}, t)\, d^0V = \int_{{}^0V} f(a,b,c,t)\, d^0V. \tag{42}$$

As an example we may mention the expression for the mass of a body:

$$m(t) = \int_{{}^0V} \rho({}^0\mathbf{r}, t)\, J({}^0\mathbf{r}, t)\, d^0V, \tag{43}$$

where m is actually due to the principle of conservation of mass a constant. J is the Jacobian determinant connected to the mapping (1), see Section 11.1.

When the integration over the space is considered to be performed at each moment of time, the integrals are seen to become functions of time only: $I = I(t)$. Thus the conventional time derivative of a function of one variable is at the same time the material time derivative experienced by the body and we can write

$$I^{\cdot} \equiv \frac{DI}{Dt} = \dot{I} \equiv \frac{dI}{dt}. \tag{44}$$

The differentiation of an integral like (42) with respect to a parameter t (here t is called a parameter with respect to the integration as the integration is with respect to the coordinates a, b, c) can performed according to mathematics as

$$\boxed{I^{\cdot} \equiv \frac{DI}{Dt} = \dot{I} \equiv \frac{d}{dt}\int_{{}^0V} f \, d^0V = \int_{{}^0V} \frac{\partial f}{\partial t} d^0V.} \tag{45}$$

10.3 REFERENCES

Bathe, K.-J. (1996): *Finite Element Procedures*, Prentice Hall, Englewoods Ciffs.

Malvern, L. E. (1969): *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall, Englewoods Ciffs.

Prager, W. (1961): *Introduction to Mechanics of Continua*, Ginn and Company, Boston.

Washizu, K. (1982): *Variational Methods in Elasticity & Plasticity*, 3rd ed., Pergamon Press, Oxford.

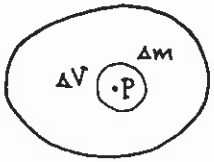
CHAPTER 11

KINETICS

The contents of this chapter are still very inadequate.

11.1 PRINCIPLE OF CONSERVATION OF MASS

The *density* (tiheys) ρ ($[\rho] = \mathrm{kg/m^3}$) of a continuum at a certain point P (Figure 11.1) at a certain moment of time is defined in principle as the limit



$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{\mathrm{d}m}{\mathrm{d}V}. \tag{1}$$

Figure 11.1 Mass Δm and volume ΔV.

Here ΔV is the volume inside a small closed surface containing point P and Δm is the corresponding mass. The volume ΔV in the definition (1) is made smaller and smaller the point P remaining inside the surface. The density is thus of the form $\rho = \rho(r,t) = \rho(x,t) = \rho(x,y,z,t)$. We obtain in addition the formal expression

$$\mathrm{d}m = \rho \,\mathrm{d}V, \tag{2}$$

which is useful in transforming the integrals over mass to integrals over volume.

Remark 11.1. As is implied above, we may use alternatively the symbols $\boldsymbol{r}$ or $\boldsymbol{x}$ or x, y, z to refer to dependence on position in the current configuration. Similarly, the notations ${}^0\boldsymbol{r}$ or $\boldsymbol{a}$ or a, b, c may be used for the same purpose in the reference configuration. □

The principle of conservation of mass is actually a purely kinematic law but as one of the axioms it is for consistency considered in this section. The total mass of a body at a moment of time t is (Figure 11.2)

$$m = \int dm = \int_V \rho \,\mathrm{d}V = \int_{V(t)} \rho(\boldsymbol{x},t)\, \mathrm{d}V. \tag{3}$$

In the reference configuration at the initial time ($t = {}^0t$)

$$m = \int dm = \int_{{}^0V} {}^0\rho \,\mathrm{d}{}^0V = \int_{{}^0V} {}^0\rho(\boldsymbol{a})\,\mathrm{d}{}^0V, \tag{4}$$

where ${}^0\rho(\boldsymbol{a})$ is thus the density at the initial time, a given quantity.

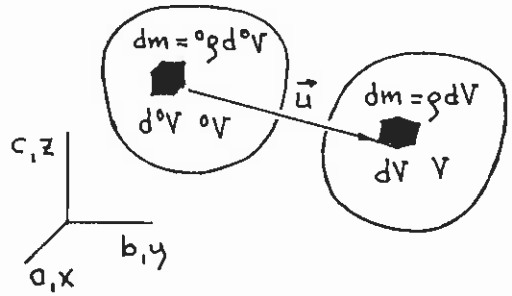


Figure 11.2 Motion of a body.

We transform the integral (3) the way shown in mathematics, e.g. Kreyszig (1967), over the reference volume ${}^0V$. The transformation is based on the mapping (10.2.1') or (10.2.11') or briefly

$$x_i = x_i(\boldsymbol{a},t) = a_i + u_i(\boldsymbol{a},t). \tag{5}$$

Integral (3) can be written

$$m = \int_{V(t)} \rho(\boldsymbol{x},t)\, \mathrm{d}V = \int_{{}^0V} \rho(\boldsymbol{x}(\boldsymbol{a},t))\, J(\boldsymbol{a},t)\,\mathrm{d}{}^0V, \tag{6}$$

where

$$J(\boldsymbol{a},t) \equiv \det\left[\frac{\partial x_i}{\partial a_j}\right] = \det\begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{bmatrix} = \det\begin{bmatrix} 1+\frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} & \frac{\partial u}{\partial c} \\ \frac{\partial v}{\partial a} & 1+\frac{\partial v}{\partial b} & \frac{\partial v}{\partial c} \\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial b} & 1+\frac{\partial w}{\partial c} \end{bmatrix} \tag{7}$$

is the so-called *Jacobian* or more precisely the *Jacobian determinant* (Jacobin determinantti).

On the basis of the principle of conservation of mass the mass m of a body is constant and thus comparing expressions (4) and (6) ($m({}^0t) = m(t)$) we obtain

$$\int_{{}^0V} ({}^0\rho - \rho J)\,\mathrm{d}{}^0V = 0. \tag{8}$$

Because this equation must be valid for any subdomain of ${}^0V$, we obtain the following *local form of the principle of conservation of mass*:

$$\boxed{{}^0\rho = \rho J.} \tag{9}$$

Remark 11.2. The line of thought used in the step between formulas (8) and (9) should be considered in more detail.

Let us consider a global form

$$\int_{{}^0V} f(\boldsymbol{a},t)\,\mathrm{d}^0V = 0, \tag{10}$$

valid for a finite body, where the integrand f (above $f = {}^0\rho - \rho J$) is a *continuous* function with respect to position and which may in addition depend on time. If equation (10) is valid for any subdomain Δ^0V of the original domain ${}^0V$ or if (even if (10) has been written first for a certain body, we remember that the axioms of mechanics are valid for an arbitrary body and thus also for any subbody of the original body under study)

$$\int_{\Delta^0V} f\,\mathrm{d}^0V = 0, \tag{11}$$

the local form

$$f = 0 \quad \text{in } {}^0V. \tag{12}$$

follows from this. The proof is based on an opposite proposition. Let us assume that f is different from zero and say positive at some interior point P of ${}^0V$. As a continuous function f is then positive even in some neighbourhood Δ^0V of P and the left hand side of (11) becomes positive. The proposition is thus wrong. The result obtained is valid also for vector- and tensor-valued functions. □

Formula (9) can be arrived at in an alternative way. The mass $\mathrm{d}m$ of a continuum element is constant (Figure 11.2)

$$\mathrm{d}m = {}^0\rho\,\mathrm{d}^0V = \rho\,\mathrm{d}V, \tag{13}$$

or

$$\frac{\mathrm{d}V}{\mathrm{d}^0V} = \frac{{}^0\rho}{\rho}. \tag{14}$$

From mathematics, the Jacobian determinant

$$J = \frac{\mathrm{d}V}{\mathrm{d}^0V}. \tag{15}$$

Combining (14) and (15) gives (9).

11.2 PRINCIPLE OF BALANCE OF MOMENTUM

11.2.1 Small displacements

Euler stress. The *stress vector* or *traction* (jännitysvektori, traktio) $\boldsymbol{t}$ acting on a surface element with an external unit normal vector $\boldsymbol{n}$ is defined by (Figure 11.3)

$$\boxed{\boldsymbol{t} = \lim_{\Delta S \to 0} \frac{\Delta \boldsymbol{F}}{\Delta S} = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}S}.} \tag{1}$$

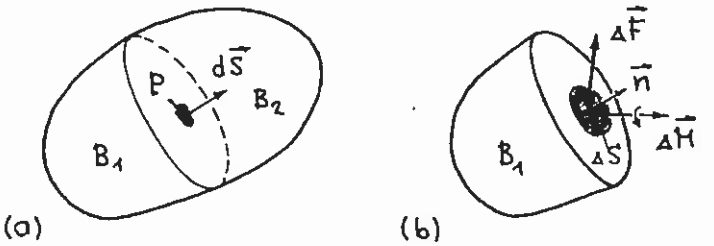


Figure 11.3 (a) Continuum divided into two parts. **(b)** Surface ΔS and the force system $\Delta \boldsymbol{F}$, $\Delta \boldsymbol{M}$ acting on it.

The stress vectors and their components acting on the faces of a rectangular parallelepided have the symbols shown in Figure 11. 4.

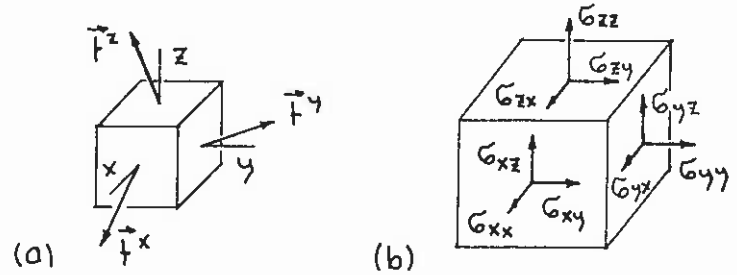


Figure 11.4 (a) Stress vectors corresponding to three different surface elements. **(b)** Stress components.

We have

$$\boxed{t^j = \sigma_{ji} i_i\,,} \tag{2}$$

or

$$\begin{aligned} t^x &= \sigma_{xx} i + \sigma_{xy} j + \sigma_{xz} k, \\ t^y &= \sigma_{yx} i + \sigma_{yy} j + \sigma_{yz} k, \\ t^z &= \sigma_{zx} i + \sigma_{zy} j + \sigma_{zz} k. \end{aligned} \tag{2'}$$

These stress concepts are referred to as *Euler stresses* (Eulerin jännitys) or *Cauchy stresses* (Cauchyn jännitys). It is essential that these stresses are defined by dividing the actual force acting on a surface element by the actual area of surface element.

Remark 11.3. In a somewhat analogous manner as discussed in Remark 10.4, engineering literature often employs the notations $\sigma_{xx} \to \sigma_x$, etc. and $\sigma_{xy} \to \tau_{xy}$, etc. so that instead of (2') we have

$$\begin{aligned} t^x &= \sigma_x i + \tau_{xy} j + \tau_{xz} k, \\ t^y &= \tau_{yx} i + \sigma_y j + \tau_{yz} k, \\ t^z &= \tau_{zx} i + \tau_{zy} j + \sigma_z k. \end{aligned} \tag{2''}$$ □

Equations of motion and traction-stress relations. The local forms of the principle of the balance of momentum are found to be

$$\rho b + \frac{\partial t^j}{\partial x_j} = \rho a \qquad \text{in } V \tag{3}$$

and

$$t = n_j t^j \qquad \text{on } S. \tag{4}$$

Here b is the body force intensity due to external forces per unit mass and n is the external unit normal vector to the body surface.

It should be noticed that the formulas above have been written for the current configuration of the body. In the small displacement theory we make no difference between the current and reference configuration and we can operate with the Euler stress. We can thus directly write down the approriate equations of motion

$$\boxed{{}^0\rho b + \frac{\partial t^j}{\partial a_j} = {}^0\rho a} \qquad \text{in } {}^0V \tag{5}$$

and the traction-stress relations

$$\boxed{t = {}^0n_j t^j} \qquad \text{on } {}^0S. \tag{6}$$

In the component forms we may used either the coordinate symbols a, b, c or x, y, z as agreed.

11.2.2 Large displacements

Kirchhoff stress. In connection with the large displacement theory the stress concepts become complicated. The Euler stress is no more suitable as the stress should be associated with given material particles and not with spatial points. This is for instance necessary to be able to employ the appropriate constitutive relation in an inhomogeneous body. We borrow again strongly from Washizu (1982).

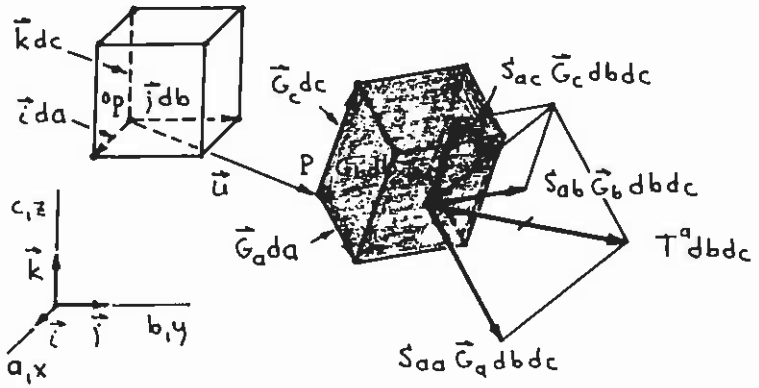


Figure 11.5 Surface force $T^a dbdc$ acting on a face of a parallelepiped.

Figure 11.5 corresponds to Figure 10.2 now with certain additional notation. A resultant surface force $T^a dbdc$ acts on the face of the deformed parallelepided, originally perpendicular to the a-axis. The stress T^a, defined in this way, is not the conventional stress or traction on the surface as in the expression $T^a dbdc$, $dbdc$ is the corresponding area in the reference configuration and not the actual area $|G_b db \times G_c dc|$. The quantity can be called *pseudo-stress* (pseudojännitys) to emphasize this. We shall denote the vector components of T^a in the directions of the lattice vectors G_a, G_b, G_c as shown in Figure 11.5, so that

$$T^a = S_{aa}G_a + S_{ab}G_b + S_{ac}G_c\,,$$
$$T^b = S_{ba}G_a + S_{bb}G_b + S_{bc}G_c\,, \qquad (7)$$
$$T^c = S_{ca}G_a + S_{cb}G_b + S_{cc}G_c\,.$$

The two other expressions refer similarly to the other two faces of the parallelepiped. In index notation, we have concisely

$$\boxed{T^j = S_{ji}G_i\,.} \qquad (7')$$

S_{ji} is the index representation in rectangular cartesian coordinates of the *second Piola-Kirchhoff stress tensor* (toisen lajin Piola-Kirchhoffin jännitystensori) or here in the following briefly the *Kirchhoff stress tensor* (Kirchhoffin jännitystensori).

Equations of motion and traction-stress relations. The local forms of the principle of the balance of momentum are found to be

$$\boxed{{}^0\rho b + \frac{\partial T^j}{\partial a_j} = {}^0\rho a} \quad \text{in } {}^0V \qquad (8)$$

and

$$\boxed{T = {}^0n_j t^j} \quad \text{on } {}^0S. \qquad (9)$$

11.3 PRINCIPLE OF BALANCE OF MOMENT OF MOMENTUM

11.3.1 Small displacements

The local form of the principle of balance of moment of momentum is that the Euler stress tensor is symmetric:

$$\boxed{\sigma_{ij} = \sigma_{ji}} \qquad (1)$$

or

$$\sigma_{bc} = \sigma_{cb}\,, \quad \sigma_{ca} = \sigma_{ac}\,, \quad \sigma_{ab} = \sigma_{ba}\,. \qquad (1')$$

11.3.2 Large displacements

The local form of the principle of balance of moment of momentum is that the Kirchhoff stress tensor is symmetric:

$$\boxed{S_{ij} = S_{ji}} \qquad (2)$$

or

$$S_{bc} = S_{cb}\,, \quad S_{ca} = S_{ac}\,, \quad S_{ab} = S_{ba}\,. \qquad (2')$$

11. 4 PRINCIPLE OF BALANCE OF ENERGY (missing)

11.5 BOUNDARY AND INITIAL CONDITIONS

11.5.1 Mechanical conditions

Basic case. The boundary ${}^0S$ of the body (Figure 11.6) is divided into two non-overlapping parts ${}^0S_u$ and ${}^0S_t$, forming the whole boundary so that using set theory notations, ${}^0S_u \cup {}^0S_t = {}^0S$ and ${}^0S_u \cap {}^0S_t = \emptyset$. Parts ${}^0S_u$ and ${}^0S_t$ do not need to consist of simply connected regions as is the case in Figure 11.6.

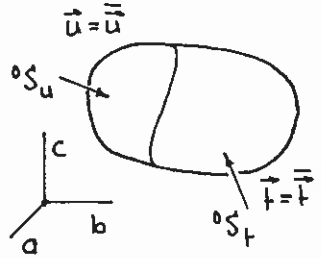


Figure 11.6 Mechanical boundary conditions.

The mechanical boundary conditions are conventionally of two kinds. The displacement is given on boundary ${}^0S_u$:

$$\boxed{u = \bar{u}} \quad \text{on } {}^0S_u \qquad (1)$$

or

$$u_a = \bar{u}_a\,,$$
$$u_b = \bar{u}_b\,, \qquad (1')$$
$$u_c = \bar{u}_c\,.$$

The traction is given on boundary ${}^0S_t$:

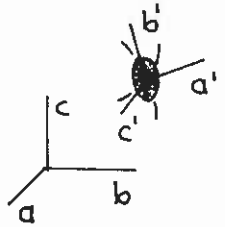
$$\boxed{t = \bar{t}} \quad \text{on } {}^0S_t \qquad (2)$$

or

$$
\begin{aligned}
t_a &= \bar{t}_a\,, \\
t_b &= \bar{t}_b\,, \\
t_c &= \bar{t}_c\,.
\end{aligned}
\tag{2'}
$$

The bar above a symbol refers to a given value.

Generalization. The basic case described above is convenient for notational reasons. However, the more general situation for each boundary point is as follows (Figure 11.7):



$$
\boxed{\begin{aligned}
u_{a'} &= \bar{u}_{a'} \quad \text{or} \quad t_{a'} = \bar{t}_{a'}\,, \\
u_{b'} &= \bar{u}_{b'} \quad \text{or} \quad t_{b'} = \bar{t}_{b'}\,, \\
u_{c'} &= \bar{u}_{c'} \quad \text{or} \quad t_{c'} = \bar{t}_{c'}\,.
\end{aligned}}
\tag{3}
$$

Figure 11.7 Local rotated coordinate system.

From each row either (but not both) of the conditions is to be selected. Here at every point on the boundary a rectangular cartesian coordinate system $a'b'c'$ is erected so that usually one axis (in the figure the a'-axis) coincide with the normal direction to the boundary. The quantities equipped with dashes refer to the components of vectors $\boldsymbol{u}$ and $\boldsymbol{t}$ in the local coordinate directions.

Remark 11.4. Boundary conditions have been considered above from the point of view of small displacements. In the case of large displacements the conditions are of the similar type but more complicated.

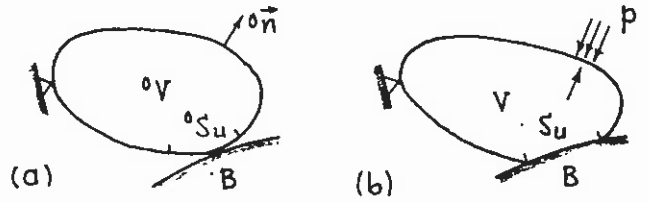


Figure 11.8 (a) Reference and (b) current configuration in a contact problem.

As an example we may consider Figure 11.8. The surface ${}^0S_u$, where the displacements are presribed, is not known in advance. □

Initial conditions. In statics we naturally have to give the initial configuration of the body to be able to predict its response to slowly varying exitations. In dynamics we have to give additionally the initial velocity field.

11.5.2 Thermal conditions (missing)

11.6 REFERENCES

Kreyszig, E. (1967): *Advanced Engineering Analysis*, 2nd ed., Wiley, New York

Washizu, K. (1982): *Variational Methods in Elasticity & Plasticity*, 3rd ed., Pergamon Press, Oxford.

CHAPTER 12

VIRTUAL WORK

12.1 SMALL DISPLACEMENTS

12.1.1 General

The mathematics literature dealing with the finite element method takes as its starting point instead differential equations so-called *weak forms* (heikko muoto) or weak formulations. These are scalar equations obtained from the governing differential equations and boundary and initial conditions by certain manipulations with so-called *weighting functions* or *test functions* (painofunktio, testifunktio). The ancient principle of virtual work is in fact from the mathematical point of view a weak form, where the virtual displacement has the role of the weighting function. In the mathematics literature, however, usually no demands on the weighting functions to be infinitesimal are made. Why is this done in connection with the principle of virtual work? Some answers to this question are given in Remarks 12.1 and 12.5.

12.1.2 Principle of virtual work

We try to present the derivation of the principle of virtual work using similar steps as in Section 5.2 for the particle system to emphasize the analogous features. To simplify the notation, we however first derive the result in the static case and only later include dynamics through the inertia forces the way explained in Remark 4.5.

According to Section 11.2.1 the governing equilibrium equations are in the body

$$ {}^0\rho \boldsymbol{b} + \frac{\partial \boldsymbol{t}^j}{\partial a_j} = 0 \qquad \text{in } {}^0V \tag{1}$$

and on the body surface

$$ \boldsymbol{t} = {}^0n_j \boldsymbol{t}^j \qquad \text{on } {}^0S. \tag{2}$$

We simplify the notation by dropping the left superscrifts in this section. The manipulation is as follows. We multiply (1) (scalar product) by an arbitrary vector

$$ \boldsymbol{w}(\boldsymbol{a}) = w_i(\boldsymbol{a})\boldsymbol{i}_i \tag{3}$$

and integrate the resulting equation over the body volume to obtain an equation

$$ \int_V (\rho \boldsymbol{b} \cdot \boldsymbol{w} + \frac{\partial \boldsymbol{t}^j}{\partial a_j} \cdot \boldsymbol{w}) \mathrm{d}V = 0. \tag{4}$$

This equation can be considered as some kind of weighted average equilibrium equation for the body the quantity $\boldsymbol{w}$ acting as the weighting function. The term due to the internal forces can be transformed into a more useful form by integration by parts. We obtain

$$ \begin{aligned} \int_V \frac{\partial \boldsymbol{t}^j}{\partial a_j} \cdot \boldsymbol{w} \mathrm{d}V &= -\int_V \boldsymbol{t}^j \cdot \frac{\partial \boldsymbol{w}}{\partial a_j} \mathrm{d}V + \int_S n_j \boldsymbol{t}^j \cdot \boldsymbol{w} \mathrm{d}S \\ &= \int_S \boldsymbol{t} \cdot \boldsymbol{w} \mathrm{d}S - \int_V \boldsymbol{t}^j \cdot \frac{\partial \boldsymbol{w}}{\partial a_j} \mathrm{d}V. \end{aligned} \tag{5}$$

Integration by parts formula (B.3.1a) and formula (2) have been made use of. The integration by parts formula referred to does not apply directly to the scalar product of two vector functions but it is not difficult to show that the formula is valid in an analogous form also in this case. This manipulation is similar to the one performed in Example 5.1 to change the virtual work expression of two pairwise forces into a more useful form. Now our equation is

$$ \int_V \rho \boldsymbol{b} \cdot \boldsymbol{w} \mathrm{d}V + \int_S \boldsymbol{t} \cdot \boldsymbol{w} \mathrm{d}S - \int_V \boldsymbol{t}^j \cdot \frac{\partial \boldsymbol{w}}{\partial a_j} \mathrm{d}V = 0. \tag{6}$$

We make the interpretation

$$ \begin{aligned} \boldsymbol{w} = \delta \boldsymbol{r} = \delta \boldsymbol{u} = \delta u_k \boldsymbol{i}_k &= \delta u_1 \boldsymbol{i}_1 + \delta u_2 \boldsymbol{i}_2 + \delta u_3 \boldsymbol{i}_3 \\ &= \delta u \boldsymbol{i} + \delta v \boldsymbol{j} + \delta w \boldsymbol{k}. \end{aligned} \tag{7}$$

Quantity $\delta \boldsymbol{r}(\boldsymbol{a}) = \delta \boldsymbol{u}(\boldsymbol{a})$ is the virtual displacement, that is, the variation of the position vector $\boldsymbol{r}$ or the displacement vector $\boldsymbol{u}$. Equation (6) obtains the form

$$ \boxed{\int_V \rho \boldsymbol{b} \cdot \delta \boldsymbol{u} \mathrm{d}V + \int_S \boldsymbol{t} \cdot \delta \boldsymbol{u} \mathrm{d}S - \int_V \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} \mathrm{d}V = 0.} \tag{8}$$

This is the *principle of virtual work* (statics) for a continuum in the case of small displacements. We can write also similarly as in (5.2.10):

$$ \delta' W \equiv \delta' W_{\text{ext}} + \delta' W_{\text{int}} = 0 \tag{9}$$

with

$$ \begin{aligned} \delta' W_{\text{ext}} &= \int_V \rho \boldsymbol{b} \cdot \delta \boldsymbol{u} \mathrm{d}V + \int_S \boldsymbol{t} \cdot \delta \boldsymbol{u} \mathrm{d}S, \\ \delta' W_{\text{int}} &= -\int_V \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} \mathrm{d}V. \end{aligned} \tag{10}$$

The integrand $\boldsymbol{\sigma}:\delta\boldsymbol{\varepsilon}$ in the internal virtual work expression is called *double-dot product* of the tensors $\boldsymbol{\sigma}$ and $\delta\boldsymbol{\varepsilon}$ and it means in index notation in rectangular cartesian coordinates, Malvern (1969),

$$\begin{aligned}\boldsymbol{\sigma}:\delta\boldsymbol{\varepsilon} = \sigma_{ij}\delta\varepsilon_{ij} &= \sigma_{11}\delta\varepsilon_{11} + \sigma_{12}\delta\varepsilon_{12} + \sigma_{13}\delta\varepsilon_{13} + \\ &+ \sigma_{21}\delta\varepsilon_{21} + \sigma_{22}\delta\varepsilon_{22} + \sigma_{23}\delta\varepsilon_{23} + \\ &+ \sigma_{31}\delta\varepsilon_{31} + \sigma_{32}\delta\varepsilon_{32} + \sigma_{33}\delta\varepsilon_{33}\,. \end{aligned} \tag{11}$$

The terms

$$\delta\varepsilon_{ij} = \delta\frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right) = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_j}{\partial a_i}\right). \tag{12}$$

are called *virtual strains* (virtuaalinen venymä). The detailed derivation concerning the internal virtual work is performed in Example 12.1.

Example 12.1. We consider the term

$$t^j \cdot \frac{\partial w}{\partial a_j} = t^j \cdot \frac{\partial \delta u}{\partial a_j} \tag{a}$$

in (6). Summation convention is used here and in the following. We have

$$t^j = \sigma_{ji} i_i, \qquad \delta u = \delta u_k i_k\,. \tag{b}$$

The first expression is from (11.2.2). Thus

$$\begin{aligned} t^j \cdot \frac{\partial \delta u}{\partial a_j} &= \sigma_{ji} i_i \cdot \frac{\partial \delta u_k}{\partial a_j} i_k = \sigma_{ji}\frac{\partial \delta u_k}{\partial a_j} i_i \cdot i_k = \sigma_{ji}\frac{\partial \delta u_k}{\partial a_j}\delta_{ik} \\ &= \sigma_{ji}\frac{\partial \delta u_i}{\partial a_j} = \frac{1}{2}\sigma_{ij}\frac{\partial \delta u_i}{\partial a_j} + \frac{1}{2}\sigma_{ji}\frac{\partial \delta u_i}{\partial a_j} = \frac{1}{2}\sigma_{ij}\frac{\partial \delta u_i}{\partial a_j} + \frac{1}{2}\sigma_{ij}\frac{\partial \delta u_j}{\partial a_i} \\ &= \sigma_{ij}\frac{1}{2}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_j}{\partial a_i}\right). \end{aligned} \tag{c}$$

On the second line use have been made of the symmetry of the stress tensor and summation indices have been changed. The expression of the small strain is from (10.2.40)

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right). \tag{d}$$

Its variation is seen to appear in (c). Thus finally

$$t^j \cdot \frac{\partial \delta u}{\partial a_j} = \sigma_{ij}\delta\varepsilon_{ij}\,. \tag{e}$$

Remark 12.1. If we let the weighting function w remain finite and take no specific interpretation, we have from formula (c) of Example 12.1 the result

$$t^j \cdot \frac{\partial w}{\partial a_j} = \sigma_{ij}\frac{1}{2}\left(\frac{\partial w_i}{\partial a_j} + \frac{\partial w_j}{\partial a_i}\right). \tag{13}$$

Especially, if we consider w to be some finite displacement u, we obtain

$$t^j \cdot \frac{\partial u}{\partial a_j} = \sigma_{ij}\varepsilon_{ij}\,, \tag{14}$$

where we have used the shorthand notation

$$\varepsilon_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right). \tag{15}$$

This consideration shows that it is not necessary to consider the weighting function to be infinitesimal. The essential thing is that *the weighting function appears linearly in the governing equations irrespective of the linearity or not of the equations*; only the type of distribution of the weighting functions has an effect on the final weak form. Formulas (14) and (15) somewhat explain the basis of the unit dummy load method discussed in Remark 5.11. The interpretation of the weighting as a variation is, however, often convenient. It for instance makes possible the step to the principle of stationary potential energy. □

Remark 12.2. A continuum problem usually have kinematical constraints in the form of boundary conditions

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \quad \text{on } S_u \tag{16}$$

as discussed in Section 11.5. When applying *kinematically admissible virtual displacements* we must thus take

$$\delta\boldsymbol{u} = \boldsymbol{0} \quad \text{on } S_u\,. \tag{17}$$

The unknown tractions $\boldsymbol{t}$ (constraint forces) on S_u then disappear from the formulation and the principle of virtual work reads

$$\boxed{\int_V \rho\boldsymbol{b}\cdot\delta\boldsymbol{u}\,\mathrm{d}V + \int_{S_t} \boldsymbol{t}\cdot\delta\boldsymbol{u}\,\mathrm{d}S - \int_V \boldsymbol{\sigma}:\delta\boldsymbol{\varepsilon}\,\mathrm{d}V = 0} \tag{18}$$

combined with (17). This is the final normal application form of the principle of virtual work in the small displacement theory. □

Remark 12.3. The corresponding dynamics form of the virtual work equation is arrived at with the substitution

$$\boldsymbol{b} := \boldsymbol{b} - \boldsymbol{a}. \tag{19}$$

as discussed analogously in Remark 5.5. To be more precise, the motion of the body induces naturally in addition to the inertia forces also, say, viscous damping forces, which must be included in the formulation. □

A matrix form of the virtual work equation (8) is often convenient. To shorten the expressions we consider the two-dimensional case in the xy-plane. The generalization to the three-dimensional case is obvious. The notations used follow closely those employed in Zienkiewicz and Taylor (1989). We define

$$\begin{aligned}
\{u\} &= \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \{b\} = \begin{Bmatrix} b_a \\ b_b \end{Bmatrix}, \quad \{t\} = \begin{Bmatrix} t_a \\ t_b \end{Bmatrix}, \quad \{\sigma\} = \begin{Bmatrix} \sigma_a \\ \sigma_b \\ \tau_{ab} \end{Bmatrix}, \\
\{\varepsilon\} &= \begin{Bmatrix} \varepsilon_a \\ \varepsilon_b \\ \gamma_{ab} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial a \\ \partial v/\partial b \\ \partial u/\partial b + \partial v/\partial a \end{Bmatrix} = \begin{bmatrix} \partial/\partial a & 0 \\ 0 & \partial/\partial b \\ \partial/\partial b & \partial/\partial a \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = [S]\{u\}
\end{aligned} \tag{20}$$

where the *strain-displacement* operator matrix

$$[S] = \begin{bmatrix} \partial/\partial a & 0 \\ 0 & \partial/\partial b \\ \partial/\partial b & \partial/\partial a \end{bmatrix}. \tag{21}$$

Equation (8) is now with a change of sign

$$\boxed{\int_\Omega \delta\{\varepsilon\}^{\mathrm{T}} \{\sigma\} \mathrm{d}\Omega - \int_\Omega \delta\{u\}^{\mathrm{T}} \rho\{b\} \mathrm{d}\Omega - \int_\Gamma \delta\{u\}^{\mathrm{T}} \{t\} \mathrm{d}\Gamma = 0.} \tag{8'}$$

Variation of the last equation (20) gives

$$\delta\{\varepsilon\} = [S]\delta\{u\}, \tag{22}$$

so the final matrix form is

$$\boxed{\int_\Omega ([S]\delta\{u\})^{\mathrm{T}} \{\sigma\} \mathrm{d}\Omega - \int_\Omega \delta\{u\}^{\mathrm{T}} \rho\{b\} \mathrm{d}\Omega - \int_\Gamma \delta\{u\}^{\mathrm{T}} \{t\} \mathrm{d}\Gamma = 0.} \tag{8''}$$

It should be noticed that we cannot write the first integrand to the seemingly correct form $\delta\{u\}^{\mathrm{T}}[S]^{\mathrm{T}}\{\sigma\}$ as the operator $[S]^{\mathrm{T}}$ would then apply on a wrong quantity.

Remark 12.4. *The principle of virtual work is a general principle valid for any body irrespective of its material properties.* One application of the principle is to derive by it the governing equilibrium equations (or equations of motion) for a specific body configuration. A sample of this feature will be given in Section 13.3.2. The most important application is the discretized form, say, by the finite element method, of the principle leading to a model with finite degrees of freedom and to a similar situation that was considered in Chapter 5. □

12.1.3 Principle of virtual work for an elastic body

We develop the virtual work equation (8") further in the case of an elastic body. The stress-strain constitutive relation for elastic material without initial strains and stresses is

$$\boxed{\{\sigma\} = [D]\{\varepsilon\}} \tag{23}$$

where $[D]$ is a symmetric positive definite *stress-strain* matrix. In particular, for isotropic material in the *plane stress* case (tasojännitystila),

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \tag{24}$$

and in the *plane strain* case (tasovenymätila)

$$[D] = \begin{bmatrix} 2G+\lambda & \lambda & 0 \\ \lambda & 2G+\lambda & 0 \\ 0 & 0 & G \end{bmatrix} \tag{25}$$

where E is *Young's modulus* (kimmokerroin), ν *Poisson's ratio* (Poissonin vakio), G the *shear modulus* (liukukerroin) and λ the *Lame parameter* (Lamen parametri). Only two of the parameters are independent and one can write many dependencies between them, such as

$$\begin{aligned}
G &= \frac{E}{2(1+\nu)}, \\
\lambda &= \frac{2G\nu}{1-2\nu}.
\end{aligned} \tag{26}$$

From (20),

$$\{\sigma\} = [D]\{\varepsilon\} = [D][S]\{u\}, \tag{27}$$

and the virtual work equation (8") obtains the specific form

$$\boxed{\int_\Omega ([S]\delta\{u\})^{\mathrm{T}}[D][S]\{u\}\mathrm{d}\Omega - \int_\Omega \delta\{u\}^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \int_\Gamma \delta\{u\}^{\mathrm{T}}\{t\}\mathrm{d}\Gamma = 0.} \tag{28}$$

12.1.4 Sensitized principle of virtual work

Equilibrium equations (1) are in detail in the two-dimensional case (substitute expressions (11.2.2') into (1))

$$\begin{aligned} &\frac{\partial\sigma_a}{\partial a} + \frac{\partial\tau_{ba}}{\partial b} + \rho b_a = 0, \\ &\frac{\partial\tau_{ab}}{\partial a} + \frac{\partial\sigma_b}{\partial b} + \rho b_b = 0, \end{aligned} \tag{29}$$

or

$$\begin{bmatrix} \partial/\partial a & 0 & \partial/\partial b \\ 0 & \partial/\partial b & \partial/\partial a \end{bmatrix} \begin{Bmatrix} \sigma_a \\ \sigma_b \\ \tau_{ab} \end{Bmatrix} + \rho \begin{Bmatrix} b_a \\ b_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{30}$$

or

$$\{R(\{\sigma\})\} \equiv \{L(\{\sigma\})\} + \rho\{b\} \equiv [E]\{\sigma\} + \rho\{b\} = \{0\}, \tag{31}$$

where the *equilibrium* operator matrix

$$[E] = \begin{bmatrix} \partial/\partial a & 0 & \partial/\partial b \\ 0 & \partial/\partial b & \partial/\partial a \end{bmatrix}. \tag{32}$$

It is seen that here $[E] = [S]^{\mathrm{T}}$. The least-squares functional corresponding to equations (31) is (see Section D.2.8)

$$\Pi(\{\sigma\}) = \frac{1}{2}\int_\Omega \{R\}^{\mathrm{T}}[\tau]\{R\}\mathrm{d}\Omega. \tag{33}$$

The symmetric matrix $[\tau]$ is a sensitizing parameter matrix discussed in Section D.2.9. We consider here just the case with one sensitizing integral. Demanding (33) to have a stationary value gives the equation (see Section D.2.11)

$$\int_\Omega \{L(\{\delta\sigma\})\}^{\mathrm{T}}[\tau]\{R(\{\sigma\})\}\mathrm{d}\Omega = 0. \tag{34}$$

As discussed in Section D.2.11, a sensitized weak form — here a sensitized principle of virtual work — is obtained as a linear combination of (8") and (34):

$$\boxed{\begin{aligned} &\int_\Omega ([S]\delta\{u\})^{\mathrm{T}}\{\sigma\}\mathrm{d}\Omega - \int_\Omega \delta\{u\}^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \int_\Gamma \delta\{u\}^{\mathrm{T}}\{t\}\mathrm{d}\Gamma + \\ &+\int_\Omega ([E]\{\delta\sigma\})^{\mathrm{T}}[\tau]([E]\{\sigma\} + \rho\{b\}))\mathrm{d}\Omega = 0. \end{aligned}} \tag{35}$$

The expression $\{\delta\sigma\}$ is to interpreted here finally as the virtual change of stresses due to a virtual displacement $\delta\{u\}$.

12.1.5 Sensitized principle of virtual work for an elastic body

For an elastic material we have the expression (27):

$$\{\sigma\} = [D][S]\{u\} \tag{36}$$

and thus

$$\delta\{\sigma\} = [D][S]\delta\{u\}. \tag{37}$$

The virtual work equation (35) obtains the specific form

$$\boxed{\begin{aligned} &\int_\Omega ([S]\delta\{u\})^{\mathrm{T}}[D][S]\{u\}\mathrm{d}\Omega - \int_\Omega \delta\{u\}^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \int_\Gamma \delta\{u\}^{\mathrm{T}}\{t\}\mathrm{d}\Gamma + \\ &+\int_\Omega ([E][D][S]\delta\{u\})^{\mathrm{T}}[\tau]([E][D][S]\{u\} + \rho\{b\}))\mathrm{d}\Omega = 0. \end{aligned}} \tag{38}$$

Again it should be noticed that the operator matrices are to be applied on the (total) quantities on the right-hand sides of them as far as indicated by parentheses.

12.2 LARGE DISPLACEMENTS

12.2.1 Principle of virtual work

The governing equilibrium equations are in the body

$$^0\rho b + \frac{\partial T^j}{\partial a_j} = \mathbf{0} \qquad \text{in } {}^0V \tag{1}$$

and on the body surface

$$T = {}^0n_j T^j \qquad \text{on } {}^0S. \tag{2}$$

These are exactly of the same form as equations (1) and (2) in Section 12.1, only the lower case traction symbols replaced here with the capital ones. Repeating the manipulation with the weighting function

$$w(a) = w_i(a)i_i \tag{3}$$

produces thus the counterpart of equation (6) in Section 12.1:

$$\int_{{}^0V} {}^0\rho \boldsymbol{b} \cdot \boldsymbol{w}\, \mathrm{d}^0V + \int_{{}^0S} \boldsymbol{T} \cdot \boldsymbol{w}\, \mathrm{d}^0S - \int_{{}^0V} \boldsymbol{T}^j \cdot \frac{\partial \boldsymbol{w}}{\partial a_j}\, \mathrm{d}^0V = 0. \tag{4}$$

The interpretation

$$\begin{aligned} \boldsymbol{w} = \delta\boldsymbol{r} = \delta\boldsymbol{u} = \delta u_k \boldsymbol{i}_k &= \delta u_1 \boldsymbol{i}_1 + \delta u_2 \boldsymbol{i}_2 + \delta u_3 \boldsymbol{i}_3 \\ &= \delta u\boldsymbol{i} + \delta v\boldsymbol{j} + \delta w\boldsymbol{k} \end{aligned} \tag{5}$$

leads to the *principle of virtual work* (statics) for a continuum in the case of large displacements:

$$\boxed{\int_{{}^0V} {}^0\rho \boldsymbol{b} \cdot \delta\boldsymbol{u}\, \mathrm{d}^0V + \int_{{}^0S} \boldsymbol{T} \cdot \delta\boldsymbol{u}\, \mathrm{d}^0S - \int_{{}^0V} \boldsymbol{S} : \delta\boldsymbol{E}\, \mathrm{d}^0V = 0.} \tag{6}$$

Similarly as in Section 12.2, this can be written

$$\delta' W \equiv \delta' W_{\text{ext}} + \delta' W_{\text{int}} = 0 \tag{7}$$

with

$$\begin{aligned} \delta' W_{\text{ext}} &= \int_{{}^0V} {}^0\rho \boldsymbol{b} \cdot \delta\boldsymbol{u}\, \mathrm{d}^0V + \int_{{}^0S} \boldsymbol{T} \cdot \delta\boldsymbol{u}\, \mathrm{d}^0S, \\ \delta' W_{\text{int}} &= -\int_{{}^0V} \boldsymbol{S} : \delta\boldsymbol{E}\, \mathrm{d}^0V. \end{aligned} \tag{8}$$

The double-dot product is in rectangular cartesian coordinates

$$\begin{aligned} \boldsymbol{S} : \delta\boldsymbol{E} = S_{ij}\delta E_{ij} = {} & S_{11}\delta E_{11} + S_{12}\delta E_{12} + S_{13}\delta E_{13} + \\ & + S_{21}\delta E_{21} + S_{22}\delta E_{22} + S_{23}\delta E_{23} + \\ & + S_{31}\delta E_{31} + S_{32}\delta E_{32} + S_{33}\delta E_{33}. \end{aligned} \tag{9}$$

The terms

$$\begin{aligned} \delta E_{ij} &= \delta \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right) \\ &= \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_j}{\partial a_i} + \frac{\partial \delta u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j} + \frac{\partial u_k}{\partial a_i}\frac{\partial \delta u_k}{\partial a_j}\right) \end{aligned} \tag{10}$$

are virtual Green strains. The detailed derivation concerning internal virtual work is performed in Example 12.2.

Example 12.2. We consider the term

$$\boldsymbol{T}^j \cdot \frac{\partial \boldsymbol{w}}{\partial a_j} = \boldsymbol{T}^j \cdot \frac{\partial \delta\boldsymbol{u}}{\partial a_j} \tag{a}$$

in (4). We have

$$\boldsymbol{T}^j = S_{ji}\boldsymbol{G}_i, \qquad \boldsymbol{G}_i = \boldsymbol{i}_i + \frac{\partial u_l}{\partial a_i}\boldsymbol{i}_l, \qquad \delta\boldsymbol{u} = \delta u_k \boldsymbol{i}_k. \tag{b}$$

The first and second forms in (b) are given in (11.2.7') and (10.2.23'). Thus

$$\begin{aligned} \boldsymbol{T}^j \cdot \frac{\partial \delta\boldsymbol{u}}{\partial a_j} &= S_{ji}\boldsymbol{G}_i \cdot \frac{\partial \delta u_k}{\partial a_j}\boldsymbol{i}_k = S_{ji}\frac{\partial \delta u_k}{\partial a_j}\boldsymbol{G}_i \cdot \boldsymbol{i}_k = S_{ji}\frac{\partial \delta u_k}{\partial a_j}\left(\boldsymbol{i}_i \cdot \boldsymbol{i}_k + \frac{\partial u_l}{\partial a_i}\boldsymbol{i}_l \cdot \boldsymbol{i}_k\right) \\ &= S_{ji}\frac{\partial \delta u_k}{\partial a_j}\left(\delta_{ik} + \frac{\partial u_l}{\partial a_i}\delta_{lk}\right) = S_{ji}\frac{\partial \delta u_k}{\partial a_j}\left(\delta_{ik} + \frac{\partial u_k}{\partial a_i}\right) \\ &= S_{ji}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_k}{\partial a_j}\frac{\partial u_k}{\partial a_i}\right) \\ &= \frac{1}{2}S_{ij}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_k}{\partial a_j}\frac{\partial u_k}{\partial a_i}\right) + \frac{1}{2}S_{ji}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_k}{\partial a_j}\frac{\partial u_k}{\partial a_i}\right) \\ &= \frac{1}{2}S_{ij}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_k}{\partial a_j}\frac{\partial u_k}{\partial a_i}\right) + \frac{1}{2}S_{ij}\left(\frac{\partial \delta u_j}{\partial a_i} + \frac{\partial \delta u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right) \\ &= S_{ij}\frac{1}{2}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_j}{\partial a_i} + \frac{\partial \delta u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j} + \frac{\partial u_k}{\partial a_i}\frac{\partial \delta u_k}{\partial a_j}\right). \end{aligned} \tag{c}$$

The steps used should be rather obvious. Symmetry of the Kirchhoff stress has been made use of. The expression for the Green strain from (10.2.36) is

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right). \tag{d}$$

Its variation δE_{ij} is seen to appear in (c) and thus

$$\boldsymbol{T}^j \cdot \frac{\partial \delta\boldsymbol{u}}{\partial a_j} = S_{ij}\delta E_{ij}. \tag{e}$$

Remark 12.5. We repeat here the discussion of Remark 12.1. If we let the weighting function $\boldsymbol{w}$ remain finite and take no specific interpretation, we have from formula (c) of Example 12.2 the result

$$\boldsymbol{T}^j \cdot \frac{\partial \boldsymbol{w}}{\partial a_j} = S_{ij}\frac{1}{2}\left(\frac{\partial w_i}{\partial a_j} + \frac{\partial w_j}{\partial a_i} + \frac{\partial w_k}{\partial a_i}\frac{\partial u_k}{\partial a_j} + \frac{\partial u_k}{\partial a_i}\frac{\partial w_k}{\partial a_j}\right). \tag{11}$$

The large displacement theory is seen to be present via the current displacement state $\boldsymbol{u}$. We again see that there is no absolute need to consider $\boldsymbol{w}$ as infinitesimal and as the variation of $\boldsymbol{u}$. If we consider $\boldsymbol{w}$ to be the current finite displacement $\boldsymbol{u}$, we obtain

$$T^j \cdot \frac{\partial u}{\partial a_j} = S_{ij} \frac{1}{2} \left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i} \frac{\partial u_k}{\partial a_j} + \frac{\partial u_k}{\partial a_i} \frac{\partial u_k}{\partial a_j} \right). \tag{12}$$

Now the term multiplying S_{ij} is not the Green strain corresponding to the displacement field and for instance the unit dummy load method does not work any more. □

How to deal with kinematical boundary conditions and with dynamics should be obvious from Remarks 12.2 and 12.3. We do not consider the sensitized form of the principle of virtual work in the large displacement case here.

12.3 REFERENCES

Malvern, L. E. (1969): *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall, Englewoods Ciffs.

Zienkiewicz, O. C. and R.L. Taylor (1989): *The Finite Element Method*, 4th ed. Vol. 1, McGraw-Hill, London.

CHAPTER 13

APPLICATIONS IN STATICS

Only the small displacement theory is considered. The finite element system equations are derived in the two-dimensional elastic case. A plane arch is analysed as a demonstration example. A more detailed consideration is given for a straight elastic beam.

13.1 FINITE ELEMENT METHOD

We develop in some detail the discrete system equations emerging with the use of the finite element method for an elastic body. The virtual work equation is (12.1.28):

$$\int_\Omega ([S]\delta\{u\})^{\mathrm{T}}[D][S]\{u\}\mathrm{d}\Omega - \int_\Omega \delta\{u\}^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \int_\Gamma \delta\{u\}^{\mathrm{T}}\{t\}\mathrm{d}\Gamma = 0. \tag{1}$$

In the so-called *displacement formulation* (siirtymäformulaatio) using the finite element method, the displacement field is expressed by the finite element approximation

$$\boxed{\{u(x)\} \approx \{\tilde{u}(x)\} = [N(x)]\{a\}} \tag{2}$$

where $[N]$ is a given shape function matrix and $\{a\}$ is a column matrix of undetermined parameters or here of the so-called *nodal displacements* (solmusiirtymät). Some details of this has been explained in Section D.3.4. Based on (2), a variation gives the virtual displacement field

$$\delta\{u\} = [N]\delta\{a\}. \tag{3}$$

Substitution of (2) and (3) in (1) gives

$$\begin{aligned}
&\int_\Omega ([S][N]\delta\{a\})^{\mathrm{T}}[D][S][N]\{a\}\mathrm{d}\Omega + \\
&-\int_\Omega ([N]\delta\{a\})^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \int_\Gamma ([N]\delta\{a\})^{\mathrm{T}}\{t\}\mathrm{d}\Gamma = 0, \\
&\delta\{a\}^{\mathrm{T}}\int_\Omega ([S][N])^{\mathrm{T}}[D]([S][N])\mathrm{d}\Omega\{a\} + \\
&-\delta\{a\}^{\mathrm{T}}\int_\Omega [N]^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \delta\{a\}^{\mathrm{T}}\int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma = 0, \\
&\delta\{a\}^{\mathrm{T}}\int_\Omega [B]^{\mathrm{T}}[D][B]\mathrm{d}\Omega\{a\} + \\
&-\delta\{a\}^{\mathrm{T}}\int_\Omega [N]^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \delta\{a\}^{\mathrm{T}}\int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma = 0.
\end{aligned} \tag{4}$$

The standard shorthand notation

$$[B] = [S][N] \tag{5}$$

has been used. The steps in obtaining the final form (4) are rather obvious, for instance, the quantities $\{a\}$ and $\delta\{a\}$ do not depend on position and can be taken outside the integrals.

The left-hand side of (4) is the negative virtual work expression for the discretized body having the form

$$-\delta' W \equiv \delta\{a\}^{\mathrm{T}}(-\{Q\}) = \delta\{a\}^{\mathrm{T}}(-\{Q\}_{\mathrm{int}} - \{Q\}_{\mathrm{ext}}) \tag{6}$$

where the column vectors of generalized forces are

$$\begin{aligned}
\{Q\}_{\mathrm{int}} &= -\int_\Omega [B]^{\mathrm{T}}[D][B])\mathrm{d}\Omega\{a\}, \\
\{Q\}_{\mathrm{ext}} &= \int_\Omega [N]^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega + \int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma.
\end{aligned} \tag{7}$$

Thus the final system equations are

$$-\{Q\}_{\mathrm{int}} - \{Q\}_{\mathrm{ext}} = \{0\} \tag{8}$$

or

$$\boxed{[K]\{a\} = \{b\}} \tag{9}$$

with

$$\begin{aligned}
[K] &= \int_\Omega [B]^{\mathrm{T}}[D][B]\mathrm{d}\Omega, \\
\{b\} &= \int_\Omega [N]^{\mathrm{T}}\{f\}\mathrm{d}\Omega + \int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma.
\end{aligned} \tag{10}$$

To avoid possible confusion, we have denoted the body force intensity per unit volume finally by $\{f\} = \rho\{b\}$. The coefficient matrix $[K]$ is seen to be symmetric and it is usually called in the displacement formulation as the *stiffness matrix* (jäykkyysmatriisi).

13.2 SENSITIZED FINITE ELEMENT METHOD

Again only the elastic body case is considered. The sensitizing term in principle (12.1.38) is of the form

$$\int_\Omega ([E][D][S]\delta\{u\})^{\mathrm{T}}[\tau]([E][D][S]\{u\} + \rho\{b\})\mathrm{d}\Omega. \tag{1}$$

Substitution of expressions (2) and (3) gives

$$\int_\Omega ([E][D][S][N]\delta\{a\})^{\mathrm{T}}[\tau]([E][D][S][N]\{a\}+\rho\{b\})\mathrm{d}\Omega,$$

$$\delta\{a\}^{\mathrm{T}}\int_\Omega ([E][D][B])^{\mathrm{T}}[\tau]([E][D][B]\{a\}+\rho\{b\})\mathrm{d}\Omega,$$

$$\delta\{a\}^{\mathrm{T}}\int_\Omega ([E][D][B])^{\mathrm{T}}[\tau]([E][D][B])\mathrm{d}\Omega\{a\}+$$
$$+\delta\{a\}^{\mathrm{T}}\int_\Omega ([E][D][B])^{\mathrm{T}}[\tau]\rho\{b\}\mathrm{d}\Omega. \qquad (2)$$

Proceeding similarly as in Section 13.1, we find that the system equations are now

$$\boxed{[K]\{a\}=\{b\}} \qquad (3)$$

with

$$[K]=\int_\Omega [B]^{\mathrm{T}}[D][B])\mathrm{d}\Omega+\int_\Omega ([E][D][B])^{\mathrm{T}}[\tau]([E][D][B])\mathrm{d}\Omega,$$
$$\{b\}=\int_\Omega [N]^{\mathrm{T}}\{f\}\mathrm{d}\Omega+\int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma-\int_\Omega ([E][D][B])^{\mathrm{T}}[\tau]\{f\}\mathrm{d}\Omega. \qquad (4)$$

Remark 13.1. Terms like $[E][D][B]$ become very involved if the material properties depend on position. Thus in practice we assume some constant representative value for $[D]$ in an element to simplify the formulas. This theme is discussed also in Section D.3.5. □

13.3 ARCH

13.3.1 Kinematics

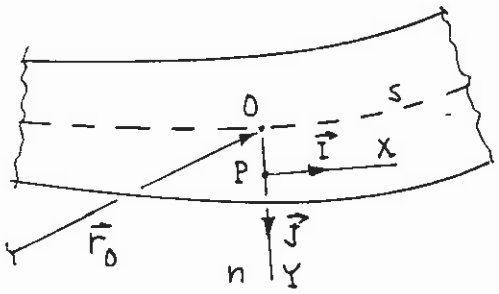


Figure 13.1 Curved beam.

A plane arch or curved plane beam is considered using the notation of Figure 13.1. Lagrangian rectangular cartesian coordinates are denoted here simply by x and y. We employ the method of local cartesian coordinates described in Appendix C. The curved axis of the beam is taken to be a α-coordinate line and on this line α is associated with the arc length s. The β-coordinate lines are straight and $\beta = n$ where n is the perpendicular distance from the beam axis. The direction of coordinate n is 90° clockwise from the positive direction of s. The purpose is to derive first the expressions for the strains according to the assumptions of beam theory and based on that then the corresponding equilibrium equations using the principle of virtual work.

The position vector of a generic point P is

$$\boldsymbol{r}(s,n)=\boldsymbol{r}_0(s)+n\boldsymbol{e}_n(s). \qquad (1)$$

From curve theory $\mathrm{d}\boldsymbol{r}_0/\mathrm{d}s=\boldsymbol{e}_s$ and from the well-known Frenet formulas, e.g. Kreyszig (1967, p. 273),

$$\frac{\mathrm{d}\boldsymbol{e}_s}{\mathrm{d}s}=-\frac{1}{R}\boldsymbol{e}_n, \qquad \frac{\mathrm{d}\boldsymbol{e}_n}{\mathrm{d}s}=\frac{1}{R}\boldsymbol{e}_s. \qquad (2)$$

The curvature $1/R$ of the beam axis is here positive if the center of curvature is on the negative side of the n-axis.

Differentiation of expression (1) gives thus

$$\frac{\partial \boldsymbol{r}}{\partial s}=\frac{\mathrm{d}\boldsymbol{r}_0}{\mathrm{d}s}+n\frac{\mathrm{d}\boldsymbol{e}_n}{\mathrm{d}s}=\boldsymbol{e}_s+n\frac{\boldsymbol{e}_s}{R}=(1+\frac{n}{R})\boldsymbol{e}_s,$$
$$\frac{\partial \boldsymbol{r}}{\partial n}=\boldsymbol{e}_n. \qquad (3)$$

The scale factors are

$$h_s=\left|\frac{\partial \boldsymbol{r}}{\partial s}\right|=1+\frac{n}{R}, \qquad h_n=\left|\frac{\partial \boldsymbol{r}}{\partial n}\right|=1. \qquad (4)$$

At the local origin (see formulas (C.2.1) and (C.2.2))

$$\boldsymbol{I}=\boldsymbol{e}_s, \quad \boldsymbol{J}=\boldsymbol{e}_n, \qquad (5)$$

and

$$\frac{\partial}{\partial X}=(1+\frac{n}{R})^{-1}\frac{\partial}{\partial s}, \qquad \frac{\partial}{\partial Y}=\frac{\partial}{\partial n}. \qquad (6)$$

We use here the kinematical assumption according to the *Timoshenko beam theory*, e.g., Fung (1965, p. 322): the material fibers originally perpendicular to the beam axis are assumed to remain straight and inextensible but not necessarily perpendicular to the beam axis after the deformation. From Figure 13.1 we find for point P (small displacements)

$$u(s,n) = u_s e_s + u_n e_n = [\hat{u}(s) - n\theta(s)]e_s(s) + \hat{v}(s)e_n(s). \tag{7}$$

Quantities $\hat{u}$ and $\hat{v}$ are the displacement components of the origin O of the cross-section in the s- and n-directions, respectively, and θ is the rotation (positive clockwise) of the cross-section fiber.

The relevant strain components in beam theory are (see Example C.1, formula (o))

$$\begin{aligned} \varepsilon_s &= \varepsilon_X = \frac{\partial u}{\partial X} \bullet I = (1+\frac{n}{R})^{-1} \frac{\partial u}{\partial s} \bullet e_s, \\ \gamma_{sn} &= \gamma_{XY} = \frac{\partial u}{\partial Y} \bullet I + \frac{\partial u}{\partial X} \bullet J = \frac{\partial u}{\partial n} \bullet e_s + (1+\frac{n}{R})^{-1} \frac{\partial u}{\partial s} \bullet e_n. \end{aligned} \tag{8}$$

From (7), the derivatives

$$\begin{aligned} \frac{\partial u}{\partial s} &= (\frac{\mathrm{d}\hat{u}}{\mathrm{d}s} - n\frac{\mathrm{d}\theta}{\mathrm{d}s})e_s + (\hat{u} - n\theta)(-\frac{e_n}{R}) + \frac{\mathrm{d}\hat{v}}{\mathrm{d}s}e_n + \hat{v}\frac{e_s}{R}, \\ \frac{\partial u}{\partial n} &= -\theta e_s \end{aligned} \tag{9}$$

and substitution in (8) gives

$$\boxed{\begin{aligned} \varepsilon_s &= (1+\frac{n}{R})^{-1}(\frac{\mathrm{d}\hat{u}}{\mathrm{d}s} - n\frac{\mathrm{d}\theta}{\mathrm{d}s} + \frac{\hat{v}}{R}), \\ \gamma_{sn} &= (1+\frac{n}{R})^{-1}(-\frac{\hat{u}-n\theta}{R} + \frac{\mathrm{d}\hat{v}}{\mathrm{d}s}) - \theta. \end{aligned}} \tag{10}$$

To derive these exact results with confidence based on some figures representing the kinematics would be very difficult.

In the *Bernoulli-Euler beam theory* (later shortly Bernoulli beam theory) it is assumed that the shearing strain is zero, that is, the material fibers originally perpendicular to the beam axis remain perpendicular to the deformed beam axis. The latter expression in (10) can be put in the form

$$\gamma_{sn} = (1+\frac{n}{R})^{-1}(\frac{\mathrm{d}\hat{v}}{\mathrm{d}s} - \frac{\hat{u}}{R} - \theta). \tag{11}$$

By demanding $\gamma_{sn} = 0$ we obtain the constraint

$$\theta = \frac{\mathrm{d}\hat{v}}{\mathrm{d}s} - \frac{\hat{u}}{R}. \tag{12}$$

Substitution of this in the first of (10) gives the Bernoulli beam strain expression

$$\begin{aligned} \varepsilon_s &= (1+\frac{n}{R})^{-1}[\frac{\mathrm{d}\hat{u}}{\mathrm{d}s} - n\frac{\mathrm{d}}{\mathrm{d}s}(\frac{\mathrm{d}\hat{v}}{\mathrm{d}s} - \frac{\hat{u}}{R}) + \frac{\hat{v}}{R}] \\ &= (1+\frac{n}{R})^{-1}[\frac{\mathrm{d}\hat{u}}{\mathrm{d}s} - n(\frac{\mathrm{d}^2\hat{v}}{\mathrm{d}s^2} - \frac{\mathrm{d}\hat{u}/\mathrm{d}s}{R} + \frac{\hat{u}}{R^2}\frac{\mathrm{d}R}{\mathrm{d}s}) + \frac{\hat{v}}{R}] \end{aligned} \tag{13}$$

or

$$\boxed{\varepsilon_s = \frac{\mathrm{d}\hat{u}}{\mathrm{d}s} + (1+\frac{n}{R})^{-1}[\frac{\hat{v}}{R} - n(\frac{\hat{u}}{R^2}\frac{\mathrm{d}R}{\mathrm{d}s} + \frac{\mathrm{d}^2\hat{v}}{\mathrm{d}s^2})].} \tag{14}$$

The expressions obtained can be approximated for shallow arches by developing them in truncated power series in n/R.

The formulas simplify considerably in the case of a straight beam, where $1/R = 0$. In the Timoshenko theory, formulas (10) become

$$\boxed{\begin{aligned} \varepsilon_x &= \frac{\mathrm{d}\hat{u}}{\mathrm{d}x} - y\frac{\mathrm{d}\theta}{\mathrm{d}x}, \\ \gamma_{xy} &= \frac{\mathrm{d}\hat{v}}{\mathrm{d}x} - \theta. \end{aligned}} \tag{15}$$

In the Bernoulli theory, from (14):

$$\boxed{\varepsilon_x = \frac{\mathrm{d}\hat{u}}{\mathrm{d}x} - y\frac{\mathrm{d}^2\hat{v}}{\mathrm{d}x^2}.} \tag{16}$$

In these formulas, s and n have been replaced by the more conventional symbols x and y for a straight beam.

13.3.2 Equilibrium equations

Figure 13.2 shows some of the notations which appear later in this section.The principle of virtual work is

$$\delta' W_{\text{int}} + \delta' W_{\text{ext}} = 0. \tag{17}$$

In the plane stress case considered here the non-zero stress components are σ_x, σ_y, τ_{xy} and we have

$$\delta' W_{\text{int}} = -\int_V (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy})\mathrm{d}V,$$
$$\delta' W_{\text{ext}} = \int_V (f_x \delta u + f_y \delta v)\mathrm{d}V + \int_S (t_x \delta u + t_y \delta v)\mathrm{d}S. \tag{18}$$

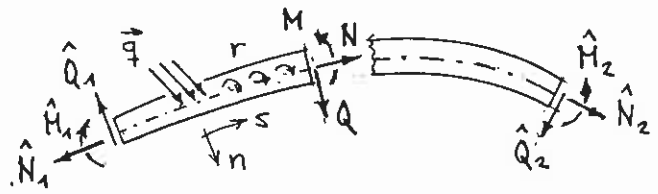


Figure 13.2 Some notations.

The body force components per unit volume have been denoted by $f_x(=\rho b_x)$ and $f_y(=\rho b_y)$.The integrals and also the integrands in (18) are scalar invariants, that is, their values do not depend on the coordinate system used for their evaluation. Further, according to conventional beam theory, the essentially non-zero stress components at a generic point P (Figure 13.1) are $\sigma_X = \sigma_s$ and $\tau_{XY} = \tau_{sn}$. Thus

$$\begin{aligned}
&\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy} = \sigma_X \delta\varepsilon_X + \tau_{XY}\delta\gamma_{XY} = \sigma_s \delta\varepsilon_s + \tau_{sn}\delta\gamma_{sn}\,,\\
&f_x \delta u + f_y \delta v = f_s \delta u_s + f_n \delta u_n\,,\\
&t_x \delta u + t_y \delta v = f_s \delta u_s + t_n \delta u_n\,.
\end{aligned} \tag{19}$$

We have also assumed the virtual displacements to take place in the gobal xy-plane.

The volume element

$$\mathrm{d}V = \mathrm{d}X\mathrm{d}Y\mathrm{d}Z = h_\alpha \mathrm{d}\alpha\, h_\beta \mathrm{d}\beta\, \mathrm{d}Z = (1+\frac{n}{R})\mathrm{d}s\mathrm{d}n\mathrm{d}Z = (1+\frac{n}{R})\mathrm{d}A\mathrm{d}s, \tag{20}$$

where

$$\mathrm{d}A = \mathrm{d}Z\mathrm{d}n \tag{21}$$

is now the area element of the cross-section.

The virtual work of the internal forces obtains the form

$$\delta' W_{\text{int}} = -\int_s [\int_A (\sigma_s \delta\varepsilon_s + \tau_{sn}\delta\gamma_{sn})(1+\frac{n}{R})\mathrm{d}A]\mathrm{d}s\,. \tag{22}$$

From (10) and (11),

$$\begin{aligned}
\delta\varepsilon_s &= (1+\frac{n}{R})^{-1}(\frac{\mathrm{d}\delta\hat{u}}{\mathrm{d}s} - n\frac{\mathrm{d}\delta\theta}{\mathrm{d}s} + \frac{\delta\hat{v}}{R}),\\
\delta\gamma_{sn} &= (1+\frac{n}{R})^{-1}(\frac{\mathrm{d}\delta\hat{v}}{\mathrm{d}s} - \frac{\delta\hat{u}}{R} - \delta\theta)
\end{aligned} \tag{23}$$

and substitution in (22) gives

$$\begin{aligned}
\delta' W_{\text{int}} &= -\int_s \{\int_A [\sigma_s(\frac{\mathrm{d}\delta\hat{u}}{\mathrm{d}s} - n\frac{\mathrm{d}\delta\theta}{\mathrm{d}s} + \frac{\delta\hat{v}}{R}) + \tau_{sn}(\frac{\mathrm{d}\delta\hat{v}}{\mathrm{d}s} - \frac{\delta\hat{u}}{R} - \delta\theta)]\mathrm{d}A\}\mathrm{d}s\\
&= -\int_s [N(\frac{\mathrm{d}\delta\hat{u}}{\mathrm{d}s} + \frac{\delta\hat{v}}{R}) - M\frac{\mathrm{d}\delta\theta}{\mathrm{d}s} + Q(\frac{\mathrm{d}\delta\hat{v}}{\mathrm{d}s} - \frac{\delta\hat{u}}{R} - \delta\theta)]\mathrm{d}s\,.
\end{aligned} \tag{24}$$

Here the standard definitions

$$\begin{aligned}
N &= \int_A \sigma_s \mathrm{d}A\,,\\
Q &= \int_A \tau_{sn}\mathrm{d}A\,,\\
M &= \int_A \sigma_s n\,\mathrm{d}A
\end{aligned} \tag{25}$$

for the stress resultants — *normal force* (normaalivoima) N, *shearing force* (leikkausvoima) Q, *bending moment* (taivutusmomentti) M — of the cross-section have been employed. It should be noticed that the terms containing virtual displacements in (24) depend only on s and they can thus be taken outside the integral over the cross-section. Formula (24) is useful if, say, a numerical solution by the finite element method is attempted. When deriving the relevant beam theory equilibrium equations further manipulation is needed.

Here a one-dimensional integration by parts formula (see formula (B.1.1a))

$$\int_s g\frac{\mathrm{d}h}{\mathrm{d}s}\mathrm{d}s = -\int_s \frac{\mathrm{d}g}{\mathrm{d}s}h\,\mathrm{d}s + \Big|_{s_1}^{s_2} gh \tag{26}$$

with obvious meaning is used. This applied to (24) to remove the derivatives on the virtual quantities gives

$$\begin{aligned}
\delta' W_{\text{int}} &= \int_s [(\frac{\mathrm{d}N}{\mathrm{d}s} + \frac{Q}{R})\delta\hat{u} + (\frac{\mathrm{d}Q}{\mathrm{d}s} - \frac{N}{R})\delta\hat{v} + (-\frac{\mathrm{d}M}{\mathrm{d}s} + Q)\delta\theta]\mathrm{d}s +\\
&\quad -\Big|_{s_1}^{s_2}[N\delta\hat{u} + Q\delta\hat{v} - M\delta\theta]\,.
\end{aligned} \tag{27}$$

The virtual work of the external forces

$$\delta' W_{ext} = \int_V (f_x \delta u + f_y \delta v) dV + \int_S (t_x \delta u + t_y \delta v) dS.$$
$$= \int_V (f_s \delta u_s + f_n \delta u_n) dV +$$
$$+ \int_{A_1} (t_s \delta u_s + t_n \delta u_n) dA + \int_{A_2} (t_s \delta u_s + t_n \delta u_n) dA. \quad (28)$$

Here we have for simplicity assumed that non-zero tractions exist only on the end cross-sections surfaces A_1 and A_2 of the total beam boundary surface S. From (7)

$$\delta u_s = \delta \hat{u} - n \delta \theta,$$
$$\delta u_n = \delta \hat{v}. \quad (29)$$

Thus

$$\delta' W_{ext} = \int_s \{ \int_A [f_s (\delta \hat{u} - n \delta \theta) + f_n \delta \hat{v}] (1 + \frac{n}{R}) dA \} ds +$$
$$+ \int_{A_1} [t_s (\delta \hat{u} - n \delta \theta) + t_n \delta \hat{v}] dA + \int_{A_2} [t_s (\delta \hat{u} - n \delta \theta) + t_n \delta \hat{v}] dA. \quad (30)$$

We define

$$q_s = \int_A f_s (1 + \frac{n}{R}) dA,$$
$$q_n = \int_A f_n (1 + \frac{n}{R}) dA, \quad (31)$$
$$r = -\int_A f_s n (1 + \frac{n}{R}) dA.$$

These are the body force components and couple per unit beam axis emerging naturally from the virtual work expression. We further denote

$$n_s = \boldsymbol{n} \cdot \boldsymbol{e}_s, \quad (32)$$

where $\mathbf{n}$ is the outward pointing unit normal vector on the end cross-sections giving $n_s(s_1) = -1$ and $n_s(s_2) = 1$. We define

$$\hat{N}_i = \int_{A_i} t_s n_s \, dA,$$
$$\hat{Q}_i = \int_{A_i} t_n n_s \, dA, \quad (33)$$
$$\hat{M}_i = \int_{A_i} t_s n n_s \, dA$$

for $i = 1,2$ giving the stress resultants from possible tractions t_s and t_n on the end cross-sections.

Expression (30) obtains thus finally the form

$$\delta' W_{ext} = \int_s (q_s \delta \hat{u} + q_n \delta \hat{v} + r \delta \theta) ds + \Big|_{s_1}^{s_2} (\hat{N} \delta \hat{u} + \hat{Q} \delta \hat{v} - \hat{M} \delta \theta). \quad (34)$$

Here the quantities $\hat{N}$, $\hat{Q}$, $\hat{M}$ should be equipped with the indices appropriate at $s = s_1$ and $s = s_2$. Substitution of (27) and (34) into the virtual work equation (17) gives

$$\int_s [(\frac{dN}{ds} + \frac{Q}{R} + q_s) \delta \hat{u} + (\frac{dQ}{ds} - \frac{N}{R} + q_n) \delta \hat{v} + (-\frac{dM}{ds} + Q + r) \delta \theta] ds +$$
$$+ \Big|_{s_1}^{s_2} [(-N + \hat{N}) \delta \hat{u} + (-Q + \hat{Q}) \delta \hat{v} + (M - \hat{M}) \delta \theta] = 0. \quad (35)$$

Because the variations $\delta \hat{u}$, $\delta \hat{v}$, $\delta \theta$ can be taken arbitrarily, the equilibrium equations

$$\boxed{\begin{aligned} &\frac{dN}{ds} + \frac{Q}{R} + q_s = 0, \\ &\frac{dQ}{ds} - \frac{N}{R} + q_n = 0, \\ &-\frac{dM}{ds} + Q + r = 0 \end{aligned}} \quad (36)$$

on $s_1 < s < s_2$ and the force boundary conditions

$$\boxed{\begin{aligned} &-N + \hat{N} = 0, \\ &-Q + \hat{Q} = 0, \\ &M - \hat{M} = 0 \end{aligned}} \quad (37)$$

at $s = s_1$ and $s = s_2$ are arrived at. Actually, if some kinematic boundary conditions in the form $\hat{u} = \text{given}$, $\hat{v} = \text{given}$, $\theta = \text{given}$, exist, the corresponding force boundary conditions (37) must be replaced by them.

If the Bernoulli theory with $\gamma_{sn} = 0$ or $\theta = d\hat{v}/d\hat{s} - \hat{u}/R$ is used as the starting point, the basic displacement expression (7) is replaced by

$$u(s,n) = u_s e_s + u_n e_n = \{\hat{u}(s) - n[\frac{d\hat{v}(s)}{ds} - \frac{\hat{u}(s)}{R(s)}]\} e_s(s) + \hat{v}(s) e_n(s). \quad (38)$$

and equations (22) and (23) are replaced by

$$\delta' W_{int} = -\int_s [\sigma_s \delta \varepsilon_s (1 + \frac{n}{R}) dA] ds, \quad (39)$$

CHAPTER 13

APPLICATIONS IN STATICS

Only the small displacement theory is considered. The finite element system equations are derived in the two-dimensional elastic case. A plane arch is analysed as a demonstration example. A more detailed consideration is given for a straight elastic beam.

13.1 FINITE ELEMENT METHOD

We develop in some detail the discrete system equations emerging with the use of the finite element method for an elastic body. The virtual work equation is (12.1.28):

$$\int_\Omega ([S]\delta\{u\})^{\mathrm{T}}[D][S]\{u\}\mathrm{d}\Omega - \int_\Omega \delta\{u\}^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \int_\Gamma \delta\{u\}^{\mathrm{T}}\{t\}\mathrm{d}\Gamma = 0. \tag{1}$$

In the so-called *displacement formulation* (siirtymäformulaatio) using the finite element method, the displacement field is expressed by the finite element approximation

$$\boxed{\{u(x)\} \approx \{\tilde{u}(x)\} = [N(x)]\{a\}} \tag{2}$$

where $[N]$ is a given shape function matrix and $\{a\}$ is a column matrix of undetermined parameters or here of the so-called *nodal displacements* (solmusiirtymät). Some details of this has been explained in Section D.3.4. Based on (2), a variation gives the virtual displacement field

$$\delta\{u\} = [N]\delta\{a\}. \tag{3}$$

Substitution of (2) and (3) in (1) gives

$$\begin{aligned}
&\int_\Omega ([S][N]\delta\{a\})^{\mathrm{T}}[D][S][N]\{a\}\mathrm{d}\Omega + \\
&-\int_\Omega ([N]\delta\{a\})^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \int_\Gamma ([N]\delta\{a\})^{\mathrm{T}}\{t\}\mathrm{d}\Gamma = 0, \\
&\delta\{a\}^{\mathrm{T}}\int_\Omega ([S][N])^{\mathrm{T}}[D]([S][N])\mathrm{d}\Omega\{a\} + \\
&-\delta\{a\}^{\mathrm{T}}\int_\Omega [N]^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \delta\{a\}^{\mathrm{T}}\int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma = 0, \\
&\delta\{a\}^{\mathrm{T}}\int_\Omega [B]^{\mathrm{T}}[D][B]\mathrm{d}\Omega\{a\} + \\
&-\delta\{a\}^{\mathrm{T}}\int_\Omega [N]^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \delta\{a\}^{\mathrm{T}}\int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma = 0.
\end{aligned} \tag{4}$$

The standard shorthand notation

$$[B] = [S][N] \tag{5}$$

has been used. The steps in obtaining the final form (4) are rather obvious, for instance, the quantities $\{a\}$ and $\delta\{a\}$ do not depend on position and can be taken outside the integrals.

The left-hand side of (4) is the negative virtual work expression for the discretized body having the form

$$-\delta' W \equiv \delta\{a\}^{\mathrm{T}}(-\{Q\}) = \delta\{a\}^{\mathrm{T}}(-\{Q\}_{\mathrm{int}} - \{Q\}_{\mathrm{ext}}) \tag{6}$$

where the column vectors of generalized forces are

$$\begin{aligned}
\{Q\}_{\mathrm{int}} &= -\int_\Omega [B]^{\mathrm{T}}[D][B])\mathrm{d}\Omega\{a\}, \\
\{Q\}_{\mathrm{ext}} &= \int_\Omega [N]^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega + \int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma.
\end{aligned} \tag{7}$$

Thus the final system equations are

$$-\{Q\}_{\mathrm{int}} - \{Q\}_{\mathrm{ext}} = \{0\} \tag{8}$$

or

$$\boxed{[K]\{a\} = \{b\}} \tag{9}$$

with

$$\begin{aligned}
[K] &= \int_\Omega [B]^{\mathrm{T}}[D][B]\mathrm{d}\Omega, \\
\{b\} &= \int_\Omega [N]^{\mathrm{T}}\{f\}\mathrm{d}\Omega + \int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma.
\end{aligned} \tag{10}$$

To avoid possible confusion, we have denoted the body force intensity per unit volume finally by $\{f\} = \rho\{b\}$. The coefficient matrix $[K]$ is seen to be symmetric and it is usually called in the displacement formulation as the *stiffness matrix* (jäykkyysmatriisi).

13.2 SENSITIZED FINITE ELEMENT METHOD

Again only the elastic body case is considered. The sensitizing term in principle (12.1.38) is of the form

$$\int_\Omega ([E][D][S]\delta\{u\})^{\mathrm{T}}[\tau]([E][D][S]\{u\} + \rho\{b\})\mathrm{d}\Omega. \tag{1}$$

$$\delta\varepsilon_s = (1+\frac{n}{R})^{-1}[\frac{\mathrm{d}\delta\hat{u}}{\mathrm{d}s} - n\frac{\mathrm{d}}{\mathrm{d}s}(\frac{\mathrm{d}\delta\hat{v}}{\mathrm{d}s} - \frac{\delta\hat{u}}{R}) + \frac{\delta\hat{v}}{R}]. \tag{40}$$

Now more integrations by parts are needed to arrive finally at the equilibrium equations. There are found

$$\boxed{\begin{aligned} &\frac{\mathrm{d}N}{\mathrm{d}s} + \frac{1}{R}(\frac{\mathrm{d}M}{\mathrm{d}s} - r) + q_s = 0, \\ &\frac{\mathrm{d}^2M}{\mathrm{d}s^2} - \frac{N}{R} + q_n - \frac{\mathrm{d}r}{\mathrm{d}s} = 0 \end{aligned}} \tag{41}$$

and the force boundary conditions

$$\boxed{\begin{aligned} &-N + \hat{N} - \frac{1}{R}\left(M - \hat{M}\right) = 0, \\ &-\frac{\mathrm{d}M}{\mathrm{d}s} + \hat{Q} + r = 0, \\ &M - \hat{M} = 0. \end{aligned}} \tag{42}$$

Equations (41) can be derived directly from (36) by employing the relationship $Q = \mathrm{d}M/\mathrm{d}s - r$ obtained from the last of them. Altogether, it seems that the Timoshenko beam theory is a cleaner starting point. If the Bernoulli theory is used, the modifications needed can be performed just on the final equilibrium equations.

Remark 13.2. The Bernoulli theory contains the kinematical constraint $\gamma_{sn} = 0$ or $\theta = \mathrm{d}\hat{v}/\mathrm{d}\hat{s} - \hat{u}/R$. This means that the corresponding shearing stress τ_{sn} and (integrated from it) the shearing force Q are constraint forces. In this section we have used kinematically admissible virtual displacements as we have taken variations of the actual (assumed) displacements (7) or (38). When applying kinematically admissible virtual displacements, according to statement (5.3.1) the constraint forces disappear from the formulation. This is seen to happen also here in connection of the Bernoulli theory as Q does not appear any more in (41). □

13.4 ELASTIC BEAM

13.4.1 Kinematics

For now on, we consider straight beams only. Most of the expressions needed are obtained from Section 13.3 by simplifications. Figure 13.3 shows the main notations used.

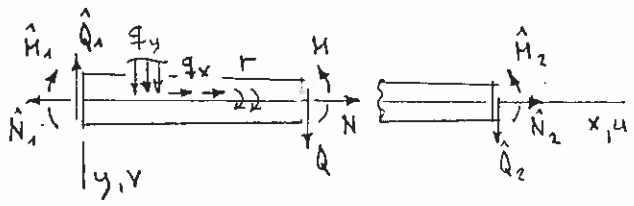


Figure 13.3 Straight beam.

We employ for generality the Timoshenko beam model and take the Bernoulli model as a special case. The functions, describing wholly the displacement state of the beam are $u(x)$, $v(x)$, $\theta(x)$ The relevant strain expressions are (formula (13.3.15))

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} - y\frac{\mathrm{d}\theta}{\mathrm{d}x} = \hat{\varepsilon} + y\kappa, \tag{1}$$

and

$$\gamma = \frac{\mathrm{d}v}{\mathrm{d}x} - \theta. \tag{2}$$

The hats from the symbols of the displacement components of the beam axis and the subscripts from the strain symbols have been dropped for simplicity of notation. The quantity

$$\hat{\varepsilon} \equiv \frac{\mathrm{d}u}{\mathrm{d}x}. \tag{3}$$

is the axial strain at the beam axis. The quantity

$$\kappa \equiv -\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{4}$$

is called (roughly) the *change of curvature* (käyristymä) of the beam axis. Formulas (1) and (2) show that in the Timoshenko model the axial strain is distributed linearly in the cross-sectional fiber direction and that the shearing strain is constant. In the Bernoulli model the latter form of (1) is still valid with the change of curvature given by

$$\kappa \equiv -\frac{\mathrm{d}^2v}{\mathrm{d}x^2} \tag{5}$$

and the shearing strain is zero.

13.4.2 Constitutive relation

Assuming that all stress components but σ_x and τ_{xy} are zero and employing the generalized Hooke's law for an isotropic elastic material gives the constitutive relations

$$\begin{aligned} \sigma_x &= E\varepsilon_x = E\varepsilon, \\ \tau_{xy} &= G\gamma_{xy} = G\gamma, \end{aligned} \tag{6}$$

where E is Young's modulus and G the shear modulus of the material. The stress resultants from (13.3.25) obtain the forms

$$\begin{aligned} N &= \int_A \sigma_x \mathrm{d}A = \int_A E\varepsilon \mathrm{d}A = \int_A E(\hat{\varepsilon} + y\kappa)\mathrm{d}A = \\ &= (\int_A E\mathrm{d}A)\hat{\varepsilon} + (\int_A Ey\mathrm{d}A)\kappa = EA\hat{\varepsilon} + ES\kappa, \\ Q &= \int_A \tau_{xy}\mathrm{d}A = \int_A G\gamma \mathrm{d}A = (\int_A G\mathrm{d}A)\gamma = GA\gamma, \\ M &= \int_A \sigma_x y\,\mathrm{d}A = \int_A E\varepsilon y\,\mathrm{d}A = \int_A E(\hat{\varepsilon} + y\kappa)y\,\mathrm{d}A \\ &= (\int_A Ey\mathrm{d}A)\hat{\varepsilon} + (\int_A Ey^2\,\mathrm{d}A)\kappa = ES\hat{\varepsilon} + EI\kappa. \end{aligned} \tag{7}$$

We have used the shorthand notations

$$EA \equiv \int_A E\mathrm{d}A, \quad ES \equiv \int_A Ey\mathrm{d}A, \quad GA \equiv \int_A G\mathrm{d}A, \quad EI \equiv \int_A Ey^2\,\mathrm{d}A. \tag{8}$$

These left hand side notations have to be understood in the general case (if E and G depend on y; layered beam) as double letter symbols. EA is called the *axial stiffness* (vetojäykkyys), GA the *shearing stiffness* (leikkausjäykkyys) and EI the *bending stiffness* (taivutusjäykkyys) ($[EA] = [GA] = \mathrm{N}$, $[EI] = \mathrm{Nm}^2$). Only if E and G are constant over the cross-section, we have actually products of E and A, G and A and E and I, the last quantity being the cross-sectional *second moment* (pintaneliömomentti, pintahitausmomentti))

$$I \equiv \int_A y^2\,\mathrm{d}A. \tag{9}$$

In the following we assume for simplicity that the beam axis has been selected so that $ES = 0$. (This is always possible at a given cross-section but the axis defined in this way may not be any more straight for an inhomogeneous beam.). The constitutive relations for the stress resultants are now

$$N = EA\hat{\varepsilon}, \quad Q = GA\gamma, \quad M = EI\kappa. \tag{10}$$

When a beam problem has been solved say by a displacement formulation, we know the quantities $u(x)$, $v(x)$, $\theta(x)$. From these we may calculate $\hat{\varepsilon}$, γ, κ

and then using (10) the stress resultants. The stresses can be evaluated alternatively from the stress resultants (see (1), (2), (6) (10)):

$$\sigma = E\frac{N}{(EA)} + Ey\frac{M}{(EI)}, \qquad \tau = G\frac{Q}{GA}. \tag{11}$$

If E and G are constants on a cross-section, the conventional formulas

$$\sigma = \frac{N}{A} + y\frac{M}{I}, \qquad \tau = \frac{Q}{A}. \tag{12}$$

are arrived at. Thus in this case the normal stress varies linearly over the depth and the shearing stress is constant.

Remark 13.3. The latter formula (12) shows clearly that we are dealing with an approximate theory. No external tractions act usually in the x-direction on the upper and lower surfaces of a beam. Thus we should have there $\tau_{xy} = \tau_{yx} = 0$. In reality the cross-sections of a beam do not remain completely plane but *warping* (käyristyminen) takes place. This fact is accounted for in practice by amending the second formula (10) into the form

$$Q = kGA\gamma, \tag{13}$$

where k is the so-called *shear correction factor* (liukumakorjauskerroin) ($[k] = -$). Also the inverse value is often called with this name. Literature contains procedures to determine k. For instance the value $k = 5/6$ is used for a rectangular cross-section for a homogeneous material. Shearing strain in formulas like (2) and (13) should be considered as some kind of average shear which takes in an overall way into into account the flexibility of the beam with respect to the shearing force. *The detailed shearing stresses should not in fact be calculated using formulas* (11) *or* (12). More accurate results are obtained by assuming the values for the normal stress σ correct and making then use of the equations of equilibrium for a continuum, e.g., Oden (1967). □

13.4.3 Virtual work equation

We write the virtual work equation in the form

$$-\delta' W_{\mathrm{int}} - \delta' W_{\mathrm{ext}} = 0 \tag{14}$$

and collect the appropriate expressions from (13.1.24) and (13.1.34) to give

$$\begin{aligned} -\delta' W_{\mathrm{int}} &= \int_s [N(\frac{\mathrm{d}\delta\hat{u}}{\mathrm{d}s} + \frac{\delta\hat{v}}{R}) - M\frac{\mathrm{d}\delta\theta}{\mathrm{d}s} + Q(\frac{\mathrm{d}\delta\hat{v}}{\mathrm{d}s} - \frac{\delta\hat{u}}{R} - \delta\theta)]\mathrm{d}s \\ &= \int_x [N\frac{\mathrm{d}\delta u}{\mathrm{d}x} - M\frac{\mathrm{d}\delta\theta}{\mathrm{d}x} + Q(\frac{\mathrm{d}\delta v}{\mathrm{d}x} - \delta\theta)]\mathrm{d}x \end{aligned}$$

$$= \int_a^b (N\delta\hat{\varepsilon} + M\delta\kappa + Q\delta\gamma)\,\mathrm{d}x \tag{15}$$

and

$$\begin{aligned} -\delta' W_{\text{ext}} &= -\int_s (q_s\delta\hat{u} + q_n\delta\hat{v} + r\delta\theta)\,\mathrm{d}s - \Big|_{s_1}^{s_2} (\hat{N}\delta\hat{u} + \hat{Q}\delta\hat{v} - \hat{M}\delta\theta) \\ &= -\int_x (q_x\delta u + q_y\delta v + r\delta\theta)\,\mathrm{d}x - \Big|_{x_1}^{x_2} (\hat{N}\delta u + \hat{Q}\delta v - \hat{M}\delta\theta) \\ &= -\int_a^b (q_x\delta u + q_y\delta v + r\delta\theta)\,\mathrm{d}x - \Big|_a^b (\hat{N}\delta u + \hat{Q}\delta v - \hat{M}\delta\theta). \end{aligned} \tag{16}$$

The steps used arrive at the final forms should be rather obvious.

It is realized that here is now no coupling between "stretching" and "bending" and we in fact can write two separate virtual work equations (consider what follows by taking first just the variation $\delta u \neq 0$):

$$\boxed{\int_a^b N\delta\hat{\varepsilon}\,\mathrm{d}x - \int_a^b q_x\delta u\,\mathrm{d}x - \Big|_a^b \hat{N}\delta u = 0} \tag{17}$$

and (we put $q_y = q$)

$$\boxed{\int_a^b (M\delta\kappa + Q\delta\gamma)\,\mathrm{d}x - \int_a^b (q\delta v + r\delta\theta)\mathrm{d}x - \Big|_a^b (\hat{Q}\delta v - \hat{M}\delta\theta) = 0.} \tag{18}$$

We concentrate in the following only on beam bending and use (18) as the starting point. Summarising, we have in (18)

$$\delta\kappa = -\frac{\mathrm{d}\delta\theta}{\mathrm{d}x}, \qquad \delta\gamma = \frac{\mathrm{d}\delta v}{\mathrm{d}x} - \delta\theta. \tag{19}$$

Further, kinematical boundary conditions concerning functions $v(x)$ and $\theta(x)$ must be taken into account so that the corresponding variations are set to zero. For an elastic beam, expressions

$$Q = kGA\gamma, \quad M = EI\kappa \tag{20}$$

are finally introduced.

In the Bernoulli beam model the shearing strain γ vanishes, $\theta \to \mathrm{d}v/\mathrm{d}x$ and $\kappa = -\mathrm{d}^2 v/\mathrm{d}x^2$ and the virtual work equation (18) takes the form

$$\int_a^b M\delta\kappa\,\mathrm{d}x - \int_a^b (q\delta v + r\frac{\mathrm{d}\delta v}{\mathrm{d}x})\mathrm{d}x - \Big|_a^b (\hat{Q}\delta v - \hat{M}\frac{\mathrm{d}\delta v}{\mathrm{d}x}) = 0, \tag{21}$$

in which

$$\delta\kappa = -\frac{\mathrm{d}^2\delta v}{\mathrm{d}x^2}. \tag{22}$$

The kinematical boundary conditions concern function $v(x)$.

13.4.4 Finite element method

We will consider only the Timoshenko beam case. To proceed in the fashion usual in the finite element literature, we introduce matrix notation similarly as in Chapter 12. We define

$$\begin{aligned} \{u\} &= \begin{Bmatrix} v \\ \theta \end{Bmatrix}, \quad \{f\} = \begin{Bmatrix} q \\ r \end{Bmatrix}, \quad \{t\} = \begin{Bmatrix} V \\ B \end{Bmatrix}, \quad \{\sigma\} = \begin{Bmatrix} Q \\ M \end{Bmatrix}, \\ \{\varepsilon\} &= \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} = \begin{Bmatrix} \mathrm{d}v/\mathrm{d}x - \theta \\ -\mathrm{d}\theta/\mathrm{d}x \end{Bmatrix} = \begin{bmatrix} \mathrm{d}/\mathrm{d}x & -1 \\ 0 & -\mathrm{d}/\mathrm{d}x \end{bmatrix} \begin{Bmatrix} v \\ \theta \end{Bmatrix} = [S]\{u\} \end{aligned} \tag{23}$$

with the strain-displacement operator matrix

$$[S] = \begin{bmatrix} \mathrm{d}/\mathrm{d}x & -1 \\ 0 & -\mathrm{d}/\mathrm{d}x \end{bmatrix}. \tag{24}$$

The notation V and B refer to the shearing force and the bending moment at the beam ends with such sign changes that V is positive in the positive y-axis direction and B positive in the clockwise direction. The virtual work equation (18) can now be written as

$$\int_\Omega ([S]\delta\{u\})^{\mathrm{T}}\{\sigma\}\mathrm{d}\Omega - \int_\Omega \delta\{u\}^{\mathrm{T}}\{f\}\mathrm{d}\Omega - \int_\Gamma \delta\{u\}^{\mathrm{T}}\{t\}\mathrm{d}\Gamma = 0. \tag{25}$$

Further, in the elastic case, employing expressions (20),

$$\{\sigma\} = [D]\{\varepsilon\} \tag{26}$$

with

$$[D] = \begin{bmatrix} kGA & 0 \\ 0 & EI \end{bmatrix}. \tag{27}$$

Taking the finite element approximation

$$\{u\} \approx \{\tilde{u}\} = [N]\{a\}, \tag{28}$$

we can write down without further elaboration the system equations by copying formulas (13.1.3) and (13.1.4):

$$[K]\{a\} = \{b\} \tag{29}$$

with

$$\begin{aligned} [K] &= \int_\Omega [B]^{\mathrm{T}}[D][B]\mathrm{d}\Omega, \\ \{b\} &= \int_\Omega [N]^{\mathrm{T}}\{f\}\mathrm{d}\Omega + \int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma. \end{aligned} \tag{30}$$

Example 13.1. We develop some finite element expressions in some detail to have more illuminating formulas.

We employ two-noded linear element approximation both for $v(x)$ and $\theta(x)$. The element shape functions are thus the same as used in Example D.22:

$$N_1 = 1-\xi, \quad N_2 = \xi \tag{a}$$

where $\xi = x/h$ $(0 \le \xi \le 1)$ is a dimensionless elementwise coordinate, x is a local elementwise coordinate measured from the left-hand end of the element and h is the length of the element.

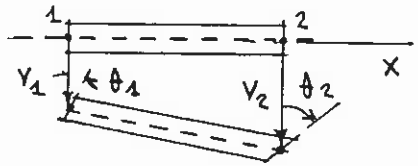


Figure (a)

We do not use any indices referring to an element to simplify the notation. The element nodal displacements are selected as shown exaggerated in Figure (a) and listed in the order

$$\{a\} = [v_1\ \theta_1\ v_2\ \theta_2]^{\mathrm{T}}. \tag{b}$$

The element approximation

$$\{\tilde{u}\} = [N]\{a\} \tag{c}$$

is thus in more detail

$$\sim \begin{Bmatrix} \tilde{v} \\ \tilde{\theta} \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1-\xi & 0 & 1-\xi & 0 \\ 0 & \xi & 0 & \xi \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}. \tag{d}$$

We can employ the general formulas developed earlier by considering at this phase the element to represent the whole system under study. Matrix $[B]$ defined by formula (13.1.5) is here

$$\begin{aligned} [B] = [S][N] &= \begin{bmatrix} \mathrm{d}/\mathrm{d}x & -1 \\ 0 & -\mathrm{d}/\mathrm{d}x \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{d}N_1/\mathrm{d}x & -N_1 & \mathrm{d}N_2/\mathrm{d}x & -N_2 \\ 0 & -\mathrm{d}N_1/\mathrm{d}x & 0 & -\mathrm{d}N_2/\mathrm{d}x \end{bmatrix} \\ &= \begin{bmatrix} -1/h & -(1-\xi) & 1/h & -\xi \\ 0 & 1/h & 0 & -1/h \end{bmatrix}. \end{aligned} \tag{e}$$

Further,

$$\begin{aligned} [D][B] &= \begin{bmatrix} kGA & 0 \\ 0 & EI \end{bmatrix} \begin{bmatrix} -1/h & -(1-\xi) & 1/h & -\xi \\ 0 & 1/h & 0 & -1/h \end{bmatrix} \\ &= \begin{bmatrix} -kGA/h & -kGA(1-\xi) & kGA/h & -kGA\xi \\ 0 & EI/h & 0 & -EI/h \end{bmatrix} \end{aligned} \tag{f}$$

and

$$\begin{aligned} [B]^{\mathrm{T}}[D][B] &= \begin{bmatrix} -1/h & 0 \\ -(1-\xi) & 1/h \\ 1/h & 0 \\ -\xi & -1/h \end{bmatrix} \begin{bmatrix} -kGA/h & -kGA(1-\xi) & kGA/h & -kGA\xi \\ 0 & EI/h & 0 & -EI/h \end{bmatrix} \\ &= \begin{bmatrix} \frac{kGA}{h^2} & \frac{kGA}{h}(1-\xi) & -\frac{kGA}{h^2} & \frac{kGA}{h}\xi \\ \frac{kGA}{h}(1-\xi) & kGA(1-\xi)^2 + \frac{EI}{h^2} & -\frac{kGA}{h}(1-\xi) & kGA(1-\xi)\xi - \frac{EI}{h^2} \\ -\frac{kGA}{h^2} & -\frac{kGA}{h}(1-\xi) & \frac{kGA}{h^2} & -\frac{kGA}{h}\xi \\ \frac{kGA}{h}\xi & kGA(1-\xi)\xi - \frac{EI}{h^2} & -\frac{kGA}{h}\xi & kGA\xi^2 + \frac{EI}{h^2} \end{bmatrix}. \end{aligned} \tag{g}$$

Integration of this over the element domain gives the element coefficient matrix (element stiffness matrix)

$$\begin{aligned} [K] &= \int_{\Omega^e} [B]^{\mathrm{T}}[D][B]\mathrm{d}\Omega \\ &= \frac{EI}{h} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + kGAh \begin{bmatrix} 1/h^2 & 1/2h & -1/h^2 & 1/2h \\ 1/2h & 1/3 & -1/2h & 1/6 \\ -1/h^2 & -1/2h & 1/h^2 & -1/2h \\ 1/2h & 1/6 & -1/2h & 1/3 \end{bmatrix}. \end{aligned} \tag{h}$$

It has been decomposed in two parts representing the "bending and shearing behaviour". The multipliers EI/h and $kGAh$ have been selected so that they have the same physical dimension and thus the ratio

$$\hat{\varepsilon}_h = \frac{EI}{kGAh^2} \tag{i}$$

is dimensionless. It is an elementwise measure between the bending and shearing stiffnesses.

The matrix product

$$[N]^T\{f\}=\begin{bmatrix}1-\xi & 0\\ 0 & 1-\xi\\ \xi & 0\\ 0 & \xi\end{bmatrix}\begin{Bmatrix}q\\ r\end{Bmatrix}=\begin{Bmatrix}(1-\xi)q\\ (1-\xi)r\\ \xi q\\ \xi r\end{Bmatrix} \tag{j}$$

and integration gives the element contribution to the right-hand side

$$\{b\}=\int_{\Omega^e}[N]^T\{f\}\,d\Omega=\frac{h}{2}\begin{Bmatrix}q\\ r\\ q\\ r\end{Bmatrix}, \tag{k}$$

where we have assumed constant q and r in the element.

For future application we record a result obtained with numerical integration. When the numerical integration is performed on purpose so that certain terms are evaluated inaccurately, the procedure is called *reduced integration* (redusoitu integrointi) and sometimes also as *selective integration* (selektiivinen integrointi). We apply here the simple one point integration formula

$$\int_0^h f(x)dx \approx hf(h/2) \tag{l}$$

to evaluate the stiffness matrix from the integrand (g) There is obtained

$$[K]=\frac{EI}{h}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}+kGAh\begin{bmatrix}1/h^2 & 1/2h & -1/h^2 & 1/2h\\ 1/2h & 1/4 & -1/2h & 1/4\\ -1/h^2 & -1/2h & 1/h^2 & -1/2h\\ 1/2h & 1/4 & -1/2h & 1/4\end{bmatrix}. \tag{m}$$

Only the elements corresponding to quadratic terms in (g) have changed their values.

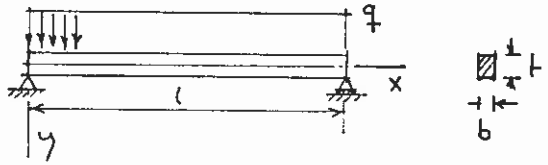


Figure (b)

A simply supported beam under constant loading q ($r=0$) has been analysed (Figure (b)). The cross section of the beam is rectangular and the material is isotropic with Poisson's ratio $\nu=1/3$. Taking $k=5/6$, we obtain

$$EI=E\frac{bt^3}{12}=\frac{1}{12}Ebt^3,$$
$$kGA=\frac{5}{6}\frac{E}{2(1+1/3)}bt=\frac{5}{16}Ebt. \tag{n}$$

We take here $t=l/10$ and obtain for the global dimensionless ratio the value

$$\hat{\varepsilon}=\frac{EI}{kGAl^2}=\frac{1}{375}. \tag{o}$$

The exact solution for the vertical deflection is found to be, Dym and Shames (1973),

$$v=\frac{ql^4}{EI}\{\frac{1}{384}[5-6(1-\xi)^2+(1-\xi)^4]+\frac{1}{8}\hat{\varepsilon}[1-(1-\xi)^2]\}, \tag{p}$$

where $\xi=x/(l/2)$ is here a global dimensionless coordinate. A finite element solution has been obtained by four equal length ($h=l/4$) elements. The elementwise ratio

$$\hat{\varepsilon}_h=\frac{EI}{kGAh^2}=\frac{16}{375}. \tag{q}$$

Figure (c) shows the exact vertical deflection and the result by the finite element method.

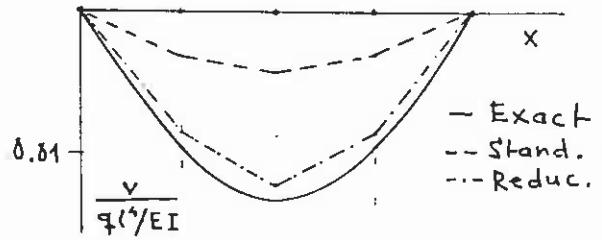


Figure (c)

The accuracy of the standard finite element solution (Stand.) is seen to be very poor. In this connection the term "*locking*" (lukkiutuminen) is commonly used. For a slender beam it is found that a very fine mesh is needed to obtain good enough results for practice. A slender ($\hat{\varepsilon}$ is small) Timoshenko beam acts in fact nearly as a Bernoulli beam and a detailed study shows that the standard discrete model puts too much emphasis on the satisfaction of just the Bernoulli kinematic constraint $dv/dx-\theta=0$. A common remedy is to use reduced integration. Intuitively, reduced integration makes the model more flexible. In the example case this is also seen to be the case (Reduc.). The error in the deflection at the center of the beam is still 9.7%. There is a large literature concerning the application of reduced and selective integration for plates and shells and for nearly incompressible material, e.g., Zienkiewicz and Taylor.

13.4.5 Sensitized finite element method

The Timoshenko beam equilibrium equations are (see equations (13.3.36))

$$\frac{dQ}{dx}+q=0,$$
$$Q-\frac{dM}{dx}+r=0 \tag{31}$$

or in matrix notation

$$\begin{bmatrix} \mathrm{d}/\mathrm{d}x & 0 \\ 1 & -\mathrm{d}/\mathrm{d}x \end{bmatrix} \begin{Bmatrix} Q \\ M \end{Bmatrix} + \begin{Bmatrix} q \\ r \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{32}$$

Thus the equilibrium operator matrix

$$[E] = \begin{bmatrix} \mathrm{d}/\mathrm{d}x & 0 \\ 1 & -\mathrm{d}/\mathrm{d}x \end{bmatrix} \tag{33}$$

is here not equal to the transpose of the strain-displacement operator matrix (24) as was the case in the two-dimensional general theory considered in Chapter 12.

The system equations are applying formulas (13.2.3) and (13.2.4):

$$[K]\{a\} = \{b\} \tag{34}$$

with

$$\begin{aligned} [K] &= \int_\Omega [B]^{\mathrm{T}}[D][B])\mathrm{d}\Omega + \int_\Omega ([E][D][B])^{\mathrm{T}}[\tau]([E][D][B])\mathrm{d}\Omega, \\ \{b\} &= \int_\Omega [N]^{\mathrm{T}}\{f\}\mathrm{d}\Omega + \int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma - \int_\Omega ([E][D][B])^{\mathrm{T}}[\tau]\{f\}\mathrm{d}\Omega. \end{aligned} \tag{35}$$

A suitable sensitizing parameter matrix (assumed constant in an element) has been determined in Freund and Salonen (1998) for the case of two-noded linear elements using the patch test similarly as explained in Section D.3.5 to have the value

$$[\tau] = -\frac{1}{12EI/h^2 + kGA} \begin{bmatrix} EI/kGA & 0 \\ 0 & 1 \end{bmatrix}, \tag{36}$$

where h is the element length.

Example 13.2. We extend the study of Example 13.2 by including the sensitizing terms into the formulation.

According to formulas (35), the contributions to the coefficient matrix and to the right-hand side from sensitizing are

$$[K]_s = \int_\Omega ([E][D][B])^{\mathrm{T}}[\tau]([E][D][B])\mathrm{d}\Omega \tag{a}$$

and

$$\{b\}_s = -\int_\Omega ([E][D][B])^{\mathrm{T}}[\tau]\{f\}\mathrm{d}\Omega. \tag{b}$$

Again, the two-noded element of Example 13.1 is considered. From Example 13.1,

$$[D][B] = \begin{bmatrix} -kGA/h & -kGA(1-\xi) & kGA/h & -kGA\xi \\ 0 & EI/h & 0 & -EI/h \end{bmatrix}. \tag{c}$$

The equilibrium matrix (33) is

$$[E] = \begin{bmatrix} \mathrm{d}/\mathrm{d}x & 0 \\ 1 & -\mathrm{d}/\mathrm{d}x \end{bmatrix} \tag{d}$$

and assuming constant data (see Remark 13.1)

$$\begin{aligned} [E][D][B] &= \begin{bmatrix} \mathrm{d}/\mathrm{d}x & 0 \\ 1 & -\mathrm{d}/\mathrm{d}x \end{bmatrix} \begin{bmatrix} -kGA/h & -kGA(1-\xi) & kGA/h & -kGA\xi \\ 0 & EI/h & 0 & -EI/h \end{bmatrix} \\ &= \begin{bmatrix} 0 & kGA/h & 0 & -kGA/h \\ -kGA/h & -kGA(1-\xi) & kGA/h & -kGA\xi \end{bmatrix}. \end{aligned} \tag{e}$$

It is seen that the bending stiffness has vanished from this matrix. The sensitizing parameter matrix from (36) is

$$[\tau] = -\frac{1}{12EI/h^2 + kGA} \begin{bmatrix} EI/kGA & 0 \\ 0 & 1 \end{bmatrix}. \tag{f}$$

Thus, using shorthand notation

$$c = -\frac{1}{12EI/h^2 + kGA} = -\frac{1/(kGA)}{12EI/(kGAh^2) + 1} = -\frac{1/(kGA)}{12\hat{\varepsilon}_h + 1}, \tag{g}$$

$$\begin{aligned} ([E][D][B])^{\mathrm{T}}[\tau] &= c \begin{bmatrix} 0 & -kGA/h \\ kGA/h & -kGA(1-\xi) \\ 0 & kGA/h \\ -kGA/h & -kGA\xi \end{bmatrix} \begin{bmatrix} EI/kGA & 0 \\ 0 & 1 \end{bmatrix} \\ &= c \begin{bmatrix} 0 & -kGA/h \\ EI/h & -kGA(1-\xi) \\ 0 & kGA/h \\ -EI/h & -kGA\xi \end{bmatrix}. \end{aligned} \tag{h}$$

Further,

$$([E][D][B])^{\mathrm{T}}[\tau]([E][D][B]) =$$

$$= c \begin{bmatrix} 0 & -kGA/h \\ EI/h & -kGA(1-\xi) \\ 0 & kGA/h \\ -EI/h & -kGA\xi \end{bmatrix} \begin{bmatrix} 0 & kGA/h & 0 & -kGA/h \\ -kGA/h & -kGA(1-\xi) & kGA/h & -kGA\xi \end{bmatrix}$$

$$= c\begin{bmatrix} \frac{(kGA)^2}{h^2} & \frac{(kGA)^2}{h}(1-\xi) & \cdots \\ \frac{(kGA)^2}{h}(1-\xi) & \frac{EI\cdot kGA}{h^2}+(kGA)^2(1-\xi)^2 & \cdots \\ -\frac{(kGA)^2}{h^2} & -\frac{(kGA)^2}{h}(1-\xi) & \cdots \\ \frac{(kGA)^2}{h}\xi & -\frac{EI\cdot kGA}{h^2}+(kGA)^2\xi(1-\xi) & \cdots \end{bmatrix}$$

$$\begin{bmatrix} -\frac{(kGA)^2}{h^2} & \frac{(kGA)^2}{h}\xi \\ -\frac{(kGA)^2}{h}(1-\xi) & -\frac{EI\cdot kGA}{h^2}+\frac{(kGA)^2}{h}\xi(1-\xi) \\ \frac{(kGA)^2}{h^2} & -\frac{(kGA)^2}{h}\xi \\ -\frac{(kGA)^2}{h}\xi & \frac{EI\cdot kGA}{h^2}+\frac{(kGA)^2}{h}\xi^2 \end{bmatrix} \qquad \text{(i)}$$

Performing the integration over the element domain, we obtain the sensitising element stiffness matrix

$$[K]_s = c(kGA)^2\begin{bmatrix} 1/h & 1/2 & -1/h & 1/2 \\ 1/2 & EI/(kGAh)+h/3 & -1/2 & -EI/(kGAh)+h/6 \\ -1/h & -1/2 & 1/h & -1/2 \\ 1/2 & -EI/(kGAh)+h/6 & -1/2 & EI/(kGAh)+h/3 \end{bmatrix}. \qquad \text{(j)}$$

A more transparent form is finally

$$[K]_s = -\frac{EI}{(12\hat{\varepsilon}_h+1)h}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} +$$

$$-\frac{kGAh}{12\hat{\varepsilon}_h+1}\begin{bmatrix} 1/h^2 & 1/2h & -1/h^2 & 1/2h \\ 1/2h & 1/3 & -1/2h & 1/6 \\ -1/h^2 & -1/2h & 1/h^2 & -1/2h \\ 1/2h & 1/6 & -1/2h & 1/3 \end{bmatrix}. \qquad \text{(k)}$$

The interpretation of this result is interesting. Comparison with equation (h) of Example 13.1 shows that contribution (k) is is obtained from the element stiffness matrix of Example 13.1 by multiplying with the factor $-1/(12\hat{\varepsilon}_h+1)$. Thus the final element stiffness matrix in the sensitized formulation is found by multipying the stiffness matrix of the standard formulation by the factor $12\hat{\varepsilon}_h/(12\hat{\varepsilon}_h+1)$.

Finally,

$$[\tau]\{f\} = c\begin{bmatrix} EI/kGA & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} q \\ r \end{Bmatrix} = c\begin{Bmatrix} EI/kGA\cdot q \\ r \end{Bmatrix} \qquad \text{(l)}$$

and

$$([E][D][B])^T[\tau]\{f\} = c\begin{bmatrix} 0 & -kGA/h \\ kGA/h & -kGA(1-\xi) \\ 0 & kGA/h \\ -kGA/h & -kGA\xi \end{bmatrix}\begin{Bmatrix} EI/kGA\cdot q \\ r \end{Bmatrix}$$

$$= c\begin{Bmatrix} -kGA/h\cdot r \\ EI/h\cdot q - kGA(1-\xi)r \\ kGA/h\cdot r \\ -EI/h\cdot q - kGA\xi r \end{Bmatrix}. \qquad \text{(m)}$$

Assuming again elementwise constant q and r, integration over the element domain gives

$$\{b\}_s = \frac{1/(kGA)}{12\hat{\varepsilon}_h+1}\begin{Bmatrix} -kGA\cdot r \\ EI\cdot q - kGAh/2\cdot r \\ kGA\cdot r \\ -EI\cdot q - kGAh/2\cdot r \end{Bmatrix}$$

$$= \frac{\hat{\varepsilon}_h}{12\hat{\varepsilon}_h+1}qh^2\begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix} + \frac{1}{12\hat{\varepsilon}_h+1}rh\begin{Bmatrix} -1/h \\ -1/2 \\ 1/h \\ -1/2 \end{Bmatrix}. \qquad \text{(n)}$$

It is interesting to note that the non-zero generalised forces due to a constant q are associated in the standard formulation with the vertical displacements (see formula (k), Example 13.1) but here they are associated with the rotational displacements (see formula (n).

The sensitized finite element solution of the problem in Example 13.1 with four equal length elements gave a more accurate solution than the reduced integration finite element result shown in Figure (c), Example 13.1. The error in the deflection at the center of the beam was 3.2%.

13.5 REFERENCES

Freund, J. and E.-M. Salonen (1998): Sensitizing the Timoshenko beam and the Reissner Mindlin plate finite element solution, Report no. 50, *Laboratory of Theoretical and Applied Mechanics, Helsinki University of Technology*.

Dym, C. L. and I. H. Shames (1973): *Solid Mechanics, A Variational Approach*, Mc-Graw-Hill, Tokyo.

Fung, Y. C. (1965): *Foundations of Solid Mechanics*, Prentice-Hall, Englewoods Cliffs.

Kreyszig, E. (1967): *Advanced Engineering Mathematics*, 2nd ed., Wiley, New York.

Oden, J. T. (1967): *Mechanics of Elastic Structures*, Mc-Graw-Hill, New York.

Zienkiewicz, O. C. and R. L. Taylor (1991): *The Finite Element Method*, 4th ed. Vol. 2, McGraw-Hill, London.

CHAPTER 14

APPLICATIONS IN DYNAMICS

The inertia force method is employed to get the governing equations in the dynamic case. This is demonstrated first in connection with general small displacement finite element equations both in the standard and in the sensitized form. In the case of an elastic beam more detailed formulas are given.

14.1 FINITE ELEMENT METHOD

The static form of the virtual work equation for an elastic body with small displacements using matrix notation is (12.1.28):

$$\int_\Omega ([S]\delta\{u\})^{\mathrm{T}}[D][S]\{u\}\mathrm{d}\Omega - \int_\Omega \delta\{u\}^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \int_\Gamma \delta\{u\}^{\mathrm{T}}\{t\}\mathrm{d}\Gamma = 0. \tag{1}$$

We obtain the dynamic form according to Remark 12.3 by the substitution

$$\{b\} := \{b\} - \frac{\partial^2}{\partial t^2}\{u\} \tag{2}$$

or with alternative notation by

$$\{f\} := \{f\} - \rho\frac{\partial^2}{\partial t^2}\{u\}. \tag{3}$$

In the finite element method we here employ the Kantorovitch type approximation (see Section D.3.3) so instead on form (13.1.2) we write

$$\boxed{\{u(x,t)\} \approx \{\bar{u}(x,t)\} = [N(x)]\{a(t)\},} \tag{4}$$

i. e., the nodal displacements are considered functions of time. Thus they have again the role of generalized displacements of particle mechanics.

The approximation of the acceleration becomes

$$\frac{\partial^2}{\partial t^2}\{u\} \approx [N]\{\ddot{a}\} \tag{5}$$

and, accordingly, we can employ the substitution

$$\{f\} := \{f\} - \rho[N]\{\ddot{a}\} \tag{6}$$

in the system equations right-hand side expression (13.1.10):

$$\{b\} = \int_\Omega [N]^{\mathrm{T}}\{f\}\mathrm{d}\Omega + \int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma. \tag{7}$$

We have thus the alteration

$$\int_\Omega [N]^{\mathrm{T}}\{f\}\mathrm{d}\Omega := \int_\Omega [N]^{\mathrm{T}}\{f\}\mathrm{d}\Omega - \int_\Omega [N]^{\mathrm{T}}\rho[N]\mathrm{d}\Omega\{\ddot{a}\}. \tag{8}$$

It should be noticed that as the nodal accelerations $\{\ddot{a}\}$ do not depend on position, they can be taken outside the integral sign. We see that the system equations (13.1.9) are transformed to

$$\boxed{[M]\{\ddot{a}\} + [K]\{a\} = \{b\}} \tag{9}$$

with

$$\begin{aligned} [M] &= \int_\Omega [N]^{\mathrm{T}}\rho[N]\mathrm{d}\Omega, \\ [K] &= \int_\Omega [B]^{\mathrm{T}}[D][B]\mathrm{d}\Omega, \\ \{b\} &= \int_\Omega [N]^{\mathrm{T}}\{f\}\mathrm{d}\Omega + \int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma. \end{aligned} \tag{10}$$

We have obtained a linear second order ordinary differential equation system. Matrix $[M]$ is symmetric and it is called the *mass matrix* (massamatriisi). The solution of system (9) happens usually by some numerical time integration method.

14.2 SENSITIZED FINITE ELEMENT METHOD

In the sensitized static finite element system equations, the right-hand side term is (13.2.4):

$$\{b\} = \int_\Omega [N]^{\mathrm{T}}\{f\}\mathrm{d}\Omega + \int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma - \int_\Omega ([E][D][B])^{\mathrm{T}}[\tau]\{f\}\mathrm{d}\Omega. \tag{1}$$

Substitution of (14.1.6) produces the alterations

$$\begin{aligned} \int_\Omega [N]^{\mathrm{T}}\{f\}\mathrm{d}\Omega &:= \int_\Omega [N]^{\mathrm{T}}\{f\}\mathrm{d}\Omega - \int_\Omega [N]^{\mathrm{T}}\rho[N]\mathrm{d}\Omega\{\ddot{a}\}, \\ \int_\Omega ([E][D][B])^{\mathrm{T}}[\tau]\{f\}\mathrm{d}\Omega &:= \int_\Omega ([E][D][B])^{\mathrm{T}}[\tau]\{f\}\mathrm{d}\Omega + \\ &\quad - \int_\Omega ([E][D][B])^{\mathrm{T}}[\tau]\rho[N]\mathrm{d}\Omega\{\ddot{a}\}. \end{aligned} \tag{2}$$

Now the system equations (13.2.3) are thus transformed to

$$\boxed{[M]\{\ddot{a}\} + [K]\{a\} = \{b\}} \tag{3}$$

with

$$[M]=\int_\Omega [N]^{\mathrm{T}}\rho[N]\mathrm{d}\Omega-\int_\Omega([E][D][B])^{\mathrm{T}}[\tau]\rho[N]\mathrm{d}\Omega,$$
$$[K]=\int_\Omega [B]^{\mathrm{T}}[D][B]\mathrm{d}\Omega+\int_\Omega([E][D][B])^{\mathrm{T}}[\tau]([E][D][B])\mathrm{d}\Omega, \qquad (4)$$
$$\{b\}=\int_\Omega [N]^{\mathrm{T}}\{f\}\mathrm{d}\Omega+\int_\Gamma [N]^{\mathrm{T}}\{t\}\mathrm{d}\Gamma-\int_\Omega([E][D][B])^{\mathrm{T}}[\tau]\{f\}\mathrm{d}\Omega.$$

It is seen that the mass matrix is no more in general symmetric.

14.3 ELASTIC BEAM

We develop the relevant finite element formulas similarly as in Section 13.4 further in some detail.

14.3.1 Introduction

According to the Timoshenko beam theory, the beam cross-section material fiber moves as a rigid body having displacement (13.3.7) or here with the present notation:

$$\boldsymbol{u}=(u-y\theta)\boldsymbol{i}+v\boldsymbol{j}, \qquad (1)$$

where u and v are the displacement components on the beam axis. The acceleration of a generic particle is thus

$$\boldsymbol{a}=(\ddot{u}-y\ddot{\theta})\boldsymbol{i}+\ddot{v}\boldsymbol{j}. \qquad (2)$$

The notation explained in Remark 10.2 has been used to shorten the formulas. The virtual work of the inertia forces per unit length of the beam is

$$-\int_A \rho\boldsymbol{a}\cdot\delta\boldsymbol{u}\mathrm{d}A=-\int_A\rho[(\ddot{u}-y\ddot{\theta})\boldsymbol{i}+\ddot{v}\boldsymbol{j}]\cdot[(\delta u-y\delta\theta)\boldsymbol{i}+\delta v\boldsymbol{j}]\mathrm{d}A=$$
$$-\int_A\rho[(\ddot{u}-y\ddot{\theta})(\delta u-y\delta\theta)+\ddot{v}\delta v]\mathrm{d}A=$$
$$-\int_A\rho[(\ddot{u}\delta u-y\ddot{\theta}\delta u+\ddot{u}y\delta\theta+y^2\ddot{\theta}\delta\theta+\ddot{v}\delta v]\mathrm{d}A=$$
$$-(\int_A\rho\mathrm{d}A)\ddot{u}\delta u+(\int_A\rho y\mathrm{d}A)\ddot{\theta}\delta u-(\int_A\rho y\mathrm{d}A)\ddot{u}\delta\theta+$$
$$-(\int_A\rho y^2\mathrm{d}A)\ddot{\theta}\delta\theta-(\int_A\rho\mathrm{d}A)\ddot{v}\delta v. \qquad (3)$$

We assume in the following for simplicity of presentation in addition of the assumption in Section 13.4.2 that the beam axis can be been selected so that mass first moment

$$\int_A\rho y\mathrm{d}A=0. \qquad (4)$$

Now (3) can be written as

$$-m\ddot{u}\delta u-m\ddot{v}\delta v-I_\rho\ddot{\theta}\delta\theta, \qquad (5)$$

where

$$m=\int_A\rho\,\mathrm{d}A \qquad (6)$$

is the mass per unit length of the beam and

$$I_\rho=\int_A\rho y^2\mathrm{d}A \qquad (7)$$

is the cross-sectional mass moment of inertia per unit lenght of the beam. The virtual work of the inertia forces is thus finally for the transverse motion

$$\delta' W^{\mathrm{I}}=-\int(m\ddot{v}\delta v+I_\rho\ddot{\theta}\delta\theta)\mathrm{d}x. \qquad (8)$$

Comparison with Expression (13.4.18) shows that here

$$q:=q-m\ddot{v},$$
$$r:=r-I_\rho\ddot{\theta}. \qquad (9)$$

Using matrix notation, the column vector $\{f\}$ is here

$$\{f\}\equiv\begin{Bmatrix}q\\r\end{Bmatrix}:=\begin{Bmatrix}q\\r\end{Bmatrix}-\begin{bmatrix}m&0\\0&I_\rho\end{bmatrix}\begin{Bmatrix}\ddot{v}\\\ddot{\theta}\end{Bmatrix}\equiv\{f\}-[\rho]\frac{\partial^2}{\partial t^2}\{u\}. \qquad (10)$$

The scalar density ρ obtains here thus the matrix counterpart

$$[\rho]=\begin{bmatrix}m&0\\0&I_\rho\end{bmatrix}. \qquad (11)$$

14.3.2 Finite element method

We proceed to evaluate the mass matrix. From (10) and employing (14.1.5), in the finite element approximation,

$$\{f\}:=\{f\}-[\rho][N]\{\ddot{a}\} \qquad (12)$$

the mass matrix is found to be

$$[M]=\int_\Omega[N]^{\mathrm{T}}[\rho][N]\mathrm{d}\Omega \qquad (13)$$

PART III

MISCELLANEOUS APPLICATIONS AND EXTENSIONS

CHAPTER 15

MECHANISM MOTION

This chapter deals with *multibody dynamics* (monikappaledynamiikka). This means roughly dynamics applications where the system under study can be considered to consist of several physically clearly discernable parts which move with large displacements and rotations.

15.1 INTRODUCTION

Many mechanical systems in engineering can be modelled as composed of a large number of rigid bodies, joints, springs, dampers, etc. The number of differential equations describing the system may be easily of the order of several thousands and quite naturally the generation and solution of the equations must be performed in a more or less computer aided automatic manner. Although the equations are based, say, on Lagrange's equations of motion, the emphasis is quite different from the older formulations in classical mechanics. To achieve a small number of equations is no more so important, the main thing is to have a systematic and straightforward formulation directly applicable to a variety of situations. This kind of formulations as opposed to older classical ones (which might be called also as *minimal-coordinate* formalisms, Ryan (198?)) are called here multibody (mb-)-formulations. In the following, especially some features present in the software package ADAMS (Automatic Dynamic Analysis of Mechanical Systems) are described, Ryan (198?), Wielenga (1987).

15.2 MORE ON CONSTRAINTS

This far we have considered Lagrange's equations in connection with holonomic constraints and especially so that all the constraints have been taken into account from the outset by the selection of the generalized coordinates in a suitable manner. Thus in Section 4.1.2 and 5.6 the number n of generalized coordinates $q_1, q_2, \cdots, q_n$ was the same as the number of degrees of freedom *dof* of the system, i. e., $dof = n$. Now we extend the theory so that the generalized coordinates need not satisfy in advance all — in fact in the limit no one— of the constraints. It is just required that the coordinates suffice to give the position of the system.

For instance, in a system composed of several rigid bodies the generalized coordinates can be the cartesian coordinates of the center of mass of each body and the Euler angles (together six coordinates for a body) *be there any kind of joints between the bodies*. In fact, this is the selection employed in the following for rigid bodies in three dimensions. In two dimensions the two cartesian coordinates of the center of mass of the rigid body and a direction angle are used. This practice clearly increases considerably the number of generalized coordinates in a problem but for instance the interpretation of the meaning of the generalized forces becomes very simple.

The number of degrees of freedom *dof* is defined now as the difference between the number n of the selected generalized coordinates and the number m of the constraints, not satisfied in advance:

$$dof = n - m. \tag{1}$$

In Section 5.6 we had the case $m = 0$ and thus there $dof = n$.

In practice, a rather general situation is covered by the following type of *nonholonomic constraint* where the generalized velocities are present linearly:

$$\boxed{\sum_{j=1}^{n} a_j \dot{q}_j + b = 0.} \tag{2}$$

Here the terms a_j and b are at most functions of the generalized coordinates and time: $a_j = a_j(q_1, q_2, \cdots, q_n, t)$, $b = b(q_1, q_2, \cdots, q_n, t)$. Constraint (2) is according to the classification of Section 4.1.2 nonholonomic as it contains velocity type quantities. A typical *holonomic constraint* is an equation containing generalized coordinates and possibly additionally explicitly the time:

$$\boxed{\Phi(q_1, q_2, \cdots, q_n, t) = 0.} \tag{3}$$

Differentiation of this with respect to time gives an equation

$$\dot{\Phi} \equiv \sum_{j=1}^{n} \frac{\partial \Phi}{\partial q_j} \dot{q}_j + \frac{\partial \Phi}{\partial t} = 0. \tag{4}$$

This is now of type (2), where

$$a_j = \frac{\partial \Phi}{\partial q_j}, \qquad b = \frac{\partial \Phi}{\partial t}. \tag{5}$$

A holonomic constraint can thus always be transformed into the apparently nonholonomic form (2) but from form (2) we cannot always get to form (3)

with $[\rho]$ according to (11).

Example 14.1. We evaluate the mass matrix

$$[M]=\int_\Omega [N]^{\mathrm{T}}[\rho][N]\mathrm{d}\Omega \tag{a}$$

for the two-noded element.

From Example 13.1,

$$[N]=\begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix}=\begin{bmatrix} 1-\xi & 0 & 1-\xi & 0 \\ 0 & \xi & 0 & \xi \end{bmatrix} \tag{b}$$

and from (11)

$$[\rho]=\begin{bmatrix} m & 0 \\ 0 & I_\rho \end{bmatrix}. \tag{c}$$

The product

$$[\rho][N]=\begin{bmatrix} m & 0 \\ 0 & I_\rho \end{bmatrix}\begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix}=\begin{bmatrix} mN_1 & 0 & mN_2 & 0 \\ 0 & I_\rho N_1 & 0 & I_\rho N_2 \end{bmatrix} \tag{d}$$

and

$$[N]^{\mathrm{T}}[\rho][N]=\begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \end{bmatrix}\begin{bmatrix} mN_1 & 0 & mN_2 & 0 \\ 0 & I_\rho N_1 & 0 & I_\rho N_2 \end{bmatrix}$$

$$=\begin{bmatrix} N_1 mN_1 & 0 & N_1 mN_2 & 0 \\ 0 & N_1 I_\rho N_1 & 0 & N_1 I_\rho N_2 \\ N_2 mN_1 & 0 & N_2 mN_2 & 0 \\ 0 & N_2 I_\rho N_1 & 0 & N_2 I_\rho N_2 \end{bmatrix}. \tag{e}$$

Integration over the element domain assuming m and I_ρ as constants gives

$$[M]=\frac{mh}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}+\frac{I_\rho h}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}. \tag{f}$$

14.3.3 Sensitized finite element method

Here the mass matrix

$$[M]=\int_\Omega [N]^{\mathrm{T}}[\rho][N]\mathrm{d}\Omega-\int_\Omega ([E][D][B])^{\mathrm{T}}[\tau][\rho][N]\mathrm{d}\Omega, \tag{14}$$

again with $[\rho]$ according to (11).

Example 14.2. We evaluate the sensitized part of the mass matrix

$$[M]_{\mathrm{s}}=-\int_\Omega ([E][D][B])^{\mathrm{T}}[\tau][\rho][N]\mathrm{d}\Omega \tag{a}$$

for the two-noded element.

The product

$$[\tau][\rho][N]=c\begin{bmatrix} EI/kGA & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} mN_1 & 0 & mN_2 & 0 \\ 0 & I_\rho N_1 & 0 & I_\rho N_2 \end{bmatrix}$$

$$=c\begin{bmatrix} EImN_1/kGA & 0 & EImN_2/kGA & 0 \\ 0 & I_\rho N_1 & 0 & I_\rho N_2 \end{bmatrix} \tag{b}$$

and

$$([E][D][B])^{\mathrm{T}}[\tau][\rho][N]=c\begin{bmatrix} 0 & -kGA/h \\ kGA/h & -kGAN_1 \\ 0 & kGA/h \\ -kGA/h & -kGAN_2 \end{bmatrix}\cdot\begin{bmatrix} EImN_1/kGA & 0 & EImN_2/kGA & 0 \\ 0 & I_\rho N_1 & 0 & I_\rho N_2 \end{bmatrix}$$

$$=-\frac{1/(kGA)}{12\hat{\varepsilon}_h+1}\begin{bmatrix} 0 & -kGAI_\rho N_1/h & 0 & -kGAI_\rho N_2/h \\ EImN_1/h & -kGAI_\rho N_1N_1 & EImN_2/h & -kGAI_\rho N_1N_2 \\ 0 & kGAI_\rho N_1/h & 0 & kGAI_\rho N_1/h \\ -EImN_1/h & -kGAI_\rho N_2N_1 & -EImN_2/h & -kGAI_\rho N_2N_2 \end{bmatrix}$$

$$=-\frac{1}{12\hat{\varepsilon}_h+1}\begin{bmatrix} 0 & -I_\rho N_1/h & 0 & -I_\rho N_2/h \\ \hat{\varepsilon}_h mN_1h & -I_\rho N_1N_1 & \hat{\varepsilon}_h mN_2h & -I_\rho N_1N_2 \\ 0 & I_\rho N_1/h & 0 & I_\rho N_1/h \\ -\hat{\varepsilon}_h mN_1h & -I_\rho N_2N_1 & -\hat{\varepsilon}_h mN_2h & -I_\rho N_2N_2 \end{bmatrix}. \tag{c}$$

Formulas from Examples 13.2 and 14.1 have been made use of above. Integration over the element domain gives finally assuming again constant m and I_ρ,

$$[M]_{\mathrm{s}}=\frac{\hat{\varepsilon}_h mh^2}{(12\hat{\varepsilon}_h+1)}\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & +1 & 0 \end{bmatrix}+\frac{I_\rho h}{(12\hat{\varepsilon}_h+1)}\frac{1}{6}\begin{bmatrix} 0 & +3 & 0 & -3 \\ 0 & +2h & 0 & +h \\ 0 & -3 & 0 & -3 \\ 0 & +h & 0 & +2h \end{bmatrix}. \tag{d}$$

The first matrix in (d) is not symmetric. It would be interesting to consider the effect of using only its symmetric part. However, these studies are left to be done later.

$$\boxed{\delta \mathbf{r}_i = \sum_{j=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \,,} \qquad i = 1,2,\cdots,N \tag{3}$$

also in the mb-formulation. Now, however, the constraints of the system are in general initially violated and we have a case of application of kinematically inadmissible virtual displacements. For instance, based on expression (2), the virtual displacement

$$\delta \mathbf{r}(x,y) = \delta x \mathbf{i} + \delta y \mathbf{j}, \tag{4}$$

where δx and δy are arbitrary, clearly leads to a situation, where the particle no more remains on the surface of the wedge.

If we use kinematically inadmissible virtual displacements, the virtual work done by the constraint forces thus does not usually vanish. The general expression for virtual work (5.4.8):

$$\delta W = \sum_{j=1}^{n} Q_j \delta q_j \tag{5}$$

must thus be made here more precise by writing

$$Q_j = Q_j^{\mathrm{k}} + Q_j^{\mathrm{r}}, \tag{6}$$

where the superscripts k and r refer to generalized forces due to the constitutive forces and constraint forces, respectively.

Let us recall the derivation of Lagrange's equations of motion using kinematically admissible virtual displacements. The starting point was the virtual work equation (5.4.11) written here as

$$\sum_{j=1}^{n} Q_j \delta q_j = \sum_{j=1}^{n} \Big(\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \Big) \delta q_j \,, \tag{7}$$

In Section 5.6 it was shown that the expression in parenthesis on the right-hand side of (7) could be manipulated into the form

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} \,. \tag{8}$$

A reflection on the steps used shows that the manipulations were based entirely on expressions (1) and nowhere the information about the constraints being

satisfied or not was employed. We can thus write the virtual work equation (7) still similarly as earlier for instance in the form

$$\sum_{j=1}^{n} \Big(Q_j - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial K}{\partial \dot{q}_j} + \frac{\partial K}{\partial q_j} \Big) \delta q_j = 0. \tag{9}$$

As we apply here kinematically inadmissible virtual displacements, we do not care about the satisfaction of the constraints at this phase and we can thus take the variations δq_j as free and we obtain as consequence the equations of motion (a change of sign has been performed to obtain a more conventional form)

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial K}{\partial \dot{q}_j} + \frac{\partial K}{\partial q_j} - Q_j^{\mathrm{k}} - Q_j^{\mathrm{r}} = 0, \quad j = 1,2,\cdots,n. \tag{10}$$

First, in this section we have written the equations of motion in the spirit of equations (5.4.12) in the standard form "something on the left-hand side equals zero". Second, in the step between equations (9) and (10), the content of formula (6) has been introduced. Third, the final forms (10) indicate that *these equations are not directly applicable* as the basis of the formulation because the constraint forces and thus also the generalized constraint forces Q_j^{r} are *unknown quantities* of the problem. Equations (10), however, are found to give an interpretation which proves to be useful below.

15.3.2 Kinematically admissible virtual displacements

In the following we shall derive Lagrange's equations of motion employing kinematically admissible virtual displacements.

If a constraint is holonomic and represented in the form (15.2.3):

$$\Phi(q_1, q_2, \cdots, q_n, t) = 0, \tag{11}$$

the variation of both sides of it produces the equation

$$\delta\Phi \equiv \sum_{j=1}^{n} \frac{\partial \Phi}{\partial q_j} \delta q_j = 0. \tag{12}$$

Kinematically admissible variations δq_j must thus satisfy this condition. We realize again that in the principle of virtual displacements, time is freezed which fact explains the omission of the partial derivative with respect to time in (12) even in the rheonomic case.

If a constraint is nonholonomic and of type (15.2.2):

$$\sum_{j=1}^{n} a_j \dot{q}_j + b = 0, \tag{13}$$

a multiplication by the time differential $\mathrm{d}t$ gives first

$$\sum_{j=1}^{n} a_j \mathrm{d}q_j + b\mathrm{d}t = 0. \tag{14}$$

In real motion the differentials $\mathrm{d}q_j$ of the generalized coordinates must satisfy this condition. When applying kinematically admissible virtual displacements condition (14) obtains the form

$$\sum_{j=1}^{n} a_j \delta q_j = 0, \tag{15}$$

because time is freezed.

Let the total number of scalar kinematical constraint equations in a problem be m. They can consist of holonomic and nonholonomic constraints of type (11) and (13), respectively. With kinematically admissible virtual displacements they thus produce the conditions

$$\boxed{\sum_{j=1}^{n} a_{kj} \delta q_j = 0,} \quad k = 1,2,\cdots,m. \tag{16}$$

between the variations δq_j. In the case of holonomic constraints

$$\Phi_k(q_1, q_2, \cdots, q_n, t) = 0, \quad k = 1,2,\cdots,m, \tag{17}$$

the multipliers are

$$a_{kj} = \frac{\partial \Phi_k}{\partial q_j} \tag{18}$$

and in the case of nonholonomic constraints:

$$\sum_{j=1}^{n} a_{kj} \dot{q}_j + b_k = 0, \quad k = 1,2,\cdots,m, \tag{19}$$

the multipliers a_{kj} are directly given. Formulas (17) and (19) have been written for notational convenience as if the constraints could be of only of either type but they can of course be present simultaneously.

We shall now derive Lagrange's equations of motion.The starting point is again equation (9):

$$\sum_{j=1}^{n} (Q_j^{\mathrm{k}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{q}_j} + \frac{\partial K}{\partial q_j}) \delta q_j = 0. \tag{20}$$

As we have restricted the virtual displacements to be kinematically admissible, the generalized forces consist, however, now only of the constitutive parts (which are known), and this fact has been taken into account in (20). Contrary to the case represented by equation (9), there no more follows that the terms inside the parentheses on the left-hand side of (20) vanish separately as the variations δq_j are no more free but have to satisfy conditions (16). The situation is similar to that considered in Section D.1.2. One essential difference is, however, that there originally the stationarity of a certain function $f(x_1, x_2, \cdots, x_2)$ was considered and *no correspondig quantity is present here*. In spite of this, the manipulations in Section D.1.2 proceeded so that the requirement

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i = 0 \tag{21}$$

was put with respect to all differentials $\mathrm{d}x_i$ satisfying the conditions

$$\sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i} \mathrm{d}x_i = 0, \quad k = 1,2,\cdots,m. \tag{22}$$

Here the counterparts of (21) ja (22) are (20) and (16), respectively; instead of differentials we have just variations. By repeating the line of thought described in Section D.1.2, we arrive here at the modified equation

$$\sum_{j=1}^{n} (Q_j^{\mathrm{k}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{q}_j} + \frac{\partial K}{\partial q_j} + \sum_{k=1}^{m} \lambda_k a_{kj}) \delta q_j = 0. \tag{23}$$

This has been obtained by adding to the left-hand side (20) the left-hand sides of (16) each multiplied by a yet undetermined factor λ_k. Also in this connection these factors are called Lagrange multipliers. The rest of the logic goes analogously with the presentation in Section D.1.2. We arrive at the *generalized Lagrange's equations of motion* (yleistetyt Lagrangen liikeyhtälöt) (a change of sign has been performed to obtain a more conventional form)

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} - Q_j^{\mathrm{k}} - \sum_{k=1}^{m} \lambda_k a_{kj} = 0,} \quad j = 1,2,\cdots,n. \tag{24}$$

$$m\ddot{y} + mg - \lambda = 0. \tag{f}$$

The generalized force expressions $Q_x^k = 0$ and $Q_y^k = -mg$ due to gravity (constitutive force) are obvious. The governing equations of the problem are (d), (f), (a) corresponding to the unknowns $x(t)$, $y(t)$, $\lambda(t)$.

The generalixed forces due to the constraint are according to (29)

$$Q_x^r = \lambda \tan\alpha, \quad Q_y^r = \lambda. \tag{g}$$

The constraint force N is perpendicular to the inclined plane and has the components $N\sin\alpha$ and $N\cos\alpha$. The corresponding generalized forces are thus simply

$$Q_x^r = N\sin\alpha, \quad Q_y^r = N\cos\alpha. \tag{h}$$

Comparison of these with (g) gives the equations

$$\begin{aligned} N\sin\alpha &= \lambda\tan\alpha, \\ N\cos\alpha &= \lambda. \end{aligned} \tag{i}$$

Either of them gives the solution

$$N = \frac{\lambda}{\cos\alpha}. \tag{j}$$

As system (i) is *overdetermined* (ylimääräytyvä) with respect to N — two equations and only one unknown — a more systematic solution would be based on the least-squares method. We form the least-squares expression

$$I(N) = \frac{1}{2}[(N\sin\alpha - \lambda\tan\alpha)^2 + (N\cos\alpha - \lambda)^2] \tag{k}$$

and demand this to have the minimum value by writing the stationarity condition

$$\frac{dI}{dN} \equiv (N\sin\alpha - \lambda\tan\alpha)\sin\alpha + (N\cos\alpha - \lambda)\cos\alpha = 0. \tag{l}$$

This is in detail

$$N(\sin^2\alpha + \cos^2\alpha) - \lambda(\tan\alpha\sin\alpha + \cos\alpha) = 0,$$

$$N - \lambda\frac{1}{\cos\alpha} = 0. \tag{m}$$

This gives naturally again result (j) but this approach might be useful in connection with numerical procedures.

Example 15.4. Figure (a) shows a slender homogeneous bar (length l, mass m), pinned frictionlessly at point O and oscillating in plane motion under gravity. We form the equations of motion of the bar employing (1) classical formulation (2) mb-formulation.

(1) We have a one degree of freedom system. In the classical formulation angle θ is the most natural generalized coordinate:

$$q_1 = \theta. \tag{a}$$

The virtual work done by gravity is

$$\delta' W = mg\delta\bar{x} = mg\delta(\frac{l}{2}\cos\theta) = mg(-\frac{l}{2}\sin\theta\cdot\delta\theta) = -\frac{1}{2}mgl\sin\theta\cdot\delta\theta, \tag{b}$$

so the corresponding generalized force

$$Q = -\frac{1}{2}mgl\sin\theta. \tag{c}$$

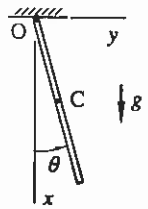


Figure (a)

The physical meaning of (c) is obvious: it represents the moment of the gravity force about point O. As the gravity force is conservative, this could have been taken care of also by the potential energy expression

$$V = mg\bar{x} = -\frac{1}{2}mgl\cos\theta, \tag{d}$$

from which $Q = -\partial V/\partial x$. The kinetic energy of the bar is

$$K = \frac{1}{2}I_O\dot{\theta}^2. \tag{e}$$

We obtain

$$\frac{\partial K}{\partial\theta} = I_O\dot{\theta}, \quad \frac{d}{dt}\frac{\partial K}{\partial\dot{\theta}} = I_O\ddot{\theta}, \quad \frac{\partial K}{\partial\theta} = 0, \tag{f}$$

and Lagrange's equation of motion becomes

$$I_O\ddot{\theta} + \frac{1}{2}mgl\sin\theta = 0, \tag{g}$$

which is the familiar rotational equation. The system equation consists thus just of (g) and we have only one unknown: $\theta(t)$.

(2) We take as generalized coordinates

$$q_1 = \bar{x}, \quad q_2 = \bar{y}, \quad q_3 = \theta. \tag{h}$$

The constraints consist of the fact that the end of the bar must stay at point O:

$$\begin{aligned} \Phi_1 &\equiv \bar{x} - \frac{l}{2}\cos\theta = 0, \\ \Phi_2 &\equiv \bar{y} - \frac{l}{2}\sin\theta = 0. \end{aligned} \tag{i}$$

The virtual work of the gravity force is simply

$$\delta' W = mg\delta\bar{x} = mg\delta\bar{x} + 0\cdot\delta\bar{y} + 0\cdot\delta\theta, \tag{j}$$

so the corresponding generalized forces are

$$Q_1^k = mg, \quad Q_2^k = 0, \quad Q_3^k = 0. \tag{k}$$

The kinetic energy of the bar is

$$K = \frac{1}{2}m(\dot{\bar{x}}^2 + \dot{\bar{y}}^2) + \frac{1}{2}\bar{I}\dot{\theta}^2. \tag{l}$$

The first generalized Lagrange's equation of motion is

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\bar{x}}} - \frac{\partial K}{\partial \bar{x}} - Q_1^k - \lambda_1\frac{\partial \Phi_1}{\partial \bar{x}} - \lambda_2\frac{\partial \Phi_2}{\partial \bar{x}} = 0, \tag{m}$$

where

$$\frac{\partial K}{\partial \dot{\bar{x}}} = m\dot{\bar{x}}, \quad \frac{d}{dt}\frac{\partial K}{\partial \dot{\bar{x}}} = m\ddot{\bar{x}}, \quad \frac{\partial K}{\partial \bar{x}} = 0, \quad \frac{\partial \Phi_1}{\partial \bar{x}} = 1, \quad \frac{\partial \Phi_2}{\partial \bar{x}} = 0 \tag{n}$$

and we obtain

$$m\ddot{\bar{x}} - mg - \lambda_1 = 0. \tag{o}$$

The second generalized Lagrange's equation of motion is

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\bar{y}}} - \frac{\partial K}{\partial \bar{y}} - Q_2^k - \lambda_1\frac{\partial \Phi_1}{\partial \bar{y}} - \lambda_2\frac{\partial \Phi_2}{\partial \bar{y}} = 0, \tag{p}$$

where

$$\frac{\partial K}{\partial \dot{\bar{y}}} = m\dot{\bar{y}}, \quad \frac{d}{dt}\frac{\partial K}{\partial \dot{\bar{y}}} = m\ddot{\bar{y}}, \quad \frac{\partial K}{\partial \bar{y}} = 0, \quad \frac{\partial \Phi_1}{\partial \bar{y}} = 0, \quad \frac{\partial \Phi_2}{\partial \bar{y}} = 1 \tag{q}$$

and we obtain

$$m\ddot{\bar{y}} - \lambda_2 = 0. \tag{r}$$

The third generalized Lagrange's equation of motion is

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\theta}} - \frac{\partial K}{\partial \theta} - Q_3^k - \lambda_1\frac{\partial \Phi_1}{\partial \theta} - \lambda_2\frac{\partial \Phi_2}{\partial \theta} = 0, \tag{s}$$

where

$$\frac{\partial K}{\partial \dot{\theta}} = \bar{I}\dot{\theta}, \quad \frac{d}{dt}\frac{\partial K}{\partial \dot{\theta}} = \bar{I}\ddot{\theta}, \quad \frac{\partial K}{\partial \theta} = 0, \quad \frac{\partial \Phi_1}{\partial \theta} = \frac{l}{2}\sin\theta, \quad \frac{\partial \Phi_2}{\partial \theta} = -\frac{l}{2}\cos\theta \tag{t}$$

and we obtain

$$\bar{I}\ddot{\theta} - \lambda_1\frac{l}{2}\sin\theta + \lambda_2\frac{l}{2}\cos\theta = 0. \tag{u}$$

The five system equations consist thus of (o), (r), (u), (i) and the unknows are $\bar{x}(t)$, $\bar{y}(t)$, $\theta(t)$, $\lambda_1(t)$, $\lambda_2(t)$. The difference between the classical formulation (equation (g)) and the present one is here quite striking.

The generalized constraint forces are according to (29)

$$Q_1^r = \lambda_1, \quad Q_2^r = \lambda_2, \quad Q_3^r = \lambda_1\frac{l}{2}\sin\theta - \lambda_2\frac{l}{2}\cos\theta. \tag{v}$$

Figure (b)

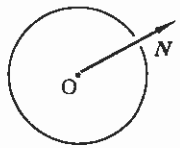


Figure (c)

Figure (b) shows schematically the traction distribution exerted on the bar surface from the pin at O. As we assume no friction, the line of action of each force differential $p\,dS$ goes through the axis of the pin and the force system from the tractions reduced at O gives a resultant (constraint force)

$$N = N_x i + N_y j \tag{w}$$

without a moment (Figure (c)). The virtual work done by this force is

$$\begin{aligned}\delta' W &= N \cdot \delta r_O = N \cdot (\delta\bar{r} + \delta'\theta \times r_{O/C}) \\ &= (N_x i + N_y j)\cdot[\delta\bar{x} i + \delta\bar{y} j + \delta\theta k \times (-\frac{l}{2}\cos\theta i - \frac{l}{2}\sin\theta j)] \\ &= (N_x i + N_y j)\cdot[(\delta\bar{x} + \frac{l}{2}\sin\theta\cdot\delta\theta) i + (\delta\bar{y} - \frac{l}{2}\cos\theta\cdot\delta\theta) j)] \\ &= N_x\delta\bar{x} + N_y\delta\bar{y} + (N_x\frac{l}{2}\sin\theta - N_y\frac{l}{2}\cos\theta)\delta\theta. \end{aligned} \tag{y}$$

Formula (5.4.19) has been made use of. The corresponding generalized forces are thus

$$Q_1^r = N_x, \quad Q_2^r = N_y, \quad Q_3^r = N_x\frac{l}{2}\sin\theta - N_y\frac{l}{2}\cos\theta. \tag{ä}$$

Comparison of these with (v) gives the equations

$$\begin{aligned} &N_x = \lambda_1, \\ &N_y = \lambda_2, \\ &N_x\frac{l}{2}\sin\theta - N_y\frac{l}{2}\cos\theta = \lambda_1\frac{l}{2}\sin\theta - \lambda_2\frac{l}{2}\cos\theta. \end{aligned} \tag{ö}$$

The two first give directly the values of the constraint force components. A least-squares formulation can be used alternatively similarly as shown in Example 15.3 in connection with a numerical procedure.

15.4 COPING WITH FRICTION

In classical formulations with Lagrange's equations hardly ever anything is said about the inclusion of Coulomb friction. This is sowewhat dishonest as friction is usually considered rather thoroughly in more elementary presentations. The reason obviously lies behind the fact that in classical formulations of Lagrange's equations the constraints are satisfied in advance and the constraint forces and in particular the normal forces N at contacts disappear and we thus cannot introduce the friction forces μN and their virtual work in the formulation. In

examples 15.3 and 15.4 it was shown that in a mb-formulation the constraint forces can be extracted from the generalized constraint forces which are available at each phase of the solution. We can thus introduce Coulomb friction into the formulation.

We discuss below some aspects of the treatment of Coulomb friction in a plane case. Although we deal here finally with Lagrange's equations of motion, the difficulties to deal with Coulomb friction in numerical schemes are quite general.

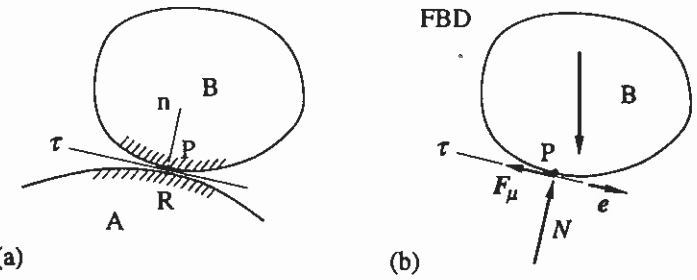


Figure 15.1 (a) Body B sliding relative to body A. (b) Free body diagram of body B.

Let us consider Figure 15.1 (a). Body B slides (or momentarily sticks on) along body A. Figure (b) shows schematically the free body diagram of body B. In the Coulomb model, the friction force vector F_μ acting on body B is given by

$$F_\mu = -\operatorname{sign}(v)\mu_k N e, \quad \text{if } v \neq 0, \tag{1}$$

$$|F_\mu| \leq \mu_s N, \quad \text{if } v = 0. \tag{2}$$

Here, the sign function is defined by

$$\operatorname{sign}(v) = \begin{cases} +1 & \text{if } v > 0, \\ -1 & \text{if } v < 0, \end{cases} \tag{3}$$

v is the *slip velocity* (liukumisnopeus) obtained in two dimensions from

$$v = (v_P - v_R) \cdot e, \tag{4}$$

where v_P and v_R are the velocities of particles P and R of bodies B and A at the contact point, e is the unit vector in the tangent plane τ to the surfaces at the contact point and N is the normal (compressive) force at the contact. The coefficients μ_k and μ_s ($\mu_s > \mu_k$) are called *kinetic* and *static coefficients of friction* (liikekitkakerroin ja lepokitkakerroin), respectively.

The difficulty lies in condition (2). In a motion, the slip velocity may for some time periods disappear and the bodies stick together. This means that for a while the number of degrees of freedom is decreased by one. We describe here one method to avoid this difficulty in numerical schemes following roughly the procedure explained in the theory manual of the finite element software package ABAQUS (Version 5.7).

The "normal situation" is taken care of by formula (1). When the magnitude of the slip velocity becomes small enough or when

$$|v| < v_{tol}, \tag{5}$$

where v_{tol} is a user given small tolerance velocity, there is danger of imminent sticking. Let the moment of time when condition (5) is found first to be satisfied be denoted as t_0. Now an elastic spring is imagined to be attached between particles P and R and we write for $t_0 \leq t < t_1$ instead of (1) or (2),

$$F_\mu = -kue, \tag{6}$$

where k is a spring constant and u the *slip* (liukuma):

$$u = (u_P - u_R) \cdot e, \tag{7}$$

where u_P and u_R are the displacements of particles P and R measured starting from time t_0 onwards. The magnitude $|F_\mu|$ evaluated from (6) is monitored and if condition

$$|F_\mu| \leq \mu_s N \tag{8}$$

is satisfied the calculation proceeds using expression (6). The possibility

$$|F_\mu| > \mu_s N \tag{9}$$

means that we have reached time t_1 and we abandon expression (6) and start to use (1).

The imaginary spring is able to simulate possible sticking by restricting the relative movement instead of zero to a small finite value. A user given small tolerance slip u_{tol} is employed to define the spring constant to have the value

$$k = \frac{\mu_s N}{u_{tol}}. \tag{10}$$

Remark 15.5. If in a contact, the value of the normal force N becomes negative, the contact has in fact vanished. Thus formulas (1) and (2) should be used only in the case $N > 0$. In a numerical scheme a negative N would direct

the friction force in an unrealistic direction. The case of contact disappearing and perhaps again reappearing possibly with some impact phenomena leads to complicated relations which are not considered here. It may be further noticed that the case depicted in Figure 15.1 is numerically far from trivial as the points in contact at an arbitary moment of time depend in a complicated way on the geometry of the surfaces. □

Example 15.5. We repeat Example 15.3 now including Coulomb friction.

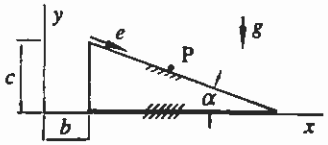


Figure (a)

We consider here only the situation covered by equation (1). The tangent unit vector e is taken as shown in Figure (a). The slip velocity is

$$v = (v_p - v_w)\cdot e, \tag{a}$$

where the velocity of the particle

$$v_p = \dot{x}i + \dot{y}j \tag{b}$$

and the velocity of the wedge (= velocity of the particle of the wedge at P),

$$v_w = \dot{b}i = -\hat{b}\cos\omega t\, i \tag{c}$$

and the unit vector

$$e = \cos\alpha\, i - \sin\alpha\, j. \tag{d}$$

We obtain

$$\begin{aligned} v &= (\dot{x}i + \dot{y}j + \hat{b}\cos\omega t\, i)\cdot(\cos\alpha\, i - \sin\alpha\, j) \\ &= \dot{x}\cos\alpha - \dot{y}\sin\alpha + \hat{b}\cos\alpha\cos\omega t. \end{aligned} \tag{e}$$

The friction force is according to (1) thus

$$F_\mu = \text{sign}(v)(-\cos\alpha\, i + \sin\alpha\, j)\mu_k N. \tag{f}$$

In Example 15.3, the expression for the normal force N was found to be

$$N = \frac{\lambda}{\cos\alpha}. \tag{g}$$

Thus the generalized forces due to the friction force are finally of the form

$$\begin{aligned} Q_x^k &= \text{sign}(v)(-\cos\alpha)\mu_k\frac{\lambda}{\cos\alpha} = -\,\text{sign}(v)\mu_k\lambda, \\ Q_y^k &= \text{sign}(v)(\sin\alpha)\mu_k\frac{\lambda}{\cos\alpha} = \text{sign}(v)\tan\alpha\cdot\mu_k\lambda, \end{aligned} \tag{h}$$

where the slip velocity is evaluated from (e). The equations of motion (d) and (f) of Example 15.3 are modified to

$$\begin{aligned} &m\ddot{x} - \lambda\tan\alpha + \text{sign}(v)\mu_k\lambda = 0. \\ &m\ddot{y} + mg - \lambda - \text{sign}(v)\tan\alpha\cdot\mu_k\lambda = 0. \end{aligned} \tag{i}$$

The constraint equation

$$\Phi(x,y,t) \equiv y + (x - \hat{b}\sin\omega t)\tan\alpha - c = 0 \tag{j}$$

naturally remains the same.

15.5 INITIAL CONDITIONS

15.5.1 Initial positions

In classical formulation the initial values for the generalized coordinates and the gneralized velocities must be given. In a mb-formulation the situation gets somewhat involved as the effect of constraints must be taken into account in the initial conditions. We consider here only holonomic constraints (15.3.31):

$$\Phi_k(q_1, q_2, \cdots, q_n, t) = 0, \quad k = 1,2,\cdots,m. \tag{1}$$

The applier gives based on the problem at the moment of time $t = 0$ some preliminary values ${}^0q_1, {}^0q_2, \cdots, {}^0q_n$ for the generalized coordinates. These do not, however, necessarily satisfy constraints (1) with good enough accuracy. More refined initial values ${}^0\overline{q}_1, {}^0\overline{q}_2, \cdots, {}^0\overline{q}_n$ can be arrived at as follows. We write with obvious notation a least-squares expression

$$C = \frac{1}{2}\{{}^0\overline{q} - {}^0q\}^T[W]\{{}^0\overline{q} - {}^0q\}, \tag{2}$$

where $[W]$ is a diagonal matrix with given positive elements (weights) and minimize (2) with respect to ${}^0\overline{q}_1, {}^0\overline{q}_2, \cdots, {}^0\overline{q}_n$ keeping (1) as constraints of the problem. This is mathematically a constrained stationarity problem which has been treated in Section D.1.2. We can thus form a modified function

$$C_L = \frac{1}{2}\{{}^0\overline{q} - {}^0q\}^T[W]\{{}^0\overline{q} - {}^0q\} + \{\lambda\}^T\{\Phi\}, \tag{3}$$

where the constraints have been taken into account via the Lagrange multiplier method. Partial differentiaton of (3) gives the system equations of type (D.1.23) from which the unknows $\{{}^0\overline{q}\}$ and $\{\lambda\}$ are determined, Wielenga (1987).

15.5.2 Initial velocities

We can treat the initial velocities quite similarly as the initial coordinates in the previous section. Differentiation of (1) gives the conditions

$$\sum_{j=1}^{n} \frac{\partial \Phi_k}{\partial q_j} \dot{q}_j + \frac{\partial \Phi_k}{\partial t} = 0, \quad k = 1,2,\cdots,m, \tag{4}$$

between the generalized velocities. The applier gives preliminary values ${}^0\dot{q}_1, {}^0\dot{q}_2, \cdots, {}^0\dot{q}_n$ for the generalized velocities. These do not necessarily satisfy conditions (4) with good enough accuracy. More refined initial velocities ${}^0\dot{\bar{q}}_1, {}^0\dot{\bar{q}}_2, \cdots, {}^0\dot{\bar{q}}_n$ can be obtained by writing first the least-squares expression

$$C = \frac{1}{2}\{{}^0\dot{\bar{q}} - {}^0\dot{q}\}^{\mathrm{T}}[W]\{{}^0\dot{\bar{q}} - {}^0\dot{q}\}, \tag{5}$$

where $[W]$ is again some diagonal matrix with positive diagonal elements. This is minimized with respect to ${}^0\dot{\bar{q}}_1, {}^0\dot{\bar{q}}_2, \cdots, {}^0\dot{\bar{q}}_n$ keeping (4) as constraints.This problem is dealt with using the Lagrange multiplier method the way explained in the previous section.

15.6 RIGID BODY IN THREE DIMENSIONAL MOTION (missing)

15.7 INCLUDING FLEXIBILITY (missing)

15.8 TIME INTEGRATION

We describe in the following some details of the type of numerical time integration solution procedures present in ADAMS, Ryan (198?). We take the mb-formulation of Example 15.4 as a demonstration case. The governing equations were found to be

$$\begin{aligned}
&m\ddot{\bar{x}} - mg - \lambda_1 = 0,\\
&m\ddot{\bar{y}} - \lambda_2 = 0,\\
&\bar{I}\ddot{\theta} - \lambda_1 \frac{l}{2}\sin\theta + \lambda_2 \frac{l}{2}\cos\theta = 0,\\
&\bar{x} - \frac{l}{2}\cos\theta = 0,\\
&\bar{y} - \frac{l}{2}\sin\theta = 0.
\end{aligned} \tag{1}$$

The first three are the equations of motion and the last two the constraint equations. Mathematically this is called *differential-algebraic equation system* (differentiaali-algebrallinen yhtälöryhmä); it is seen that the constraint equations do not contain derivatives (see Remark 15.6). By introducing new unknowns the

equations can be put in a standard form where the highest derivatives appearing are only of first order. We collect the generalized coordinates in the column vector

$$\{q\} = [q_1 = \bar{x},\ q_2 = \bar{y},\ q_3 = \theta]^{\mathrm{T}} \tag{2}$$

and denote the corresponding generalized velocities as

$$\{u\} = [u_1,\ u_2,\ u_3]^{\mathrm{T}}. \tag{3}$$

These are connected by the first order differential equations

$$\{u\} = \{\dot{q}\}. \tag{4}$$

We denote further

$$\{f\} = [\lambda_1, \lambda_2, F]^{\mathrm{T}}. \tag{5}$$

We can present equations (1) now as

$$\begin{aligned}
&m\dot{u}_1 + F - \lambda_1 = 0.\\
&m\dot{u}_2 - \lambda_2 = 0.\\
&\bar{I}\dot{u}_3 - \lambda_1 \frac{l}{2}\sin q_3 + \lambda_2 \frac{l}{2}\cos q_3 = 0,
\end{aligned} \tag{6}$$

$$\begin{aligned}
&u_1 - \dot{q}_1 = 0,\\
&u_2 - \dot{q}_2 = 0,\\
&u_3 - \dot{q}_3 = 0,
\end{aligned} \tag{7}$$

$$\begin{aligned}
&q_1 - \frac{l}{2}\cos q_3 = 0,\\
&q_2 - \frac{l}{2}\sin q_3 = 0.\\
&F + mg = 0.
\end{aligned} \tag{8}$$

The grouping of the equations correspond to that given in Ryan (198?) and presented as

$$\{M(\{q\},\{u\},\{\dot{u}\},\{f\},t)\} = \{0\}, \tag{9}$$

$$\{N(\{q\},\{u\})\} = \{0\}, \tag{10}$$

$$\{\Phi(\{q\},\{f\},t)\} = \{0\}. \tag{11}$$

Here, $\{M\}$ is the column vector of all differential dynamical equations and any user defined differential equations, $\{N\}$ is the column vector of differential kinematical equations and differential motion constraints, and $\{\Phi\}$ is the column vector of algebraic equations of joint constraints and forces. Column vector $\{f\}$ consist of Lagrange multipliers and of applied forces. In the example case, the gravity force is represented in equations (6) just by a symbol and an additional equation in (8) gives the detailed expression.

We collect the variables $\{q\}$, $\{u\}$, $\{f\}$ — called *state variables* (tilamuuttujat) — in a single column vector

$$\{y\} = [\{q\}^T, \{u\}^T, \{f\}^T]^T. \tag{12}$$

Equations (9) to (11) can now be written concisely as a single matrix equation

$$\{G(\{y\}, \{\dot{y}\}, t\} = \{0\}. \tag{13}$$

This represents a set of nonlinear equations which must, in general, be solved in an iterative way at a certain moment of time. A Newton-Raphson type procedure is employed. The equations are expanded in a Taylor series about an initial guess for $\{y\}$. At time zero, the initial guess is obtained from the initial conditions. Later, it is extrapolated from past values of $\{y\}$. Let $\{y\}_j$ and $\{\dot{y}\}_j$ denote the values of state variables and their time derivatives at the jth iteration. The Taylor series expansion of the left-hand side of (13) produces the equations

$$\{G\}_j + \left[\frac{\partial\{G\}}{\partial\{y\}}\right]_j \Delta\{y\}_j + \left[\frac{\partial\{G\}}{\partial\{\dot{y}\}}\right]_j \Delta\{\dot{y}\}_j + \cdots = \{0\}. \tag{14}$$

The subscript j for $\{G\}$ and for the rectangular matrices, called *Jacobian matrices*, indicates that these are evaluated with $\{y\}_j$ and $\{\dot{y}\}_j$ at their current values at the jth iteration and the symbols $\Delta\{y\}_j$ and $\Delta\{\dot{y}\}_j$ mean

$$\begin{aligned} \Delta\{y\}_j &= \{y\}_{j+1} - \{y\}_j, \\ \Delta\{\dot{y}\}_j &= \{\dot{y}\}_{j+1} - \{\dot{y}\}_j. \end{aligned} \tag{15}$$

In detail, in the Jacobian matrix the element on the ith row and on the kth column is $\partial G_i / \partial y_k$ for the first matrix and similarly for the second one.

Here, in the demonstration case

$$\begin{aligned} \{y\} &= [q_1, q_2, q_3, u_1, u_2, u_3, \lambda_1, \lambda_2, F]^T, \\ \{\dot{y}\} &= [\dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{u}_1, \dot{u}_2, \dot{u}_3, \dot{\lambda}_1, \dot{\lambda}_2, \dot{F}]^T, \end{aligned} \tag{16}$$

$\{G\}$ consists of the left-hand sides of (6) to (8) and we obtain

$$\left[\frac{\partial\{G\}}{\partial\{y\}}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & (J_y)_{33} & 0 & 0 & 0 & (J_y)_{37} & (J_y)_{38} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & (J_y)_{73} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & (J_y)_{83} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

and

$$\left[\frac{\partial\{G\}}{\partial\{\dot{y}\}}\right] = \begin{bmatrix} 0 & 0 & 0 & m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{I} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{18}$$

where the lengthier element expressions are given here separately as

$$\begin{aligned} (J_y)_{33} &= -\lambda_1 \frac{l}{2}\cos q_3 - \lambda_2 \frac{l}{2}\sin q_3 = -y_7 \frac{l}{2}\cos y_3 - y_8 \frac{l}{2}\sin y_3, \\ (J_y)_{37} &= -\frac{l}{2}\sin q_3 = -\frac{l}{2}\sin y_3, \\ (J_y)_{38} &= \frac{l}{2}\cos q_3 = \frac{l}{2}\cos y_3, \\ (J_y)_{73} &= \frac{l}{2}\sin q_3 = \frac{l}{2}\sin y_3, \\ (J_y)_{83} &= -\frac{l}{2}\cos q_3 = -\frac{l}{2}\cos y_3 \end{aligned} \tag{19}$$

to keep matrix (17) within the page area. It is seen that the Jacobian matrices are very *sparse* (harva). This is a general property of the calculation procedure in ADAMS which fact must naturally made effectively use of in the algorithms.

When the higher order terms in (14) indicated by ellipses are neglected, the following system is obtained

$$\left[\frac{\partial\{G\}}{\partial\{y\}}\right]_j \Delta\{y\}_j + \left[\frac{\partial\{G\}}{\partial\{\dot{y}\}}\right]_j \Delta\{\dot{y}\}_j = -\{G\}_j . \tag{20}$$

Set (20) can be transformed from being a system of algebraic-differential equations to a set of linear algebraic equations by the introduction of a numerical time integration formula. ADAMS employs the implicit Gear variable order, variable step stiff integration algorithm which is of the form

$$\begin{aligned}\{y\}_{n+1} &= \sum_{i=1}^{k} \alpha_i \{y\}_{n+1-i} - h\beta_0 \{\dot{y}\}_{n+1} \\ &= \alpha_1 \{y\}_n + \cdots + \alpha_k \{y\}_{n+1-k} - h\beta_0 \{\dot{y}\}_{n+1} .\end{aligned} \tag{21}$$

Here n represents the time step counter (t_n is the value of time after the nth integration time step), $\{y\}_{n+1}$ is the numerical approximation for $\{y(t)\}$ at $t = t_{n+1}$, h is the time step given by $h = t_{n+1} - t_n$, k is the *order* (kertaluku) of the algorithm employed, and $\alpha_i\,(i = 1, \cdots, k)$, β_0 are real numbers referred to as the Gear integration coefficients, Gear (1971).

It should be noticed that equation (20) concerns certain fixed instant of time. We now apply (21) by considering the instant $t = t_{n+1}$ to be the time under consideration. The values $\{y\}_n$, $\{y\}_{n-1}$, $\cdots$, $\{y\}_{n+1-k}$ at discrete time instants are given and fixed by the previous solutions. If we change the value of $\{y\}_{n+1}$, the value of $\{\dot{y}\}_{n+1}$ must change accordingly for the equation to hold. We thus obtain the relation

$$\Delta\{y\}_j = -h\beta_0 \Delta\{\dot{y}\}_j \tag{22}$$

or conversily

$$\Delta\{\dot{y}\}_j = -\frac{1}{h\beta_0} \Delta\{y\}_j . \tag{23}$$

We have used now the subscript j as the relation is employed in (20) to produce the result

$$\left(\left[\frac{\partial\{G\}}{\partial\{y\}}\right] - \frac{1}{h\beta_0}\left[\frac{\partial\{G\}}{\partial\{\dot{y}\}}\right]\right)_j \Delta\{y\}_j = -\{G\}_j . \tag{24}$$

This is a linear algebraic equation from which $\Delta\{y\}_j$ can be determined by standard procedures.

Following Ryan (198?), we can now summarize the predictor-corrector scheme to be used roughly as follows:

Predict

1. Predict values for $\{y\}$ and $\{\dot{y}\}$ that lie on a polynomial passing through past state variable values.

Correct

2. Evaluate $\{G\}$. If zero, the solution is acceptable and the corrector is unnecessary.

3. Evaluate the Jacobian matrix defined as

$$\left(\left[\frac{\partial\{G\}}{\partial\{y\}}\right] - \frac{1}{h\beta_0}\left[\frac{\partial\{G\}}{\partial\{\dot{y}\}}\right]\right)_j$$

4. Factorize the Jacobian matrix (In ADAMS this is performed symbolically for efficiency.)

5. Solve for $\Delta\{y\}_j$. Calculate $\{y\}_{j+1}$ and $\{\dot{y}\}_{j+1}$.

6. Repeat steps 2 through 5 until convergence criteria are satisfied.

Integration error control

7. Find best step-size and order for next step.

8. Estimate integration error. If too much, reject last step, reduce h, and go to step 1.

9. If end time has not been reached, start new time step by going back to step 1.

The description above is very cursory and additional literature should be consulted to fill in the details, e.g., Wielenga (1987), Gear (1971).

Remark 15.6. In some formulations the kinematic constraints are transformed to differntial equations by one or two differentiations with respect to time. Even linear combination of such equations with approriate weights have been used, e.g., Amirouche (1992). □

15.9 REFERENCES

Amirouche, F. M. L. (1992): *Computational Methods in Multibody Dynamics*, Prentice-Hall, Englewoods Cliffs.

Gear, C. W. (1971): *Simultaneous Numerical Solution of Differential Algebraic Systems*, IEEE Transactions on Circuit Theory, Vol. 1. pp. 89-95.

Ryan, R. R. (198?): *ADAMS Multibody System Analysis Software*, Mechanics Dynamics, Inc.,Ann Arbor, Michigan 48105, USA.

Wielenga, T. J. (1987): *Analysis Methods and Model Representation in ADAMS*, MDI Technical Paper, no.41.

APPENDIX A

MATRIX NOTATION AND SUMMATION CONVENTION

A.1 MATRIX NOTATION

Some knowledge of matrix calculus is assumed from the reader. Here we just give the main notations used in this text.

As an example, the system of linear equations

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ &\cdots \\ &a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{aligned} \tag{1}$$

can be represented in matrix notation as

$$\underset{m\times n}{[A]}\,\underset{n\times 1}{\{x\}} = \underset{m\times 1}{\{b\}} \tag{2}$$

where the $m \times n$ coefficient matrix $[A]$ is a *rectangular matrix* (suorakaidematriisi), the $n \times 1$ matrix $\{x\}$ consisting of the unknowns is a *column matrix* (sarakematriisi) and the $m \times 1$ matrix $\{b\}$ consisting of the right hand side terms is similarly a column matrix.

Indices under the matrix symbols can be used for clarity to indicate the dimensions as in (2). A column matrix is often called *column vector* (pystyvektori) or simply vector although this latter terminology may sometimes lead to confusion with a "real physical" vector.

The *transpose matrix* (transponoitu matriisi) of a matrix $[A]$ is indicated by the superscript T: $[A]^{\mathrm{T}}$. To save space, contents of a column vector can be given say as

$$\underset{n\times 1}{\{a\}} = \left[a_1\ a_2 \cdots a_n\right]^{\mathrm{T}} \tag{3}$$

or as

$$\{a\}^{\mathrm{T}} = \left[a_1\ a_2 \cdots a_n\right] \tag{4}$$

employing the transpose. The last matrix is often called *row vector* (vaakavektori).

The *inverse* (käänteismatriisi) of a *square matrix* (neliömatriisi) $[A]$ is denoted $[A]^{-1}$ and the *determinant* (determinantti) $\det[A]$.

The *scalar* or *dot product* (skalaaritulo, pistetulo)

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_y b_y \tag{5}$$

of two vectors

$$\begin{aligned} \boldsymbol{a} &= a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k}, \\ \boldsymbol{b} &= b_x \boldsymbol{i} + b_y \boldsymbol{j} + b_z \boldsymbol{k} \end{aligned} \tag{6}$$

can be represented conveniently in matrix notation as

$$\boldsymbol{a} \cdot \boldsymbol{b} = \{a\}^{\mathrm{T}}\{b\} = \{b\}^{\mathrm{T}}\{a\} \tag{7}$$

where $\{a\}$ and $\{b\}$ are the column vector equivalents of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} \triangleq \{a\} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}, \qquad \boldsymbol{b} \triangleq \{b\} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}. \tag{8}$$

The notation "$\triangleq$" means here: the quantity on the right hand side is the equivalent to the quantity on the left hand side. We cannot strictly use the equal to sign as a vector cannot be equal to a matrix although they can be associated with each other. It should be further noticed that expression (7) is a scalar and can be interpreted as a 1×1 matrix. The transposition of a 1×1 matrix does not change its value. Application of the general transposition rule for a matrix product: $([A][B])^{\mathrm{T}} = [B]^{\mathrm{T}}[A]^{\mathrm{T}}$, and remembering that $([A]^{\mathrm{T}})^{\mathrm{T}} = [A]$ explains the two forms on the right hand side of (7).

The column vector equivalent $\{c\} = [c_x\ \ c_y\ \ c_z]^{\mathrm{T}}$ of the *vector* or *cross product* (vektoritulo, scalaaritulo)

$$\begin{aligned} \boldsymbol{c} \equiv \boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\ &= (a_y b_z - a_z b_y)\boldsymbol{i} + (a_z b_x - a_x b_z)\boldsymbol{j} + (a_x b_y - a_y b_x)\boldsymbol{k} \end{aligned} \tag{9}$$

of two vectors (6) can be represented in matrix notation as

$$\boldsymbol{c} \triangleq \{c\} = [\tilde{a}]\{b\} = -[\tilde{b}]\{a\}. \tag{10}$$

Here it has been necessary to define square *skew-symmetric* (vinosymmetrinen) auxiliary matrices

$$[\tilde{a}]=\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}, \quad [\tilde{b}]=\begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix} \tag{11}$$

to achieve the result needed. Although maybe rather artificial, these short-hand notations are convenient in some matrix manipulations.

A.2 SUMMATION CONVENTION

As an example, the system of equations (A.1.1) can be written more concisely as

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i=1,2,\cdots,m. \tag{1}$$

Using the *summation convention* (summeeraussopimus) we obtain still more concisely

$$a_{ij}x_j = b_i. \tag{2}$$

Index i is called the *free index* (vapaa indeksi), which takes consecutively the values $i=1$, $i=2$, $\cdots$, $i=m$, giving the first equation, the second equation, etc. The *range* (arvoalue) of the free index — here the integers $[1,m]$ — is usually clear from the context. Index j is called the *summation index* or *dummy index* (summeerausindeksi, mykkä indeksi) and the summation convention (often called also the Einstein summation convention) is:

> An index appearing twice in a term denotes summation with respect to that index over its range. (3)

The range of the summation index is usually also clear from the context (here the integers $[1,n]$). If the summation convention is to be suppressed, the indices are enclosed in parentheses or this is specificly mentioned.

The adjective "dummy" refers to the fact that a dummy index can be replaced with an arbitrary index symbol without affecting the result. For instance, the formula

$$a_{jk}x_k = b_j \tag{4}$$

means exactly the same as formula (2). The dummy index j has been replaced by k. In fact, the free index can be similarly changed, and here i has been

replaced by j. Naturally, when changing symbols one must take care that summation is not induced by accident by using a symbol appearing already as a free index.

The main thing to check is that *the same free indices appear on both sides and in each term in a formula and that a summation index appears only twice in a term.*

As an example, in the homogeneous quadratic form

$$Q = a_{ij}x_ix_j, \tag{5}$$

i and j are summation indices and there are no free indices. Thus in detail

$$\begin{aligned} Q=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j &= a_{11}x_1x_1 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n + \\ &+ a_{21}x_2x_1 + a_{22}x_2x_2 + \cdots + a_{2n}x_2x_n + \\ &+ \cdots \\ &+ a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_nx_n . \end{aligned} \tag{6}$$

Similarly, in the generalized Hooke's law in three dimensions

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{7}$$

i and j are free indices and k and l are summation indices. Written in full, this means $3\times3=9$ scalar equations each having $3\times3=9$ summation terms on the right hand side.

Tensor analysis usually operates with indices and with the summation convention. In rectangular cartesian coordinates, conventional notations x, y, z and i, j, k are replaced with symbols like x_1, x_2, x_3 and i_1, i_2, i_3, respectively. This kind of representation is called tensor notation or indicial notation.

The scalar product of two vectors

$$\begin{aligned} \boldsymbol{a} &= a_1\boldsymbol{i}_1 + a_2\boldsymbol{i}_2 + a_3\boldsymbol{i}_3 = a_k\boldsymbol{i}_k , \\ \boldsymbol{b} &= b_1\boldsymbol{i}_1 + b_2\boldsymbol{i}_2 + b_3\boldsymbol{i}_3 = b_k\boldsymbol{i}_k \end{aligned} \tag{8}$$

is in indicial notation (on the right hand side)

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_ib_i. \tag{9}$$

The *Kronecker delta* (Kroneckerin delta) δ_{ij} is a useful manipulative device. It is defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{10}$$

For example, we see from the properties of the scalar product that the unit base vectors obey the rule

$$\boldsymbol{i}_k \cdot \boldsymbol{i}_l = \delta_{kl} = \delta_{lk} . \tag{11}$$

Summation with respect to an index of the Kronecker delta produces the following type of result

$$a_{jk}\delta_{kl} = a_{jl} \tag{12}$$

as δ_{kl} is non-zero and equals 1 only when the summation index k equals l. A further example is the formula

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \tag{13}$$

where the x's are independent variables.

The *permutation symbol* (permutaatiosymboli) ε_{ijk} is defined by

$$\begin{aligned} &\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \\ &\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1, \\ &\varepsilon_{ijk} = 0 \quad \text{if any two or three indices are the same.} \end{aligned} \tag{14}$$

In other words, the permutation symbol equals 1 when the indices follow as 1, 2, 3 or are an even permutation of them and it equals -1 if the indices are an odd permutation of 1, 2, 3. Similarly, for instance, $\varepsilon_{121} = 0$ and $\varepsilon_{333} = 0$.

Permutation symbol is useful in the indicial representation of the vector product. For instance, the unit base vectors in a right handed coordinate system obey the formula

$$\boldsymbol{i}_j \times \boldsymbol{i}_k = \varepsilon_{ijk}\boldsymbol{i}_i \tag{15}$$

As a check with $j = 1$ and $k = 1$, we obtain $\varepsilon_{i11}\boldsymbol{i}_i = \boldsymbol{0}$, with $j = 1$ and $k = 2$ $\varepsilon_{i12}\boldsymbol{i}_i = \varepsilon_{312}\boldsymbol{i}_3 = \boldsymbol{i}_3$, etc. The vector product of two vectors can be written in the form

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \varepsilon_{ijk} a_j b_k \boldsymbol{i}_i \tag{16}$$

or

$$c_i = \varepsilon_{ijk} a_j b_k \tag{17}$$

This can be checked for instance for the value 1 of the free index i. We find

$$c_1 = \varepsilon_{1jk} a_j b_k = \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2 \tag{18}$$

Thus only two non-zero contributions appear from the summations with respect to j and k. The result is seen to be in agreement with the first component in formula (A.1.9).

The vector operator *del* (nabla-operaattori), denoted by ∇,

$$\nabla = \boldsymbol{i}_1 \frac{\partial}{\partial x_1} + \boldsymbol{i}_2 \frac{\partial}{\partial x_2} + \boldsymbol{i}_3 \frac{\partial}{\partial x_3} = \boldsymbol{i}_m \frac{\partial}{\partial x_m} \tag{19}$$

can be used to produce the *gradient* (gradientti)

$$\begin{aligned} \text{grad } f \equiv \nabla f &= (\boldsymbol{i}_m \frac{\partial}{\partial x_m}) f = \boldsymbol{i}_m \frac{\partial f}{\partial x_m} = \frac{\partial f}{\partial x_m} \boldsymbol{i}_m \\ &= \frac{\partial f}{\partial x_1} \boldsymbol{i}_1 + \frac{\partial f}{\partial x_2} \boldsymbol{i}_2 + \frac{\partial f}{\partial x_3} \boldsymbol{i}_3 \end{aligned} \tag{20}$$

the *divergence* (divergenssi)

$$\begin{aligned} \text{div } f \equiv \nabla \cdot f &= (\boldsymbol{i}_m \frac{\partial}{\partial x_m}) \cdot (f_n \boldsymbol{i}_n) = \frac{\partial f_n}{\partial x_m} \boldsymbol{i}_m \cdot \boldsymbol{i}_n = \frac{\partial f_n}{\partial x_m} \delta_{mn} = \frac{\partial f_m}{\partial x_m} \\ &= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \end{aligned} \tag{21}$$

and the *curl* (roottori)

$$\begin{aligned} \text{curl } f \equiv \text{rot } f \equiv \nabla \times f &= (\boldsymbol{i}_m \frac{\partial}{\partial x_m}) \times (f_n \boldsymbol{i}_n) = \frac{\partial f_n}{\partial x_m} \boldsymbol{i}_m \times \boldsymbol{i}_n = \varepsilon_{imn} \frac{\partial f_n}{\partial x_m} \boldsymbol{i}_i \\ &= (\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}) \boldsymbol{i}_1 + (\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}) \boldsymbol{i}_2 + (\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}) \boldsymbol{i}_3 . \end{aligned} \tag{22}$$

Remark A.1. There are two main ways in tensor analysis to represent first order tensors (= vectors), second order tensors, etc; the *symbolic* or *index free representation* (symbolinen esitystapa) and the *indicial representation* (indeksiesitys). In the symbolic notation quantities are usually represented with bold letters: displacement vector $\boldsymbol{u}$, stress tensor $\boldsymbol{\sigma}$, etc. In the indicial notation we write say u_i, σ_{ij}, correspondingly. Both methods have their advantages and disadvantages depending on the application environment. It should be mentioned that in the indicial representation it is customary to call the symbol

u_i a vector, the symbol σ_{ij} a tensor, that is, we must think that the free indices take all their values and the resulting set of components represents the vector, the tensor, etc. As an example we see from (20) that the gradient vector is given in the symbolic representation by ∇f and in the indicial representation by $\partial f / \partial x_i$. □

APPENDIX B

INTEGRATION BY PARTS

Integration by parts is a basic mathematical manipulation which is used frequently in continuum mechanics and in connection with weak forms and variational calculus.

B.1 ONE DIMENSION

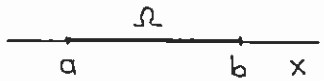


Figure B.1 One-dimensional domain $\Omega =]a,b[$ and its boundary $\Gamma = \{a,b\}$.

The one-dimensional integration by parts formula can be written as (Figure B.1))

$$\int_a^b g\frac{dh}{dx}dx = -\int_a^b \frac{dg}{dx}h dx + \Big|_a^b gh \tag{1a}$$

or using more general notation as

$$\boxed{\int_\Omega g\frac{dh}{dx}d\Omega = -\int_\Omega \frac{dg}{dx}h\,d\Omega + \Big|_a^b gh} \tag{1b}$$

Functions $g(x)$ and $h(x)$ must be continuous and at least piecewise differentiable in $\overline{\Omega} = [a,b]$.

The derivation is based on the obvious result

$$\int_a^b \frac{df}{dx}dx = \Big|_a^b f = f(b) - f(a) \tag{2}$$

due to the properties of the definite integral. Integration of the identity (product rule of differentiation)

$$\frac{d}{dx}(gh) = \frac{dg}{dx}h + g\frac{dh}{dx} \tag{3}$$

over Ω gives

$$\int_a^b \frac{d}{dx}(gh)dx = \int_a^b \frac{dg}{dx}h dx + \int_a^b g\frac{dh}{dx}dx. \tag{4}$$

Application of the rule (2) for the the first integral leads to formula (1).

B.2 TWO DIMENSIONS

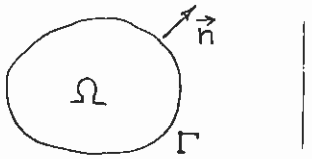


Figure B.2 Two-dimensional domain $\Omega = A$ and its boundary $\Gamma = s$.

The two-dimensional integration by parts formulas can be expressed as (Figure B. 2)

$$\begin{aligned}\int_A g\frac{\partial h}{\partial x}dA &= -\int_A \frac{\partial g}{\partial x}h\,dA + \int_s ghn_x ds,\\ \int_A g\frac{\partial h}{\partial y}dA &= -\int_A \frac{\partial g}{\partial y}h\,dA + \int_s ghn_y ds\end{aligned} \tag{1a}$$

or

$$\boxed{\begin{aligned}\int_\Omega g\frac{\partial h}{\partial x}d\Omega &= -\int_\Omega \frac{\partial g}{\partial x}h\,d\Omega + \int_\Gamma ghn_x d\Gamma,\\ \int_\Omega g\frac{\partial h}{\partial y}d\Omega &= -\int_\Omega \frac{\partial g}{\partial y}h\,d\Omega + \int_\Gamma ghn_y d\Gamma.\end{aligned}} \tag{1b}$$

Functions $g(x,y)$ and $h(x,y)$ must be continuous and at least piecewise differentiable in $\overline{\Omega} = \Omega \cup \Gamma$.

The derivation is based on the so called divergence theorem or Gauss's theorem in the plane:

$$\begin{aligned}\int_\Omega \frac{\partial f}{\partial x}d\Omega &= \int_\Gamma f n_x d\Gamma,\\ \int_\Omega \frac{\partial f}{\partial y}d\Omega &= \int_\Gamma f n_y d\Gamma.\end{aligned} \tag{2}$$

Integration of the identity

$$\frac{\partial}{\partial x}(g\,h)=\frac{\partial g}{\partial x}h+g\frac{\partial h}{\partial x} \tag{3}$$

over Ω gives

$$\int_\Omega \frac{\partial}{\partial x}(g\,h)\,\mathrm{d}\Omega=\int_\Omega \frac{\partial g}{\partial x}h\,\mathrm{d}\Omega+\int_\Omega g\frac{\partial h}{\partial x}\mathrm{d}\Omega. \tag{4}$$

Application of the first formula (2) to the first integral gives the first formula (1) and the second formula can be obtained similarly.

B.3 THREE DIMENSIONS OR MORE

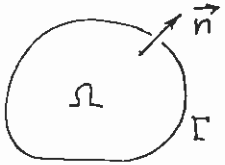


Figure B.3 Three-dimensional domain $\Omega = V$ and its boundary $\Gamma = S$.

For shortness we employ here the index notation. The formulas are direct generalizations from the two-dimensional case. The integration by parts formulas are (Figure C.3)

$$\int_V g\frac{\partial h}{\partial x_i}\mathrm{d}V=-\int_V \frac{\partial g}{\partial x_i}h\,\mathrm{d}V+\int_S ghn_i\mathrm{d}S \tag{1a}$$

or

$$\boxed{\int_\Omega g\frac{\partial h}{\partial x_i}\mathrm{d}\Omega=-\int_\Omega \frac{\partial g}{\partial x_i}h\,\mathrm{d}\Omega+\int_\Gamma ghn_i\mathrm{d}\Gamma.} \tag{1b}$$

The Gauss's theorem is

$$\int_\Omega \frac{\partial f}{\partial x_i}\mathrm{d}\Omega=\int_\Gamma f n_i\mathrm{d}\Gamma. \tag{2}$$

Remark B.1. Actually (2) represents the Gauss's formula for one component of a vector. Thus by replacing f with f_i and by letting the summation convention be valid we obtain

$$\int_\Omega \frac{\partial f_i}{\partial x_i}\mathrm{d}\Omega=\int_\Gamma f_i n_i\mathrm{d}\Gamma \tag{3a}$$

or using symbolic notation

$$\int_\Omega \nabla\cdot f\,\mathrm{d}\Omega=\int_\Gamma n\cdot f\,\mathrm{d}\Gamma. \tag{3b}$$

This form is usually called the divergence theorem. □

Remark B.2. In the one-dimensional case the formulas look a little bit untidy because there appears plus and minus signs in the boundary terms; see for instance formula (B.1.2). However, in this case the boundary consists of just two separate points and we can consider the unit outward normal vector $\boldsymbol{n}$ to have the component +1 at the right hand boundary and the component −1 at left hand boundary. With this interpretation we see that also the one-dimensional formulas are special cases of the general formulas (1) and (2). □

APPENDIX C

RECTANGULAR CURVILINEAR COORDINATES

A systematic method to generate expressions appearing in physics and valid in rectangular (orthogonal) curvilinear coordinates starting from the corresponding simple expressions in rectangular cartesian coordinates is described, Paavola and Salonen (1999). The procedure is called "*the method of local cartesian coordinates*". It can be applied for instance to derive strain expressions for curved beams. Polar coordinates are employed here as a demonstration case. In what follows we often call rectangular curvilinear coordinates and rectangular cartesian coordinates shortly curvilinear coordinates and cartesian coordinates, respectively. Summation convention is not employed in this appendix.

C.1 SOME FORMULAS

In this section some basic formulas appearing in practically any mathematics text dealing with rectangular curvilinear coordinates are reviewed. The presentation is given in two dimensions as the main points can be seen already in this case.

A cartesian coordinate system x, y with unit base vectors $\boldsymbol{i}, \boldsymbol{j}$ and a curvilinear coordinate system α, β with unit base vectors $\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta$ are considered (Figure C.1).

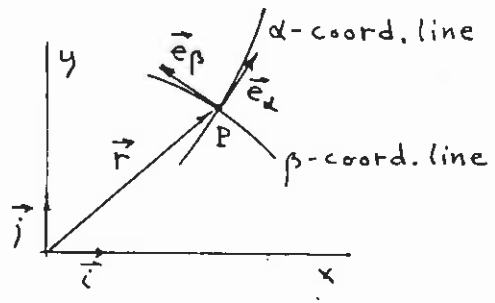


Figure C.1 Cartesian and curvilinear coordinates.

The values of the coordinates are connected by

$$x = x(\alpha, \beta), \qquad y = y(\alpha, \beta). \tag{1}$$

The position vector $\boldsymbol{r}$ can be expressed in principle as

$$\boldsymbol{r} = \boldsymbol{r}(\alpha, \beta) = x(\alpha, \beta)\boldsymbol{i} + y(\alpha, \beta)\boldsymbol{j} \tag{2}$$

or as

$$\boldsymbol{r} = \boldsymbol{r}(\alpha, \beta) = r_\alpha(\alpha, \beta)\boldsymbol{e}_\alpha(\alpha, \beta) + r_\beta(\alpha, \beta)\boldsymbol{e}_\beta(\alpha, \beta). \tag{3}$$

The partial derivatives $\partial \boldsymbol{r}/\partial\alpha$ and $\partial \boldsymbol{r}/\partial\beta$ of the position vector $\boldsymbol{r}$ with respect to the curvilinear coordinates are tangent vectors to the corresponding coordinate lines and one can thus write

$$\frac{\partial \boldsymbol{r}}{\partial \alpha} = h_\alpha \boldsymbol{e}_\alpha, \qquad \frac{\partial \boldsymbol{r}}{\partial \beta} = h_\beta \boldsymbol{e}_\beta \tag{4}$$

where the *scale factors* (skaalatekijä) $h_\alpha = |\partial \boldsymbol{r}/\partial\alpha|$, $h_\beta = |\partial \boldsymbol{r}/\partial\beta|$ can be evaluated from

$$h_\alpha = \left[\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 \right]^{1/2}, \qquad h_\beta = \left[\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2 \right]^{1/2}. \tag{5}$$

The derivatives of the unit vectors are

$$\begin{aligned} \frac{\partial \boldsymbol{e}_\alpha}{\partial \alpha} &= -\frac{1}{h_\beta}\frac{\partial h_\alpha}{\partial \beta}\boldsymbol{e}_\beta, & \frac{\partial \boldsymbol{e}_\alpha}{\partial \beta} &= \frac{1}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha}\boldsymbol{e}_\beta, \\ \frac{\partial \boldsymbol{e}_\beta}{\partial \alpha} &= \frac{1}{h_\beta}\frac{\partial h_\alpha}{\partial \beta}\boldsymbol{e}_\alpha, & \frac{\partial \boldsymbol{e}_\beta}{\partial \beta} &= -\frac{1}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha}\boldsymbol{e}_\alpha. \end{aligned} \tag{6}$$

The polar coordinates $\alpha \triangleq r$, $\beta \triangleq \theta$ (Figure C.2) are employed in what follows as a simple specific illustrative example case.

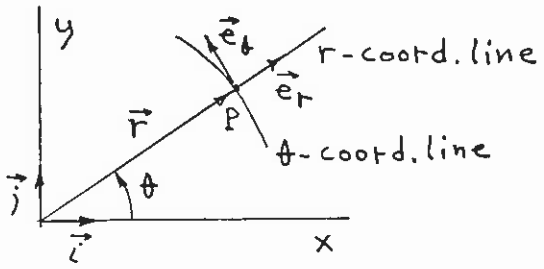


Figure C.2 Cartesian and polar coordinates.

From Figure C.2 there is obtained

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{7}$$

Corresponding to (2) and (3) we have

$$\boldsymbol{r}(r,\theta) = r\cos\theta\, \boldsymbol{i} + r\sin\theta\, \boldsymbol{j} \tag{8}$$

and

$$\boldsymbol{r}(r,\theta) = r\boldsymbol{e}_r(\theta) + 0\cdot\boldsymbol{e}_\theta(\theta). \tag{9}$$

The scale factors (5) are

$$h_r = \left[\cos^2\theta + \sin^2\theta\right]^{1/2} = 1, \qquad h_\theta = \left[r^2\sin^2\theta + r^2\cos^2\theta\right]^{1/2} = r \tag{10}$$

and the derivatives (6) are

$$\frac{\partial \boldsymbol{e}_r}{\partial r} = \boldsymbol{0}, \quad \frac{\partial \boldsymbol{e}_r}{\partial \theta} = \boldsymbol{e}_\theta, \quad \frac{\partial \boldsymbol{e}_\theta}{\partial r} = \boldsymbol{0}, \quad \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r. \tag{11}$$

In specific applications the derivatives of the unit vectors such as (11) are usually found in some simple ad hoc manner and no need for the use of the general formulas (6) arises.

C.2 LOCAL CARTESIAN COORDINATES

An auxiliary local cartesian coordinate system X, Y with unit base vectors $\boldsymbol{I}$, $\boldsymbol{J}$ is erected with its origin at a generic point P and its axes tangent to the α-, β-coordinate lines (Figure C.3). This local coordinate system can be brought to any point P but it is of utmost importance that *during a specific derivation of a result it is considered fixed* so that for instance the unit vectors $\boldsymbol{I}$ and $\boldsymbol{J}$ are then constants.

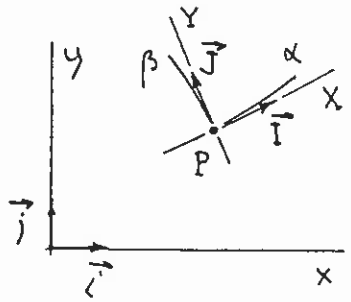


Figure C.3 Local cartesian coordinates X and Y.

At the local origin – and not elsewhere in general – clearly

$$\boxed{\boldsymbol{I} = \boldsymbol{e}_\alpha, \quad \boldsymbol{J} = \boldsymbol{e}_\beta} \tag{1}$$

and

$$\boxed{\frac{\partial}{\partial X} = \frac{1}{h_\alpha}\frac{\partial}{\partial \alpha}, \quad \frac{\partial}{\partial Y} = \frac{1}{h_\beta}\frac{\partial}{\partial \beta}.} \tag{2}$$

These are the basic application formulas. Expressions (2) can be seen to be valid from the definition of the scale factors as $dX = h_\alpha d\alpha$ and $dY = h_\beta d\beta$.

The local system can be considered as a tool which is taken temporarily in use and then discarded as its function has been fulfilled.

Example C.1. We derive the expressions for (1) the gradient, (2) the divergence, and for (3) the strains in polar coordinates.

We employ the geometry of Figure C.2. From formulas (C.1.10), (C.1.11), (1) and (2):

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \boldsymbol{e}_\theta, \quad \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r \tag{a}$$

$$\boldsymbol{I} = \boldsymbol{e}_r, \quad \boldsymbol{J} = \boldsymbol{e}_\theta, \tag{b}$$

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial Y} = \frac{1}{r}\frac{\partial}{\partial \theta}. \tag{c}$$

(1) The definition of the gradient in cartesian coordinates is

$$\text{grad}\, f \equiv \nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} = \frac{\partial f}{\partial X}\boldsymbol{I} + \frac{\partial f}{\partial Y}\boldsymbol{J} \tag{d}$$

where the latter form is expressed in the local system. Thus, due to (b) and (c), we obtain at the local origin

$$\text{grad}\, f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta. \tag{e}$$

However, as point P was arbitrary, this expression for the gradient in polar coordinates is now valid everywhere.

(2) For a vector f we have the alternative representations

$$\begin{aligned} \boldsymbol{f} &= f_x(x,y)\boldsymbol{i} + f_y(x,y)\boldsymbol{j}, \\ &= f_X(X,Y)\boldsymbol{I} + f_Y(X,Y)\boldsymbol{J}, \\ &= f_r(r,\theta)\boldsymbol{e}_r(\theta) + f_\theta(r,\theta)\boldsymbol{e}_\theta(\theta). \end{aligned} \tag{f}$$

Again it is stressed that although point P for the origin of the X,Y-system can be put anywhere during the steps to follow it and the directions of the X- and Y-axes are fixed.

The definition of the divergence in cartesian coordinates is

$$\mathrm{div} f \equiv \nabla \cdot f = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = \frac{\partial f_X}{\partial X} + \frac{\partial f_Y}{\partial Y}. \tag{g}$$

Now,

$$f_X = f \cdot I, \quad f_Y = f \cdot J \tag{h}$$

so

$$\frac{\partial f_X}{\partial X} = \frac{\partial}{\partial X}(f \cdot I) = \frac{\partial f}{\partial X} \cdot I, \quad \frac{\partial f_Y}{\partial Y} = \frac{\partial}{\partial Y}(f \cdot J) = \frac{\partial f}{\partial Y} \cdot J. \tag{i}$$

as in the local system I and J are constants. This kind of formulas for derivatives of vector components are useful in the method of local cartesian coordinates. Thus

$$\mathrm{div} f = \frac{\partial f}{\partial X} \cdot I + \frac{\partial f}{\partial Y} \cdot J \tag{j}$$

and at the local origin due to (a), (b), (c) and the last form of (f)

$$\begin{aligned} \mathrm{div} f &= \frac{\partial}{\partial r}(f_r e_r + f_\theta e_\theta) \cdot e_r + \frac{1}{r}\frac{\partial}{\partial \theta}(f_r e_r + f_\theta e_\theta) \cdot e_\theta \\ &= (\frac{\partial f_r}{\partial r} e_r + \frac{\partial f_\theta}{\partial r} e_\theta) \cdot e_r + \frac{1}{r}(\frac{\partial f_r}{\partial \theta} e_r + f_r e_\theta + \frac{\partial f_\theta}{\partial \theta} e_\theta - f_\theta e_r) \cdot e_\theta \\ &= \frac{\partial f_r}{\partial r} + \frac{1}{r}(f_r + \frac{\partial f_\theta}{\partial \theta}). \end{aligned} \tag{k}$$

Again, this a general valid result in polar coordinates.

A word of warning is in place here. As $f_r = f_X$, $f_\theta = f_Y$ at point P, a blind application of formulas (c) to expression (g) would give

$$\mathrm{div} f = \frac{\partial f_r}{\partial r} + \frac{1}{r}\frac{\partial f_\theta}{\partial \theta} \quad \text{(wrong)} \tag{l}$$

which is not correct. Differentiation is based on *comparison of the values of a certain quantity at adjacent argument values* – here at neighbouring points – and taking the limit. f_α and f_X represent different quantities outside point P and thus in general $1/h_\alpha \cdot (\partial f_\alpha / \partial \alpha) \neq \partial f_X / \partial X$ even at P. In the application of formulas (2) *the derivative must operate on the same quantity* such as f here.

(3) The displacement vector has the representations

$$\begin{aligned} u &= u_x i + u_y j \equiv ui + vj \\ &= u_X I + u_Y J \equiv UI + VJ \\ &= u_r e_r + u_\theta e_\theta . \end{aligned} \tag{m}$$

The small strain expressions in cartesian coordinates are

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{n}$$

Thus in the local system

$$\begin{aligned} \varepsilon_X &= \frac{\partial U}{\partial X} = \frac{\partial u}{\partial X} \cdot I, \\ \varepsilon_Y &= \frac{\partial V}{\partial Y} = \frac{\partial u}{\partial Y} \cdot J, \\ \gamma_{XY} &= \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} = \frac{\partial u}{\partial Y} \cdot I + \frac{\partial u}{\partial X} \cdot J. \end{aligned} \tag{o}$$

The dot product forms are obtained similarly as expressions (i). Finally, at the local origin

$$\begin{aligned} \varepsilon_r = \varepsilon_X &= \frac{\partial u}{\partial r} \cdot e_r = (\frac{\partial u_r}{\partial r} e_r + \frac{\partial u_\theta}{\partial r} e_\theta) \cdot e_r = \frac{\partial u_r}{\partial r}, \\ \varepsilon_\theta = \varepsilon_Y &= \frac{1}{r}\frac{\partial u}{\partial \theta} \cdot e_\theta = \frac{1}{r}(\frac{\partial u_r}{\partial \theta} e_r + u_r e_\theta + \frac{\partial u_\theta}{\partial \theta} e_\theta - u_\theta e_r) \cdot e_\theta = \frac{1}{r}(u_r + \frac{\partial u_\theta}{\partial \theta}), \\ \gamma_{r\theta} = \gamma_{XY} &= \frac{1}{r}\frac{\partial u}{\partial \theta} \cdot e_r + \frac{\partial u}{\partial r} \cdot e_\theta \\ &= \frac{1}{r}(\frac{\partial u_r}{\partial \theta} e_r + u_r e_\theta + \frac{\partial u_\theta}{\partial \theta} e_\theta - u_\theta e_r) \cdot e_r + (\frac{\partial u_r}{\partial r} e_r + \frac{\partial u_\theta}{\partial r} e_\theta) \cdot e_\theta \\ &= \frac{1}{r}(\frac{\partial u_r}{\partial \theta} - u_\theta) + \frac{\partial u_\theta}{\partial r}. \end{aligned} \tag{p}$$

C.3 INTEGRATION BY PARTS

Integration by parts has been considered in cartesian coordinates in Appendix B. Here we extend this for curvilinear coordinates. Only the two-dimensional case is presented.

The Gauss's theorem or the divergence theorem is (see formula (B.3.3b))

$$\int_A \nabla \cdot f \, \mathrm{d}A = \int_s n \cdot f \, \mathrm{d}s. \tag{1}$$

It is not difficult to show using the same technique as in Example C.1 that the general expression in curvilinear coordinates for the divergence is

$$\nabla \cdot f = \frac{1}{h_\alpha h_\beta}[\frac{\partial}{\partial \alpha}(h_\beta f_\alpha) + \frac{\partial}{\partial \beta}(h_\alpha f_\beta)] \quad \text{(no sum on } \alpha \text{ or } \beta\text{)} \tag{2}$$

Further,

$$\mathrm{d}A = \mathrm{d}X\mathrm{d}Y = h_\alpha \mathrm{d}\alpha \, h_\beta \mathrm{d}\beta = h_\alpha h_\beta \mathrm{d}\alpha \, \mathrm{d}\beta \tag{3}$$

and

$$n \cdot f = f_x n_x + f_y n_y = f_\alpha n_\alpha + f_\beta n_\beta . \tag{4}$$

Thus (1) is in curvilinear coordinates

$$\int_{\alpha,\beta}[\frac{\partial}{\partial\alpha}(h_\beta f_\alpha)+\frac{\partial}{\partial\beta}(h_\alpha f_\beta)]\mathrm{d}\alpha\,\mathrm{d}\beta=\int_s(f_\alpha n_\alpha+f_\beta n_\beta)\mathrm{d}s. \tag{5}$$

The α,β-notation is used to indicate that the area integral is to be taken in the α,β-plane. Defining $F_\alpha=h_\beta f_\alpha$, $F_\beta=h_\alpha f_\beta$ gives the form

$$\int_{\alpha,\beta}(\frac{\partial F_\alpha}{\partial\alpha}+\frac{\partial F_\beta}{\partial\beta})\mathrm{d}\alpha\,\mathrm{d}\beta=\int_s(F_\alpha\frac{n_\alpha}{h_\beta}+F_\beta\frac{n_\beta}{h_\alpha})\mathrm{d}s. \tag{6}$$

Finally, making the consecutive selections $F_\alpha=fg$, $F_\beta=0$ and $F_\alpha=0$, $F_\beta=fg$, where $f(\alpha,\beta)$ and $g(\alpha,\beta)$ are two functions, we arrive at the integration by parts formulas

$$\boxed{\begin{aligned}\int_{\alpha,\beta}f\frac{\partial g}{\partial\alpha}\mathrm{d}\alpha\,\mathrm{d}\beta&=-\int_{\alpha,\beta}\frac{\partial f}{\partial\alpha}g\,\mathrm{d}\alpha\,\mathrm{d}\beta+\int_s fg\frac{n_\alpha}{h_\beta}\mathrm{d}s,\\ \int_{\alpha,\beta}f\frac{\partial g}{\partial\beta}\mathrm{d}\alpha\,\mathrm{d}\beta&=-\int_{\alpha,\beta}\frac{\partial f}{\partial\beta}g\,\mathrm{d}\alpha\,\mathrm{d}\beta+\int_s fg\frac{n_\beta}{h_\alpha}\mathrm{d}s.\end{aligned}} \tag{7}$$

These may be compared with the conventional cartesian integration by parts formulas (B.2.1(a)):

$$\begin{aligned}\int_A f\frac{\partial g}{\partial x}\mathrm{d}A&=-\int_A\frac{\partial f}{\partial x}g\,\mathrm{d}A+\int_s fgn_x\,\mathrm{d}s,\\ \int_A f\frac{\partial g}{\partial y}\mathrm{d}A&=-\int_A\frac{\partial f}{\partial y}g\,\mathrm{d}A+\int_s fgn_y\,\mathrm{d}s.\end{aligned} \tag{8}$$

In polar coordinates $\mathrm{d}A=\mathrm{d}X\,\mathrm{d}Y=\mathrm{d}r\,r\,\mathrm{d}\theta=r\,\mathrm{d}r\,\mathrm{d}\theta$ and equations (7) obtain the forms

$$\begin{aligned}\int_{r,\theta}f\frac{\partial g}{\partial r}\mathrm{d}r\,\mathrm{d}\theta&=-\int_{r,\theta}\frac{\partial f}{\partial r}g\,\mathrm{d}r\,\mathrm{d}\theta+\int_s fg\frac{n_r}{r}\mathrm{d}s,\\ \int_{r,\theta}f\frac{\partial g}{\partial\theta}\mathrm{d}r\,\mathrm{d}\theta&=-\int_{r,\theta}\frac{\partial f}{\partial\theta}g\,\mathrm{d}r\,\mathrm{d}\theta+\int_s fgn_\theta\,\mathrm{d}s.\end{aligned} \tag{9}$$

Remark C.1. It should be emphasized that n_α and n_β in (7) are the components of the unit outward normal vector to the boundary curve s measured with respect to the local e_α-, e_β- directions. This should be obvious from formula (4). □

C.4 REFERENCE

Paavola, J. and E.-M. Salonen, (1999): "Coping with Curvilinear Coordinates", accepted for publication in the *IJMEE*.

APPENDIX D

VARIATIONAL CALCULUS

As emphasized earlier, we consider the principle of virtual work as a weak form and not a variational principle. In applying the principle of virtual work in its basic form we hardly need any tools of variational calculus. It is, however, good to have some knowledge of the concepts of a functional and of variational principles to be able to follow the relevant literature surrounding the principle of virtual work. The basis task of variational calculus is to find the function giving a certain functional a stationary value. Approximate discrete methods transform this problem in fact to the problem of finding the point giving an ordinary function a stationary value. Therefore it is illuminating and useful to start by considering first functions and only later functionals. The presentation to follow relies strongly on the masterful text by Lanczos (1974).

D.1 STATIONARY VALUE OF A FUNCTION

D.1.1 Stationary value of a function without constraints

Let us consider the problem to determine the *extremum value* (ääriarvo) of a function $f(x)$ on some interval $[a,b]$ of the x-axis. If the function has continuous first derivative (belongs to class C^1) the necessary condition for achieving this in the *interior* $]a,b[$ of $[a,b]$ is that the differential of the function disappears:

$$\mathrm{d}f \equiv \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = 0. \tag{1}$$

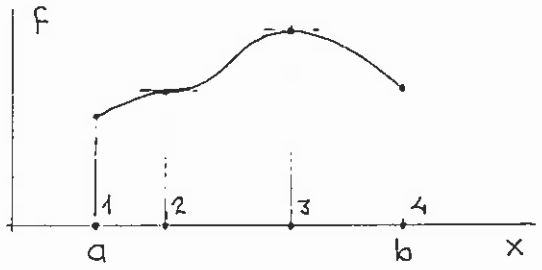


Figure D.1 Points 1 ($x = a$), 3 and 4 ($x = b$) are extremum points. Points 2 and 3 are stationary points. Local and global minimum is achieved at 1. No extremum is achieved at point 2. Local and global maximum is achieved at point 3.

This condition is not sufficient for achieving a local *extremum* (= minimum or maximum) in the interior as is seen from point 2 in Figure D.1. However, points where the rate of change of a function is zero are already of importance. The function is said to have at these points a *stationary value* (stationaarinen arvo) and the corresponding points are called *stationary points* (stationaarinen piste) or critical points. The condition for stationarity (1) is represented usually in the equivalent form

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}x} = 0.} \tag{2}$$

Similarly, a function $f(x_1, x_2, \cdots, x_n)$ of several variables is said to have a stationary value at a point if the total differential of the function vanishes with respect to all admissible differential changes $\mathrm{d}x_1, \mathrm{d}x_2, \cdots, \mathrm{d}x_n$:

$$\mathrm{d}f \equiv \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\mathrm{d}x_2 + \cdots + \frac{\partial f}{\partial x_n}\mathrm{d}x_n = 0. \tag{3}$$

If all the variables x are really independent or "free", the differentials $\mathrm{d}x$ can be chosen arbitrarily and from this follows that (3) is equivalent to the stationarity conditions

$$\boxed{\frac{\partial f}{\partial x_i} = 0,} \qquad i = 1,2,\cdots,n. \tag{4}$$

In detail, we can first put $\mathrm{d}x_1 \neq 0$, all the other $\mathrm{d}x$ to zero to obtain $(\partial f / \partial x_1)\mathrm{d}x_1 = 0$ and thus further the first equation (4), etc. The solution of the system of equations (4) gives the stationary point (points), that is, the values $x_1, x_2, \cdots, x_n$ making the function f stationary. The actual stationary value of the function is often — at least in mechanics — of less interest. If wanted, it can be obtained by substituting the values of x found into the expression of f.

To find the possibility of an extremum, it is not enough to consider just the first differential $\mathrm{d}^1 f = \mathrm{d}f$ but the behaviour of the function must be studied in more detail in the neighbourhood of a stationary point.We will consider the change of a function using a similar technique which will be employed later in connection of variational calculus.

The study of the change of the value of a function $f(x_1, x_2, \cdots, x_n)$ due to changes $\mathrm{d}x_1, \mathrm{d}x_2, \cdots, \mathrm{d}x_n$ of the arguments is reduced to the study of a function of *one variable* ε by putting

$$\mathrm{d}x_1 = \varepsilon\eta_1, \quad \mathrm{d}x_2 = \varepsilon\eta_2, \quad \cdots, \quad \mathrm{d}x_n = \varepsilon\eta_n. \tag{5}$$

The arbitrarily selected numbers η are kept fixed when the small parameter ε changes its value. It this connection it is illuminating to consider the variables $x_1, x_2, \cdots, x_n$ to represent the rectangular coordinates of a point P in a n-dimensional space. Expressions (5) can now be thought to describe a displacement into a direction whose direction cosines are the η's. Parameter ε controls the magnitude of the change.

The change of the value of the function is

$$\Delta f = f(x_1^*, x_2^*, \cdots, x_n^*) - f(x_1, x_2, \cdots, x_n) \tag{6}$$

where

$$x_1^* = x_1 + \varepsilon\eta_1, \quad x_2^* = x_2 + \varepsilon\eta_2, \quad \cdots, \quad x_n^* = x_n + \varepsilon\eta_n, \tag{7}$$

During the following study, we consider the initial point P $(x_1, x_2, \cdots, x_n)$ and the direction cosines η fixed. Then Δf is a function of only one variable:

$$\Delta f = \Delta f(\varepsilon). \tag{8}$$

Function Δf is expanded into a Taylor series around the point $\varepsilon = 0$:

$$\Delta f = \frac{\mathrm{d}\Delta f}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\varepsilon + \frac{1}{2}\frac{\mathrm{d}^2\Delta f}{\mathrm{d}\varepsilon^2}\Big|_{\varepsilon=0}\varepsilon^2 + \cdots = \frac{\mathrm{d}f^*}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\varepsilon + \frac{1}{2}\frac{\mathrm{d}^2 f^*}{\mathrm{d}\varepsilon^2}\Big|_{\varepsilon=0}\varepsilon^2 + \cdots \tag{9}$$

The last form follows from expression (6) as ε is contained only in the term $f^* \equiv f(x_1^*, x_2^*, \cdots, x_n^*)$. The derivatives are calculated using the chain rule. Thus (summation convention is employed frequently in the following)

$$\begin{aligned} \frac{\mathrm{d}f^*}{\mathrm{d}\varepsilon} &= \frac{\partial f(x_1^*, x_2^*, \cdots, x_n^*)}{\partial x_i^*}\eta_i, \\ \frac{\mathrm{d}^2 f^*}{\mathrm{d}\varepsilon^2} &= \frac{\partial^2 f(x_1^*, x_2^*, \cdots, x_n^*)}{\partial x_i^* \partial x_j^*}\eta_i\eta_j. \end{aligned} \tag{10}$$

When these are evaluated at $\varepsilon = 0$, $x_i^* \to x_i$, the "stars" can be removed and expression (9) obtains the form

$$\begin{aligned} \Delta f &= \varepsilon\frac{\partial f}{\partial x_i}\eta_i + \frac{1}{2}\varepsilon^2\frac{\partial^2 f}{\partial x_i \partial x_j}\eta_i\eta_j + \cdots \\ &= \frac{\partial f}{\partial x_i}\mathrm{d}x_i + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}\mathrm{d}x_i\mathrm{d}x_j + \cdots \\ &\equiv \mathrm{d}^1 f + \frac{1}{2}\mathrm{d}^2 f + \cdots \end{aligned} \tag{11}$$

The *first differential* (ensimmäinen differentiaali) $\mathrm{d}^1 f$ vanishes at the stationary point so that at it

$$\Delta f = \frac{1}{2}\mathrm{d}^2 f + \cdots = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}\mathrm{d}x_i\mathrm{d}x_j + \cdots \tag{12}$$

Here the derivatives are to be evaluated at the stationary point. The question about the possible local extremum is determined by the character of the *quadratic form* (neliömuoto) in the differentials $\mathrm{d}x$ represented by the *second differential* (toinen differentiaali) $\mathrm{d}^2 f$ as it governs the behaviour of the change in the small enough neighbourhood of the stationary point. (Some writers define the second differential with the factor 1/2 included.) The matrix of the quadratic form, the elements of which consist of the second derivatives, is called the *Hessian matrix* or simply the *Hessian*. The quadratic form can be (1) positive definite, (2) negative definite, (3) indefinite, (4) positive or negative semidefinite. It can be shown that the function has, correspondingly, (1) a local strict minimum, (2) a local strict maximum, (3) no extremum, (4) higher order terms must be studied before conclusions can be drawn, Salonen (1995, p. 28, 48).

Remark D.1. During the derivation of the result (11) we considered the direction cosines η fixed. However, they can be fixed in any direction and thus after obtaining formula (11) we are again free to choose the differentials $\mathrm{d}x$ any way we want. It should be further mentioned that we have of course assumed f to be smooth enough to allow the series representation used. □

Example D.1. Let us determine the possible extremum of the function

$$f(x, y) = (x-2)^2 + (y-1)^2 \tag{a}$$

where x and y are the independent variables.

The stationarity conditions (4) are

$$\begin{aligned} \frac{\partial f}{\partial x} &\equiv 2x - 4 = 0, \\ \frac{\partial f}{\partial y} &\equiv 2y - 2 = 0. \end{aligned} \tag{b}$$

The solution — the stationary point — is $x = 2$, $y = 1$. The second derivatives are

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial x} &= 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0, \\ \frac{\partial^2 f}{\partial y \partial x} &= 0, \quad \frac{\partial^2 f}{\partial y \partial y} = 2 \end{aligned} \tag{c}$$

and we have according to formula (12) simply

$$\frac{1}{2}\mathrm{d}^2 f = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x \partial x}\mathrm{d}x\,\mathrm{d}x + \frac{\partial^2 f}{\partial x \partial y}\mathrm{d}x\,\mathrm{d}y + \frac{\partial^2 f}{\partial y \partial x}\mathrm{d}y\,\mathrm{d}x + \frac{\partial^2 f}{\partial y \partial y}\mathrm{d}y\,\mathrm{d}y\right)$$
$$= \frac{1}{2}(2\cdot \mathrm{d}x\,\mathrm{d}x + 2\cdot \mathrm{d}y\,\mathrm{d}y) = (\mathrm{d}x)^2 + (\mathrm{d}y)^2. \qquad \text{(d)}$$

This is clearly positive definite and a strict minimum value 0 is achieved as is obvious directly from expression (a). In fact, here the second degree term (d) already gives the exact change of the function value from the stationary point.

Example D.2. Let us consider stationarity of the function

$$f(x_1, x_2, \cdots, x_n) = \frac{1}{2}a_{ij}x_i x_j + a_i x_i + c$$
$$= \frac{1}{2}\{x\}^{\mathrm{T}}[A]\{x\} + \{a\}^{\mathrm{T}}\{x\} + c. \qquad \text{(a)}$$

The coefficients a_{ij} of the *quadratic form* (neliömuoto)

$$Q = \frac{1}{2}a_{ij}x_i x_j = \frac{1}{2}\{x\}^{\mathrm{T}}[A]\{x\} \qquad \text{(b)}$$

the coefficients a_i of the *linear form* (lineaarimuoto)

$$L = a_i x_i = \{a\}^{\mathrm{T}}\{x\} = \{x\}^{\mathrm{T}}\{a\} \qquad \text{(c)}$$

and the constant c do not depend on the variables x. The expression is the most general one consisting of a quadratic, of a linear and of a constant term in the variables x. Summation convention is applied and corresponding matrix forms with obvious meaning of the symbols have been written down. Multiplier 1/2 is included just for convenience of representation. This example gives some practise in index manipulation and produces differentiation formulas for quadratic and linear forms.

We write the stationarity conditions (4) here first as

$$\frac{\partial f}{\partial x_k} = 0. \qquad \text{(d)}$$

(If we applied formula (4) blindly to expression (a), summation index i would appear three times in a term leading to confusion.) We obtain

$$\frac{\partial f}{\partial x_k} = \frac{1}{2}a_{ij}\frac{\partial x_i}{\partial x_k}x_j + \frac{1}{2}a_{ij}x_i\frac{\partial x_j}{\partial x_k} + a_i\frac{\partial x_i}{\partial x_k} + 0$$
$$= \frac{1}{2}a_{ij}\delta_{ik}x_j + \frac{1}{2}a_{ij}x_i\delta_{jk} + a_i\delta_{ik}$$
$$= \frac{1}{2}a_{kj}x_j + \frac{1}{2}a_{ik}x_i + a_k. \qquad \text{(e)}$$

Formulas (A.2.12) and (A.2.13) have been made use of. In (e), k is a free index and j and i are summation indices. We now transform this into a cleaner form by putting $k \to i$, $i \to j$ to obtain the formula

$$\frac{\partial f}{\partial x_i} = \frac{1}{2}a_{ij}x_j + \frac{1}{2}a_{ji}x_j + a_i. \qquad \text{(f)}$$

This is in matrix notation

$$\frac{\partial f}{\partial \{x\}} = \frac{1}{2}[A]\{x\} + \frac{1}{2}[A]^{\mathrm{T}}\{x\} + \{a\} = \frac{1}{2}([A] + [A]^{\mathrm{T}})\{x\} + \{a\}. \qquad \text{(g)}$$

The results above have been derived in fact with too much generality. As is well-known, Salonen (1995, p. 28), the matrix of a quadratic form can be taken always symmetric. We have thus instead of (f) and (g)

$$\frac{\partial f}{\partial x_i} = a_{ij}x_j + a_i \qquad \text{(h)}$$

and

$$\frac{\partial f}{\partial \{x\}} = [A]\{x\} + \{a\}. \qquad \text{(i)}$$

The stationarity conditions are

$$a_{ij}x_j = b_i \qquad \text{(j)}$$

or

$$[A]\{x\} = \{b\} \qquad \text{(k)}$$

where we have put $b_i = -a_i$. A linear system of equations is arrived at as is obvious already from the previous example.

We have obtained two useful differentiation formulas. For a quadratic form with a symmetric matrix

$$\frac{\partial Q}{\partial x_i} = a_{ij}x_j, \qquad \frac{\partial Q}{\partial \{x\}} = [A]\{x\} \qquad \text{(l)}$$

and for a linear form

$$\frac{\partial L}{\partial x_i} = a_i, \qquad \frac{\partial L}{\partial \{x\}} = \{a\}. \qquad \text{(m)}$$

The second derivatives from (h) are

$$\frac{\partial^2 f}{\partial x_i \partial x_k} = a_{ij}\frac{\partial x_j}{\partial x_k} + 0 = a_{ij}\delta_{jk} = a_{ik} \qquad \text{(n)}$$

or changing the notation, finally

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = a_{ij}. \qquad \text{(o)}$$

The study of the possible extremum must thus be based on the properties of the symmetric coefficient matrix $[A]$. Here the Hessian is thus a constant matrix.

D.1.2 Lagrange multiplier method

In a stationary problem, the arguments of a function $f(x_1, x_2, \cdots, x_n)$ are not always independent. There may be present *auxiliary conditions* or *side condi-*

tions or *constraint conditions* or shortly *constraints* (sideyhtälö, rajoitusyhtälö, rajoite) of the form

$$g_k(x_1, x_2, \cdots, x_n) = 0, \quad k = 1,2,\cdots,m. \tag{13}$$

The number of constraints m must be smaller than the number of variables n ($m < n$) for otherwise there are no independent variables left to vary. In the extreme case $m = n$, the solution $(x_1, x_2, \cdots, x_n)$ is determined directly by (13).

The stationarity condition (3) cannot be used directly any more to derive equations (4) as due to the constraints the differentials $dx_1, dx_2, \cdots, dx_n$ cannot be varied arbitrarily.

At least in principle we can consider, say, the last m variables solved from (13) and expressed in the first $n-m$ variables and substituted in the expression of f to obtain a new function $f_F(x_1, x_2, \cdots, x_{n-m})$ where the arguments are now independent or free. The stationarity conditions are then

$$\boxed{\frac{\partial f_F}{\partial x_i} = 0,} \quad i = 1,2,\cdots,n-m. \tag{14}$$

This procedure to take care of the constraints could be called the *elimination method* (eliminaatiomenettely). It has the advantage that the number of unknowns is reduced to $n-m$. However, firstly the elimination may be difficult and clumsy to perform. Secondly, the constraints are often symmetric in the variables and then there is really no good grounds in selecting some of the variables as independent and some as dependent. We therefore do not consider the elimination method here in more detail.

The *Lagrange multiplier method* or shortly *Lagrange's method* (Lagrangen kertojamenettely, Lagrangen menettely) is an alternative to the elimination method to transform a constrained stationary problem to a free or unconstrained problem. The procedure is straightforward and preserves possible symmetries but the number of unknowns increases to $n+m$ instead of reducing them to $n-m$.

The derivation of the Lagrange multiplier method proceeds as follows. The total differential

$$df = \frac{\partial f}{\partial x_i} dx_i \tag{15}$$

should vanish for all possible differentials $dx_1, dx_2, \cdots, dx_n$ which satisfy the differential type constraints

$$dg_k \equiv \frac{\partial g_k}{\partial x_i} dx_i = 0, \quad k = 1,2,\cdots,m. \tag{16}$$

obtained by taking the total differential of both sides of the original constraint equations (13). We again consider $x_1, x_2, \cdots, x_{n-m}$ as the independent variables and the dependent ones $x_{n-m+1}, \cdots, x_n$ to be represented in them. Similarly, in the new constraints (16) also the $dx_{n-m+1}, \cdots, dx_n$ should be given in terms of $x_1, x_2, \cdots, x_{n-m}$ and $dx_1, dx_2, \cdots, dx_{n-m}$. However, before doing this we modify (15) by adding to it the left hand sides of equations (16) each multiplied by a still undetermined factor λ_k. We obtain

$$df = \frac{\partial f}{\partial x_i} dx_i + \lambda_k \frac{\partial g_k}{\partial x_i} dx_i = (\frac{\partial f}{\partial x_i} + \lambda_k \frac{\partial g_k}{\partial x_i}) dx_i. \tag{17}$$

The value of the differential df has obviously not been changed by adding zeros. The elimination of the last differentials $dx_{n-m+1}, \cdots, dx_n$ can, however, now be passed by selecting the λ factors so that at the stationary point

$$\frac{\partial f}{\partial x_i} + \lambda_k \frac{\partial g_k}{\partial x_i} = 0, \quad i = n-m+1,\cdots,n \tag{18}$$

which means that the contributions in (17) due to differentials $dx_{n-m+1}, \cdots, dx_n$ vanish be their values whatsoever. This leaves expression (17) in the form

$$df = \sum_{i=1}^{m-n} (\frac{\partial f}{\partial x_i} + \lambda_k \frac{\partial g_k}{\partial x_i}) dx_i. \tag{19}$$

All the dx remaining in (19) are now free. Thus each term multiplying a dx must be separately zero for a stationary point:

$$\frac{\partial f}{\partial x_i} + \lambda_k \frac{\partial g_k}{\partial x_i} = 0, \quad i = 1,\cdots,n-m. \tag{20}$$

Equations (18) and (20) read combined

$$\frac{\partial f}{\partial x_i} + \lambda_k \frac{\partial g_k}{\partial x_i} = 0, \quad i = 1,2,\cdots,n. \tag{21}$$

Thus to find the possible stationary point we have to solve $x_1, x_2, \cdots, x_n$ and the *Lagrange multipliers* (Lagrangen kertoja) $\lambda_1, \lambda_2, \cdots, \lambda_m$ from the sets (21) and (13) of $n+m$ equations.

We can achieve a compact formulation of the Lagrange multiplier method by defining a new modified function $f_L(x_1, x_2, \cdots, x_n; \lambda_1, \lambda_2, \cdots, \lambda_m)$ — often called *Lagrange's function* (Lagrangen funktio) — defined by

$$\boxed{f_L = f + \lambda_k g_k .} \tag{22}$$

Considering the x's and λ's as independent variables, the stationarity conditions (4) give directly the governing equations

$$\begin{aligned} \frac{\partial f_L}{\partial x_i} &\equiv \boxed{\frac{\partial f}{\partial x_i} + \lambda_k \frac{\partial g_k}{\partial x_i} = 0,} \qquad i = 1,2,\cdots,n \\ \frac{\partial f_L}{\partial \lambda_k} &\equiv \boxed{g_k = 0,} \qquad k = 1,2,\cdots,m. \end{aligned} \tag{23}$$

Because equations (23) are exactly the same as equations (21) and (13) we have obtained the following interpretation of the Lagrange multiplier method: *The original constrained problem can be replaced by a free problem by defining a modified function f_L whose stationary value is achieved at the same point as with the original function f.* The stationary value of f_L is the same as that of f because g_k vanish at the stationary point.

Example D.3. Let us consider the stationarity of the function

$$f(x,y) = (x-2)^2 + (y-1)^2 \tag{a}$$

treated already in Example D.1 now, however, under the constraint

$$g(x,y) \equiv x + y - 5 = 0. \tag{b}$$

This is a problem with $n = 2$ and $m = 1$. We consider this first using the elimination method. If, say, x is taken as the independent variable, we have

$$y = 5 - x \tag{c}$$

and substituting this into (a) gives the function

$$f_F(x) = (x-2)^2 + (5-x-1)^2 = (x-2)^2 + (4-x)^2. \tag{d}$$

From the stationarity condition

$$\frac{df_F}{dx} \equiv 2x - 4 + 2x - 8 = 4x - 12 = 0 \tag{e}$$

we obtain the solution $x = 3$ and (c) gives further $y = 2$ The corresponding stationary value from (a) or (d) is 2. Figures (a) and (b) illustrate the situation. In Figure (a), the contour curves of f are circles with the centre at P:(2,1).The higher the value of the radius, the higher the value of f. The constraint is represented by the line L and the stationary point is on it at R:(3,2). Figure (b) shows the same in three dimensions. The stationary point must be searched for on the curve obtained as the intersection of the graph of f and the plane represented by the constraint equation in the xyf-space.

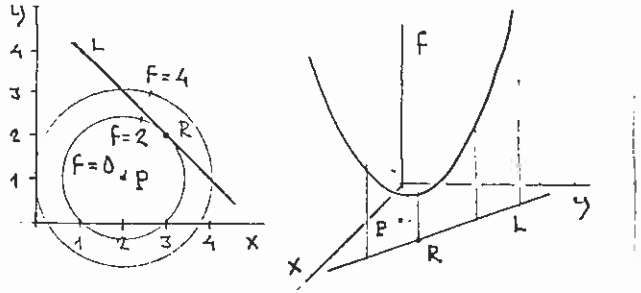


Figure (a) **Figure (b)**

Second, the Lagrange multiplier method is applied. The modified function (22) is here

$$f_L(x,y;\lambda) = f(x,y) + \lambda g(x,y) = (x-2)^2 + (y-1)^2 + \lambda(x+y-5). \tag{f}$$

Stationarity conditions (23) are

$$\begin{aligned} \frac{\partial f_L}{\partial x} &\equiv 2x - 4 + \lambda = 0, \\ \frac{\partial f_L}{\partial y} &\equiv 2y - 2 + \lambda = 0, \\ \frac{\partial f_L}{\partial \lambda} &\equiv x + y - 5 = 0. \end{aligned} \tag{g}$$

The equations are here linear and the solution is

$$x = 3, \quad y = 2, \quad \lambda = -2. \tag{h}$$

Thus the same stationary point was achieved as in the elimination method.

Example D.4. We consider the elastic system consisting of two joints (particles) and three springs in Figure (a). We determine the displacements u_1 and u_2 of the joints due to the external constant force F by the principle of stationary potential energy. The spring constant K of the middle spring is large compared to the spring constants k of the other two springs.

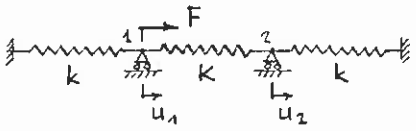


Figure (a)

The expression for the potential energy of the system is

$$V(u_1,u_2) = \frac{1}{2}ku_1^2 + \frac{1}{2}K(u_2-u_1)^2 + \frac{1}{2}ku_2^2 - Fu_1. \tag{a}$$

The stationarity conditions give the equilibrium equations

$$\frac{\partial V}{\partial u_1} \equiv ku_1 - K(u_2 - u_1) - F = 0,$$
$$\frac{\partial V}{\partial u_2} \equiv K(u_2 - u_1) + ku_2 = 0 \qquad \text{(b)}$$

or in matrix notation

$$\begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}. \qquad \text{(c)}$$

If a numerical solution is attempted, inaccurate results are obtained when $K >> k$ because the coefficient matrix is then nearly singular. (The coefficient matrix contains two nearly linearly dependent rows.) Numerical problems can be avoided by considering K as infinitely large; that is, the extension $u_2 - u_1$ as zero in which case the potential energy expression becomes

$$V(u_1, u_2) = \frac{1}{2}ku_1^2 + \frac{1}{2}ku_2^2 - Fu_1. \qquad \text{(d)}$$

(Because the string force S_2 in the middle string obviously remains bounded as $K \to \infty$, substitution of the term $u_2 - u_1 = S_2 / K$ into (a) gives (d).) We, however, now have a constrained stationary problem with the constraint

$$g(u_1, u_2) \equiv u_2 - u_1 = 0. \qquad \text{(e)}$$

and S_2 becomes a constraint force (See Section 4.1.3). As in the previous example, it is here extremely easy to employ the elimination method to produce, say, the expression

$$V_F(u_1) = ku_1^2 - Fu_1, \qquad \text{(f)}$$

the stationarity condition

$$\frac{dV_F}{du_1} \equiv 2ku_1 - F = 0 \qquad \text{(g)}$$

and the solution

$$u_1 = \frac{F}{2k}, \quad u_2 = \frac{F}{2k}. \qquad \text{(h)}$$

Using the Lagrange multiplier method we write the modified function

$$V_L(u_1, u_2; \lambda) = V(u_1, u_2) + \lambda g(u_1, u_2) = \frac{1}{2}ku_1^2 + \frac{1}{2}ku_2^2 - Fu_1 + \lambda(u_2 - u_1). \qquad \text{(i)}$$

The stationarity conditions are

$$\frac{\partial V_L}{\partial u_1} \equiv ku_1 - F - \lambda = 0,$$
$$\frac{\partial V_L}{\partial u_2} \equiv ku_2 + \lambda = 0, \qquad \text{(j)}$$
$$\frac{\partial V_L}{\partial \lambda} \equiv u_2 - u_1 = 0$$

or

$$\begin{bmatrix} k & 0 & -1 \\ 0 & k & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix} \qquad \text{(k)}$$

and the solution is

$$u_1 = \frac{F}{2k}, \quad u_2 = \frac{F}{2k}, \quad \lambda = -\frac{F}{2}. \qquad \text{(l)}$$

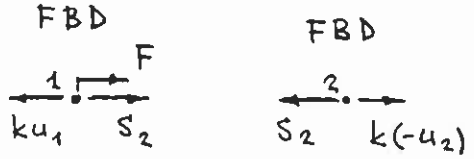


Figure (b)

By comparing the two first equations (j) with the equilibrium equations

$$\rightarrow \; -ku_1 + S_2 + F = 0,$$
$$\rightarrow \; -S_2 + k(-u_2) = 0 \qquad \text{(m)}$$

obtained from the free body diagrams of Figure (b), we see that the Lagrange multiplier has here the physical interpretation $\lambda = S_2$. It is quite common that the Lagrange multipliers associated with kinematical constraints have this kind of constraint force interpretations.

The previous examples have concerned the rather usual case where the function to be made stationary is at highest quadratic in the variables, that is, using the matrix notations introduced in Example D.2:

$$f(\{x\}) = \frac{1}{2}\{x\}^T[A]\{x\} + \{x\}^T\{a\} + c \qquad (24)$$

and where the constraints are linear:

$$\underset{m\times 1}{\{G\}} \equiv \underset{m\times n}{[G]}\underset{n\times 1}{\{x\}} + \underset{m\times 1}{\{d\}} = \underset{m\times 1}{\{0\}}. \qquad (25)$$

We develop some further formulas in matrix notation. The Lagrange function is thus

$$f_L(\{x\};\{\lambda\}) = \frac{1}{2}\{x\}^T[A]\{x\} + \{x\}^T\{a\} + c + \{\lambda\}^T([G]\{x\} + \{d\}) \qquad (26)$$

where $\{\lambda\}$ is a $m \times 1$ column vector consisting of the Lagrange multipliers.

Stationarity conditions (23) obtain the forms (Formulas (l) and (m) of example D.2 have been made use of. It should be noted that the scalar $\{\lambda\}^T[G]\{x\}$ is in the transposed form $\{x\}^T[G]^T\{\lambda\}$ where $[G]^T\{\lambda\}$ is a column vector.)

$$\frac{\partial f_L}{\partial \{x\}} \equiv [A]\{x\} + [G]^T\{\lambda\} + \{a\} = \{0\},$$
$$\frac{\partial f_L}{\partial \{\lambda\}} \equiv [G]\{x\} + \{d\} = \{0\} \tag{27}$$

or

$$\begin{bmatrix} [A] & [G]^T \\ [G] & [0] \end{bmatrix} \begin{Bmatrix} \{x\} \\ \{\lambda\} \end{Bmatrix} = - \begin{Bmatrix} \{a\} \\ \{d\} \end{Bmatrix}. \tag{28}$$

This is a linear *symmetric* system of equations. As zeros appear on the main diagonal due to the fact that the Lagrange multipliers are not present in the constraint equations, the system coefficient cannot be definite. It should be mentioned that the unknowns need of course not be listed in the order first the x, then the λ; this is just convenient in matrix notation.

D.1.3 Penalty method

Penalty functions (sakkofunktio) are used widely in optimization theory and practice to transform constrained optimization problems to unconstrained ones. This procedure to take into account the constraints is called the *penalty function method* or shortly the *penalty method* (sakkofunktiomenettely, sakkomenettely). This is thus an alternative to the Lagrange multiplier method.

In the penalty method, the problem of Section D.1.2 to make a function $f(x_1, x_2, \cdots, x_n)$ stationary under the constraints

$$g_k(x_1, x_2, \cdots, x_n) = 0, \qquad k = 1, 2, \cdots, m \tag{29}$$

is transformed to the problem of the stationarity of a modified function

$$\boxed{f_P(x_1, x_2, \cdots, x_n) = f + \frac{1}{2}(\alpha_1 g_1^2 + \alpha_2 g_2^2 + \cdots + \alpha_m g_m^2) = f + \frac{1}{2}\alpha_k g_k^2} \tag{30}$$

with no constraints. The term $1/2 \cdot \alpha_k g_k^2$ is called the penalty term or penalty function. (In optimization theory, e.g. Bazaraa, Sherali and Shetty (1993), many types of penalty functions can be used but the one given above is usual in connection with equality constraints.) The factor $1/2$ is unessential and is included just for notational convenience. The quantities α are *given* positive — when minimizing a function — multipliers which are called *weight parameters* or *penalty parameters* or *penalty numbers* (painokerroin, sakkokerroin). Their dimensions must of course be selected so that f_P becomes dimensionally

homogeneous. The stationary point is obtained as the solution of the system of equations

$$\frac{\partial f_P}{\partial x_i} \equiv \boxed{\frac{\partial f}{\partial x_i} + \sum_{k=1}^{m} \alpha_k g_k \frac{\partial g_k}{\partial x_i} = 0,} \qquad i = 1, 2, \cdots, n. \tag{31}$$

The purpose of the penalty term is to force the solution near the point where the constraints (29) are satisfied. The terminology used obtains an illustrative meaning if we consider the objective function f as a cost which is to be minimized so that the conditions or laws set by the constraint equations are simultaneously obeyed. The modified objective function f_P includes as additional costs the fines or penalties which are to be paid if the laws are violated. Each penalty is proportional to the square of the violation or crime and depends further on the weight used. Although the variables x are kept free, the stationary point of f_P finds it way near the exact stationary point when the weights are large because the slightest error in satisfying a constraint tends to increase strongly the value of the cost function f_P.

The penalty method has the advantage over the Lagrange multiplier method that the number of unknows to be determined stays at n; in latter method the number of unknowns was $n + m$. On the other hand, the penalty method is approximate. If the weights are too small the constraints are not well satisfied and the result is inaccurate. When the values of the weights are increased the results become in principle more accurate but the system of equations (31) get more and more ill-conditioned and finally no numerical solution can be achieved. When this happens depends in addition on the properties of functions f and g_k mainly on the word length used in the computer. The ill-conditioning is understood qualitatively by considering the set (31). As the weights get large, the terms $\partial f_P / \partial x_i$ disappear compared with the penalty terms and there remains equations the left-hand sides of which are linear combinations of the m expressions $g_1, g_2, \cdots, g_m$. Because $n > m$ equations are generated from them, the equations obviously cannot any more be linearly independent.

Example D.5. We consider the same elastic spring system as in example D.4 (Figure (a)).

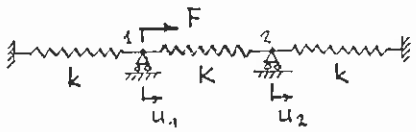


Figure (a)

We take again the case $K \to \infty$. The potential energy is thus

$$V(u_1,u_2)=\frac{1}{2}ku_1^2+\frac{1}{2}ku_2^2-Fu_1 \tag{a}$$

and the constraint is

$$g(u_1,u_2)\equiv u_2-u_1=0. \tag{b}$$

In example D.4 the Lagrange multiplier method was used and the solution

$$u_1=\frac{1}{2}\frac{F}{k},\quad u_2=\frac{1}{2}\frac{F}{k},\quad \lambda=-\frac{1}{2}F \tag{c}$$

was arrived at. We now use the penalty method. The modified function is

$$\begin{aligned}V_P(u_1,u_2)&=V(u_1,u_2)+\frac{1}{2}\alpha g^2(u_1,u_2)\\&=\frac{1}{2}ku_1^2+\frac{1}{2}ku_2^2-Fu_1+\frac{1}{2}\alpha(u_2-u_1)^2.\end{aligned} \tag{d}$$

The stationarity conditions are

$$\begin{aligned}&\frac{\partial V_P}{\partial u_1}\equiv ku_1+ku_2-F-\alpha(u_2-u_1)=0,\\&\frac{\partial V_P}{\partial u_2}\equiv ku_2+\alpha(u_2-u_1)=0\end{aligned} \tag{e}$$

or

$$\begin{bmatrix}k+\alpha & -\alpha\\ -\alpha & k+\alpha\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=\begin{Bmatrix}F\\0\end{Bmatrix} \tag{f}$$

and the analytical solution is

$$u_1=\frac{1+\alpha/k}{1+2\alpha/k}\frac{F}{k},\quad u_2=\frac{\alpha/k}{1+2\alpha/k}\frac{F}{k}. \tag{g}$$

When the penalty number $\alpha\to\infty$ the solution is seen to approach the exact solution. In practice, we naturally have to solve the equations numerically with a finite word length. As an example, in the case $\alpha=1000k$ the system of equations (f) becomes after division by k

$$\begin{bmatrix}1001 & -1000\\ -1000 & 1001\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=\frac{F}{k}\begin{Bmatrix}1\\0\end{Bmatrix}. \tag{h}$$

The corresponding analytical solution is

$$u_1=0{,}500249\cdots\frac{F}{k},\quad u_2=0{,}499750\cdots\frac{F}{k} \tag{i}$$

is already quite near the exact solution. A numerical solution of (f) with five significant digits gives

$$u_1=0{,}50050\frac{F}{k},\quad u_2=0{,}50000\frac{F}{k}, \tag{j}$$

with four significant digits still

$$u_1=0{,}5005\frac{F}{k},\quad u_2=0{,}5000\frac{F}{k}, \tag{k}$$

but with three significant digits the coefficient matrix becomes singular or

$$u_1=\infty,\quad u_2=\infty.$$

Computers use say 6 to 14 significant digits but a similar behaviour is always arrived at when the penalty parameter values are large enough.

The ill-conditioning of system (f) is here easy to see; when $\alpha\gg k$, the rows of the coefficient matrix are identical except for the sign.

The penalty number has in this example an obvious physical meaning. Comparing expression (d) with the expression (a) of Example D.4 we see that the inclusion of the penalty term means that we are analysing the spring system by the conventional principle of stationary potential energy and assume the middle spring to have a spring constant α.

We produce some matrix formulas corresponding to (24) to (28) now with the penalty formulation. With the function

$$f(\{x\})=\frac{1}{2}\{x\}^T[A]\{x\}+\{x\}^T\{a\}+c \tag{32}$$

and the constraints

$$\underset{m\times 1}{\{G\}}\equiv\underset{m\times n}{[G]}\underset{n\times 1}{\{x\}}+\underset{m\times 1}{\{d\}}=\underset{m\times 1}{\{0\}}, \tag{33}$$

the modified function is

$$\begin{aligned}f_P(\{x\})=&\frac{1}{2}\{x\}^T[A]\{x\}+\{x\}^T\{a\}+c+\\&+(([G]\{x\}+\{d\})^T[\alpha]([G]\{x\}+\{d\}).\end{aligned} \tag{34}$$

Here $[\alpha]$ is a $m\times m$ diagonal matrix with diagonal elements $\alpha_1,\alpha_2,\cdots,\alpha_m$.

Stationary conditions (31) obtain the form

$$\frac{\partial f_P}{\partial\{x\}}\equiv[A_P]\{x\}+\{a_P\}=\{0\} \tag{35}$$

where

$$\begin{aligned}&[A_P]=[A]+[G]^T[\alpha][G],\\&\{a_P\}=\{a\}+[G]^T[\alpha]\{d\}\end{aligned} \tag{36}$$

The coefficient matrix $[A_P]$ is symmetric and positive definite at least when $[A]$ is positive definite. (The diagonal matrix $[\alpha]$ is positive definite if each penalty

parameter $\alpha_k > 0$ and the matrix $[G]^T[\alpha][G]$ is then positive definite. This is seen as follows. The quadratic form $\{y\}^T[\alpha]\{y\} \geq 0$ and $= 0$ only when $\{y\} = \{0\}$. We denote $\{y\} = [G]\{x\}$ after which the previous quadratic form is $\{x\}^T[G]^T[\alpha][G]\{x\}$. It is still ≥ 0 and $= 0$ only for those values of $\{x\}$ which satisfy the system of equations $[G]\{x\} = \{0\}$. This has always (because $m < n$) also solutions $\{x\}$ differing from zero so that the matrix $[G]^T[\alpha][G]$ is only positive semidefinite. Finally, it is easy to see that the sum of a positive definite and a positive semidefinite matrix is a positive definite matrix.) Often the penalty terms make the $[A_P]$-matrix positive definite even when the original matrix $[A]$ is only positive semidefinite.

A difficulty in applying the penalty method is the selection of the penalty parameter values. Often some numerical experiments are needed for this.

Remark D.2. A constraint may contain only few of the variables which are further "geometrically near each other". We then call the constraint *local* (lokaali) and otherwise *global* (globaali). Global constraints are not very convenient to deal with by the penalty method as they tend to fill the coefficient matrix and destroy its possibly banded and sparse structure. For example, discretized forms of isoperimetric constraints (Section D.2.6) produce global constraints. □

D.1.4 Perturbed Lagrange multiplier method

We can gain further understanding of the penalty method by considering a modified function

$$\begin{aligned}\hat{f}_L(x_1, x_2, \cdots; \lambda_1, \lambda_2, \cdots) &= f_L - \frac{1}{2}\left(\frac{1}{\alpha_1}\lambda_1^2 + \frac{1}{\alpha_2}\lambda_2^2 + \cdots + \frac{1}{\alpha_m}\lambda_m^2\right) \\ &= f_L - \frac{1}{2}\frac{1}{\alpha_k}\lambda_k^2 = f + \lambda_k g_k - \frac{1}{2}\frac{1}{\alpha_k}\lambda_k^2 .\end{aligned} \tag{37}$$

This is a *perturbed* (häiritty) Lagrange function (22). The factors α_k are here given quantities. When each $|\alpha_k| \to \infty$ the perturbation term $\lambda_k^2/(2\alpha_k) \to 0$ and $\hat{f}_L \to f_L$. The stationary point of the perturbed function is obtained as the solution of the set

$$\begin{aligned}\frac{\partial \hat{f}_L}{\partial x_i} &\equiv \frac{\partial f}{\partial x_i} + \lambda_k \frac{\partial g_k}{\partial x_i} = 0, \\ \frac{\partial \hat{f}_L}{\partial \lambda_k} &\equiv g_k - \frac{1}{\alpha_{(k)}}\lambda_{(k)} = 0.\end{aligned} \tag{38}$$

This system and its solution is seen to approach the original system (23) and its solution as each $|\alpha_k| \to \infty$. But when the factors α_k are kept finite, approximate values for the Lagrange multipliers are found directly from the latter equations (38):

$$\lambda_k^{(\alpha)} = \alpha_{(k)} g_{(k)} . \tag{39}$$

The superscript (α) is used to indicate that this is an approximate value depending on the values of the factors α. Substituting (39) into (37) gives

$$\begin{aligned}\hat{f}_L^{(\alpha)}(x_1, x_2, \cdots, x_n) &\equiv \hat{f}_L(x_1, x_2, \cdots; \lambda_1^{(\alpha)}, \lambda_2^{(\alpha)}, \cdots) \\ &= f + \sum_{k=1}^{m}(\alpha_k g_k) g_k - \frac{1}{2}\sum_{k=1}^{m}\frac{1}{\alpha_k}(\alpha_k g_k)^2 = f + \frac{1}{2}\alpha_k g_k^2 ,\end{aligned} \tag{40}$$

that is, we have obtained expression (30). *Thus the use of the penalty method is equivalent to the use of the perturbed Lagrange multiplier method.* The factors α_k are the penalty parameters.

This interpretation gives immediately two important results which have not been apparent before. First, earlier the use of the penalty method was justified qualitatively in connection of a minimization (maximization) problem where the penalty terms increase (decrease) the value of a function when the weights are selected positive (negative) if the constraints are violated. Now we realize that the penalty method can be used equally well in connection with the determination of a stationary point as in the determination of an extremum point, essential is that the absolute values of the weights are large. From the computational point, of view, however, it is advantegeous still to keep the weights positive (negative) in a minimization (maximization) problem so that the coefficient matrix of the system equations would remain or possible would change positive (negative) definite.

The second important result is that we obtain from (39) approximate information on the values of the Lagrange multipliers. They often have important physical meaning which can be made use of.

In the case given by formulas (24) and (25) the perturbed Lagrange multiplier method gives instead of (28) the equation set

$$\begin{bmatrix}[A] & [G]^T \\ [G] & -[1/\alpha]\end{bmatrix}\begin{Bmatrix}\{x\} \\ \{\lambda\}\end{Bmatrix} = -\begin{Bmatrix}\{a\} \\ \{d\}\end{Bmatrix} \tag{41}$$

The matrix $[1/\alpha]$ is a $m \times m$ diagonal matrix with diagonal elements $1/\alpha_1, 1/\alpha_2, \cdots, 1/\alpha_m$.

D.1.5 Lagrange multiplier / penalty method

As before, we consider the problem to make a function $f(x_1, x_2, \cdots, x_n)$ stationary under the constraints

$$g_k(x_1, x_2, \cdots, x_n) = 0, \qquad k = 1, 2, \cdots, m. \tag{42}$$

An efficient iterative procedure can be devised which is free of most of the disadvantages of the Lagrange multiplier method and of the penalty method. We form a modified function

$$\boxed{f_{\mathrm{LP}}(x_1, x_2, \cdots, x_n) = f + \bar{\lambda}_k g_k + \frac{1}{2}\alpha_k g_k^2.} \tag{43}$$

This looks like *the Lagrange function (22) modified by the penalty terms.* However, the quantities $\bar{\lambda}_k$ are approximations to the Lagrange multipliers which are improved iteratively and which are considered fixed and given in (43). The overbar is used to remind of this. The stationarity condition of f_{LP} gives the equations

$$\frac{\partial f_{\mathrm{LP}}}{\partial x_i} \equiv \boxed{\frac{\partial f}{\partial x_i} + \sum_{k=1}^{m} \bar{\lambda}_k \frac{\partial g_k}{\partial x_i} + \sum_{k=1}^{m} \alpha_k g_k \frac{\partial g_k}{\partial x_i} = 0.} \tag{44}$$

The solution gives an approximate stationary point $\{\tilde{x}\}$. An update for the Lagrange multipliers is obtained by the formula

$$\boxed{\bar{\lambda}_k^{\mathrm{new}} = \bar{\lambda}_k + \alpha_{(k)} g_{(k)}(\{\tilde{x}\}).} \tag{45}$$

This formula is motivated as follows. The conventional Lagrange multiplier method produces the equations (the upper set of (23))

$$\frac{\partial f}{\partial x_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial x_i} = 0. \tag{46}$$

Subtracting (44) from (46) leads approximately to an equation

$$\sum_{k=1}^{m} (\lambda_k - \bar{\lambda}_k - \alpha_k g_k) \frac{\partial g_k}{\partial x_i} = 0 \tag{47}$$

which suggests the update (45). It is to be noted that (47) is not exact because the stationary points for which the quantities in (44) and (46) are evaluated generally differ. Exact justfication is presented in Bazaraa, Sherali and Shetty (1993, p. 382). This reference also contains a guide for a relevant algorithm which we roughly quote:

Initialization: Select some initial Lagrange multiplier values $\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_m$ and some values of the penalty parameters $\alpha_1, \alpha_2, \cdots, \alpha_m$. Let $\{x\}^{(0)}$ be a null column vector and denote $\mathrm{VIOL}(\{x\}^{(0)}) = \infty$. Define $\mathrm{VIOL}(\{x\}) =$ maximum $\{|g_k|; k = 1, 2, \cdots, m\}$, this being a measure of constraint violations. Put $n = 1$ and proceed to the inner loop of the algorithm.

Inner loop: Find the stationary point of f_{LP} and denote the solution $\{x\}^{(n)}$. If $\mathrm{VIOL}(\{x\}^{(n)}) = 0$, then stop with $\{x\}^{(n)}$ as the final solution. (Practically, one would terminate if $\mathrm{VIOL}(\{x\}^{(n)})$ is less than some tolerance $\varepsilon > 0$.) Otherwise, if $\mathrm{VIOL}(\{x\}^{(n)}) \le \mathrm{VIOL}(\{x\}^{(n-1)})/4$, proceed to the outer loop. On the other hand, if $\mathrm{VIOL}(\{x\}^{(n)}) > \mathrm{VIOL}(\{x\}^{(n-1)})/4$, then, for each constraint $k = 1, 2, \cdots, m$ for which $|g_k| > \mathrm{VIOL}(\{x\}^{(n-1)})/4$, replace the corresponding penalty parameter α_k by $10\alpha_k$ and repeat the inner loop step.

Outer loop: Replace $\{\bar{\lambda}\}$ by $\{\bar{\lambda}\}^{\mathrm{new}}$ where

$$\bar{\lambda}_k^{\mathrm{new}} = \bar{\lambda}_k + \alpha_{(k)} g_{(k)}(\{x\}^{(n)}), \qquad k = 1, 2, \cdots, m. \tag{48}$$

Increment n by 1, and return to the inner loop.

Remark D.3. The method described above is usually called in the literature the *augmented Lagrangian penalty method* (täydennetty Lagrangen sakkomenettely). As the procedure is some kind of mixture of the Lagrange multiplier method and the penalty method, we call it here the *Lagrange multiplier / penalty method* (Lagrangen kertoja- / sakkomenettely). The method has the merit over the Lagrange multiplier method that the number of unknowns remains at n and the merit over the pure penalty method that ill-conditioning is much reduced as the penalty parameters can be kept at reasonable values. □

Remark D.4. We can look at the Lagrange multiplier / penalty method also as follows. If we knew the exact values of the Lagrange multipliers beforehand, we could solve for the basic unknows x directly from the upper set (23). As we, however, do not know the values of the Lagrange multipliers, we introduce them iteratively into the formulation. In the limit, when the $\bar{\lambda}$'s approch the exact λ's, the actual values of the penalty parameters are not important as we in fact are solving the upper set (23) since the g's in (44) tend to zero. In mechanics, when the constraints are kinematical, the Lagrange multipliers represent constraint forces. We are thus gradually introducing the constraint forces and the values of the artificial springs (penalty parameters) have finally no effect as the constraints are satisfied. □

Example D.6. Once more the spring system of Examples D.4 and D.5 is considered, now, however, using the Lagrange multiplier / penalty method.

The modified potential energy expression is

$$V_{LP}(u_1,u_2)=\frac{1}{2}ku_1^2+\frac{1}{2}ku_2^2-Fu_1+\bar{\lambda}(u_2-u_1)+\frac{1}{2}\alpha(u_2-u_1)^2 \qquad (a)$$

and the stationarity conditions give the equations

$$\begin{aligned}&\frac{\partial V_{LP}}{\partial u_1}\equiv ku_1+ku_2-F-\bar{\lambda}-\alpha(u_2-u_1)=0,\\&\frac{\partial V_{LP}}{\partial u_2}\equiv ku_2+\bar{\lambda}+\alpha(u_2-u_1)=0\end{aligned} \qquad (b)$$

or

$$\begin{bmatrix}k+\alpha & -\alpha\\ -\alpha & k+\alpha\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=\begin{Bmatrix}F+\bar{\lambda}\\-\bar{\lambda}\end{Bmatrix}. \qquad (c)$$

We perform the manipulations in this simple case in analytical form. The solution is

$$u_1=\frac{1+\alpha/k}{1+2\alpha/k}\frac{F}{k}+\frac{1}{1+2\alpha/k}\frac{\bar{\lambda}}{k},\quad u_2=\frac{\alpha/k}{1+2\alpha/k}\frac{F}{k}-\frac{1}{1+2\alpha/k}\frac{\bar{\lambda}}{k}. \qquad (d)$$

Formula (45) gives the Lagrange multiplier update:

$$\bar{\lambda}^{new}=\bar{\lambda}+\alpha(u_2-u_1)=\bar{\lambda}-\frac{\alpha}{1+2\alpha/k}\frac{F}{k}-\frac{2\alpha}{1+2\alpha/k}\frac{\bar{\lambda}}{k}. \qquad (e)$$

We take on purpose a rather mild $\alpha=10k$ and start with $\bar{\lambda}=0$. We do not follow completely the presented algorithm and disregard the checking of the suitability the penalty parameter value and keep it simply constant. Expressions (d) and (e) obtain the detailed forms

$$u_1=\frac{11}{21}\frac{F}{k}+\frac{1}{21}\frac{\bar{\lambda}}{k},\quad u_2=\frac{10}{21}\frac{F}{k}-\frac{1}{21}\frac{\bar{\lambda}}{k} \qquad (f)$$

and

$$\bar{\lambda}^{new}=\frac{1}{21}\bar{\lambda}-\frac{10}{21}F. \qquad (g)$$

The first solution is

$$\begin{aligned}&u_1^{(1)}=\frac{11}{21}\frac{F}{k}\approx 0{,}524\frac{F}{k},\quad u_2^{(1)}=\frac{10}{21}\frac{F}{k}\approx 0{,}476\frac{F}{k},\\&\bar{\lambda}^{new}=-\frac{10}{21}F=-0{,}476F.\end{aligned} \qquad (h)$$

The second solution is

$$\begin{aligned}&u_1^{(2)}=\frac{221}{441}\frac{F}{k}\approx 0{,}501\frac{F}{k},\quad u_2^{(2)}=\frac{220}{441}\frac{F}{k}\approx 0{,}499\frac{F}{k},\\&\bar{\lambda}^{new}=-\frac{220}{441}F\approx -0{,}499F.\end{aligned} \qquad (i)$$

The third solution is

$$\begin{aligned}&u_1^{(3)}=\frac{4631}{9261}\frac{F}{k}\approx 0{,}500'05\frac{F}{k},\quad u_2^{(3)}=\frac{4630}{9261}\frac{F}{k}\approx 0{,}499'95\frac{F}{k},\\&\bar{\lambda}^{new}=-\frac{4630}{9261}F=-0{,}499'95F.\end{aligned} \qquad (j)$$

The convergence rate is seen to be quite satisfactory.

D.2 STATIONARY VALUE OF A FUNCTIONAL

D.2.1 Functional

Functional (funktionaali) is an operation associating a real number Π for each member ϕ of a set S the members consisting of functions of agreed kind, that is, it is a mapping $\Pi : S \rightarrow R$.

Usually the mapping is effected via a definite integral. An example:

$$\Pi(\phi) = \int_a^b \sqrt{1 + (\frac{d\phi}{dx})^2}\, dx. \tag{1}$$

Here S is the set of functions $\phi(x)$ defined on the x-axis interval $[a,b]$. The functions must satisfy the conditions $\phi(a) = \alpha$ and $\phi(b) = \beta$ were α and β are given and to be so smooth that the integral can be evaluated.

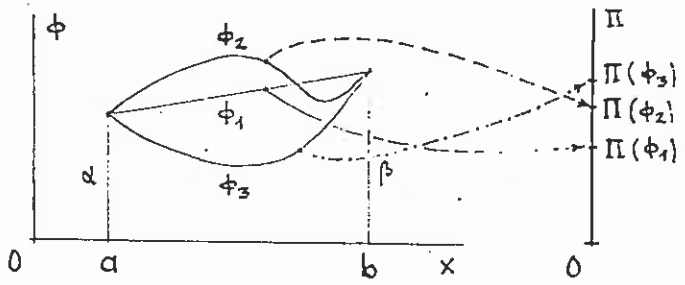


Figure D.2 Mappings of three functions.

The integral clearly represents the length of a curve passing through the points (a,α) and (b,β). Figure D.2 shows schematically the mapping of three functions $\phi(x)$ to real numbers Π. Because of the geometric interpretation, it is clear that Π obtains the minimum value when $\phi(x)$ describes a straight line.

Functional (1) is a special case of a more general situation

$$\Pi(\phi) = \int_a^b f(x, \phi, \frac{d\phi}{dx})\, dx \tag{2}$$

where the integrand depends on the independent variable x, on the *argument function* (argumenttifunktio) $\phi(x)$ and on its derivative $d\phi/dx$. In the example case (1) x and ϕ are missing.

A case, where all the above variables are present, is

$$V(v) = \int_0^l [\frac{1}{2} S (\frac{dv}{dx})^2 - qv]\, dx. \tag{3}$$

This is the expression for the potential energy of a stretched string on the interval $[0,l]$ of the x-axis. S is the string force, assumed constant for small lateral displacement $v(x)$ of the string. The possible explicit dependence on x comes through the given function $q(x)$ representing the lateral loading per unit length of the string.

Remark D.5. A functional differs from an ordinary function in the respect that the value of the functional depends on the overall behaviour of the argument in a domain and not as is the case with an ordinary function on the pointwise value of the argument. Obviously to emphasize this difference, the argument function is often represented in the literature enclosed in brackets, say as $\Pi[\phi]$. □

A functional can contain higher order derivatives than the first, several argument functions and several independent variables and terms from the boundary of the domain.

When we pick a certain argument function, its derivatives can be evaluated and it can be fed in the functional expression to give as the output a certain number. The main task of *variational calculus* (variaatiolaskenta) is to determine that argument function giving the functional an extremum value. The necessary condition for this is that the functional obtains a *stationary value* (stationaarinen arvo). This means that for "small" changes of the argument function the changes of the functional are zero. The corresponding argument function is called the *stationary function* (stationaarinen funktio).

From the stationary condition so called *Euler equation(s)* or *Euler-Lagrange equation(s)* (Euler-Lagrangen yhtälö(t)) with their boundary conditions can be derived.

If we are able to find a functional, for which the stationary condition gives the differential equations (DE) and boundary conditions (BC) we want to solve, we have obtained a convenient way to perform a discretization of the problem, that is, the numerical solution can be based instead on DE and BC on the functional. This latter possibility offers considerable advantages.

For any differential equation set a corresponding functional unfortunately does not exist. A notably example are the Navier Stokes momentum equations. This fact constraints the usefulness of the variational formulation.

To study the changes of functionals and functions when the argument function experiences changes we need a way of thinking different from conventional differential calculus. Certain notations and calculation rules have been developed for this purpose which are considered next, Lanczos (1974).

D.2.2 Variational notation

We again consider just the case of one independent variable x and one argument function $\phi(x)$. The function can experience two kinds of changes. The infinitesimal change $\mathrm{d}\phi$ is due to the infinitesimal chance $\mathrm{d}x$ of x. The infinitesimal change $\delta\phi(x)$, however, is effected by the change of moving from the curve $\phi(x)$ to an infinitesimally near neighbouring curve $\phi^*(x) = \phi(x) + \delta\phi(x)$. In variational calculus this latter type of change is considered. The *variational symbol* (variaatiomerkki) δ is customarily used instead of the symbol d to tell the difference. The quantity $\delta\phi$ is called the *variation* (variaatio) — in more detail the first variation — of function ϕ and the new function ϕ^* is called the *varied function* or modified function or comparison function (varioitu funktio).

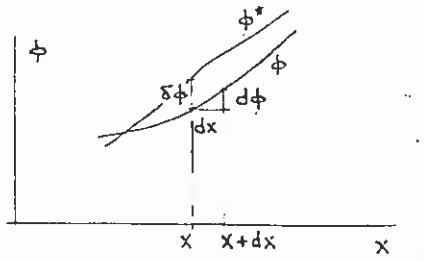


Figure D.3 Differential and variation.

If $\phi(x)$ is the actual function representing some quantity, the generation of a varied function means that some kind of mathematical thought experiment is performed to obtain comparison results: what would be the outcome if instead of ... ?

Remark D.6. In mechanics the most common example of the variation of a function is probably the concept of virtual displacement. It is usually defined to be an infinitely small imagined displacement which is thought to take place with time "freezed". This definition is seen to equivalent to the concept of the variation of a function. (x means now the time and ϕ is one space coordinate of a particle. Freezing time means that we move in vertical and not in horizontal direction in Figure D.3.) □

The expressions containing the argument function and the functional in particular obtain changes due the variation of the argument function. These changes are also called variations and the δ-symbol is again used. Table D.1 contains some calculation rules of variational calculus. These rules are valid also in the case of several independent variables. The formulas are quite analogous to the corresponding differentiation expressions.

Table D.1 Some rules of variational calculus

$\delta(f_1 + f_2) = \delta f_1 + \delta f_2$	(1)	Variation of sum
$\delta(kf) = k\delta f$	(2)	Transfer rule for a constant
$\delta(f_1 \cdot f_2) = \delta f_1 \cdot f_2 + f_1 \cdot \delta f_2$	(3)	Variation of product
$\delta(f^n) = nf^{n-1}\delta f$	(4)	Variation of power function
$\delta(\mathrm{d}\phi/\mathrm{d}x) = \mathrm{d}(\delta\phi)/\mathrm{d}x$	(5)	Variation of derivative
$\delta\int f\,\mathrm{d}x = \int \delta f\,\mathrm{d}x$	(6)	Variation of definite integral

The stationarity condition of a functional Π is represented in the form

$$\boxed{\delta\Pi = 0} \tag{4}$$

or the variation of the functional must be zero with respect to arbitrary admissible variation of the argument function(s). The content of condition (1) is called a *variational principle* (variaatioperiaate). Perhaps the most well known variational principle of mechanics is the principle of stationary (or minimum) potential energy: when the potential energy of a conservative system obtains a stationary value, the corresponding configuration of the system is the equilibrium position.

The argument functions competing in a functional must obey some smoothness conditions (so that the functional can be evaluated) and in general some boundary conditions. In this way defined argument functions are called *admissible functions* (luvallinen funktio). The boundary conditions demanded in advance from the admissible functions are called *essential boundary conditions* (oleellinen reunaehto). The stationarity condition gives as consequences the Euler-Lagrange differential equations and the so called *natural* or free or additional *boundary conditions* (luonnollinen reunaehto). These features are described in the following with some applications.

We now consider in more detail some of the concepts introduced above. For the mathematical manipulations to follow, it is useful to present the variation in the form

$$\delta\phi(x) = \varepsilon\eta(x) \tag{5}$$

which resembles formulas (D.1.5). Here $\eta(x)$ is an arbitrary smooth enough function and ε is a parameter which is allowed to approach zero. By considering the function $\eta(x)$ fixed during the derivation of a result, the

variation can be thought to be a function of only *one variable* ε. To see more clearly the similarities with Section D.1.1, we may consider Figure D.4.

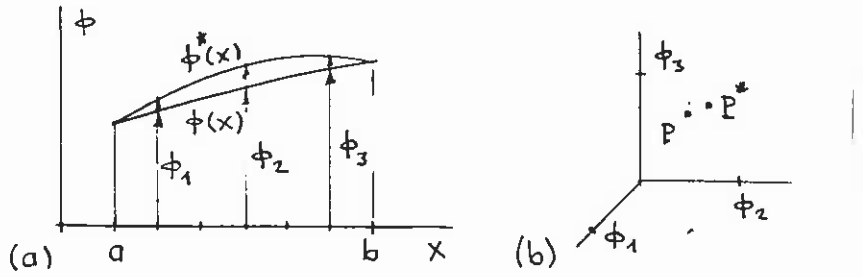


Figure D.4 (a) Function $\phi(x)$ and the varied function $\phi^*(x)$. **(b)** Function $\phi(x)$ and the varied function $\phi^*(x)$ roughly as points P and P^*.

A very crude discrete information of a function $\phi(x)$ is obtained say by dividing the interval $[a,b]$ into three equal length subintervals and by measuring the function values at their midpoints. The function is then represented by three values: ϕ_1,ϕ_2,ϕ_3. We now associate a cartesian coordinate for each subinterval and put the function values along these coordinate lines. A point P or vector $\boldsymbol{\phi}:(\phi_1,\phi_2,\phi_3)$ represents the function in this new three-dimensional space. Similarly, the modified function $\phi^*(x)=\phi(x)+\delta\phi(x)$ or $\phi^*(x)=\phi(x)+\varepsilon\eta(x)$ is also represented as a point P^* or vector $\boldsymbol{\phi}^*:(\phi_1^*,\phi_2^*,\phi_3^*)$ in the three dimensional space. The changes in the discrete values are

$$\delta\phi_1=\varepsilon\eta_1,\quad \delta\phi_2=\varepsilon\eta_2,\quad \delta\phi_3=\varepsilon\eta_3. \tag{6}$$

These formulas clearly resemble (D.1.5). (In Section D.1.1 we just denoted the changes by the d-symbol.) The selected shape of $\eta(x)$ somehow tells the direction to which we are moving from pont P. Function $\eta(x)$ can thus be considered to represent in some generalized sense the direction cosines of the change direction and ε again controls the magnitude of the change. If we increase the number of subintervals and proceed similarly, we cannot any more draw a picture of the space generated but we can still speak about the set of discrete numbers as a vector in a *finite dimensional space* (äärellisdimensioinen avaruus). Continuing without limit leads us to speak about a function as a vector in an *infinite dimensional space* (ääretöndimensionen avaruus) or shortly in a *function space* (funktioavaruus). In mathematics, a topic called *functional analysis* (funktionaalianalyysi), deals much with these latter concepts.

When a function ϕ is replaced by the varied function ϕ^*, all expressions containing ϕ also usually obtain some changes. As an example, let us consider the change of the function $f(x,\phi,\phi')$:

$$\Delta f=f(x,\phi^*,\phi^{*\prime})-f(x,\phi,\phi')=f^*-f \tag{7}$$

where

$$\begin{aligned}\phi^*&=\phi+\delta\phi=\phi+\varepsilon\eta,\\ \phi^{*\prime}&=\phi'+(\delta\phi)'=\phi'+\varepsilon\eta'\end{aligned} \tag{8}$$

and where the dash refers shortly to the derivative with respect to x. We proceed similarly as in Section D.1.1. When x, ϕ and η are kept fixed during the derivations to follow, Δf is a function of only one variable ε:

$$\Delta f=\Delta f(\varepsilon). \tag{9}$$

Function Δf is expanded into a Taylor series around the point $\varepsilon=0$:

$$\Delta f=\frac{\mathrm{d}\Delta f}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\varepsilon+\frac{1}{2}\frac{\mathrm{d}^2\Delta f}{\mathrm{d}\varepsilon^2}\Big|_{\varepsilon=0}\varepsilon^2+\cdots=\frac{\mathrm{d}f^*}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\varepsilon+\frac{1}{2}\frac{\mathrm{d}^2f^*}{\mathrm{d}\varepsilon^2}\Big|_{\varepsilon=0}\varepsilon^2+\cdots \tag{10}$$

The derivatives are calculated using the chain rule. Thus

$$\begin{aligned}\frac{\mathrm{d}f^*}{\mathrm{d}\varepsilon}&=\frac{\partial f(x,\phi^*,\phi^{*\prime})}{\partial\phi^*}\eta+\frac{\partial f(x,\phi^*,\phi^{*\prime})}{\partial\phi^{*\prime}}\eta',\\ \frac{\mathrm{d}^2f^*}{\mathrm{d}\varepsilon^2}&=\frac{\partial^2 f(x,\phi^*,\phi^{*\prime})}{\partial\phi^*\partial\phi^*}\eta\eta+\frac{\partial^2 f(x,\phi^*,\phi^{*\prime})}{\partial\phi^*\partial\phi^{*\prime}}\eta\eta'+\\ &\quad+\frac{\partial^2 f(x,\phi^*,\phi^{*\prime})}{\partial\phi^{*\prime}\partial\phi^*}\eta'\eta+\frac{\partial^2 f(x,\phi^*,\phi^{*\prime})}{\partial\phi^{*\prime}\partial\phi^{*\prime}}\eta'\eta'.\end{aligned} \tag{11}$$

When these are evaluated at $\varepsilon=0$, $\phi^*\to\phi$, $\phi^{*\prime}\to\phi'$, the stars can be removed and there remains

$$\begin{aligned}\frac{\mathrm{d}f^*}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}&=\frac{\partial f}{\partial\phi}\eta+\frac{\partial f}{\partial\phi'}\eta',\\ \frac{\mathrm{d}^2f^*}{\mathrm{d}\varepsilon^2}\Big|_{\varepsilon=0}&=\frac{\partial^2 f}{\partial\phi^2}\eta^2+2\frac{\partial^2 f}{\partial\phi\partial\phi'}\eta\eta'+\frac{\partial^2 f}{\partial\phi'^2}\eta'^2.\end{aligned} \tag{12}$$

The change in f can thus finally be represented in the form

$$\Delta f = \delta^1 f + \frac{1}{2}\delta^2 f + \cdots \tag{13}$$

where

$$\boxed{\delta f \equiv \frac{\mathrm{d}f^*}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\varepsilon} = \frac{\partial f}{\partial \phi}\varepsilon\eta + \frac{\partial f}{\partial \phi'}\varepsilon\eta' \tag{14}$$

and

$$\delta^2 f \equiv \frac{\mathrm{d}^2 f^*}{\mathrm{d}\varepsilon^2}\Big|_{\varepsilon=0}\varepsilon^2 = \frac{\partial^2 f}{\partial \phi^2}\varepsilon^2\eta^2 + 2\frac{\partial^2 f}{\partial \phi \partial \phi'}\varepsilon^2\eta\eta' + \frac{\partial^2 f}{\partial \phi'^2}\varepsilon^2\eta'^2. \tag{15}$$

The combination terms $\delta^1 f \equiv \delta f$ in Δf, containing ε in the first power is called the *first variation* (ensimmäinen variaatio) of f, the combination $\delta^2 f$, containing ε in the second power is called the *second variation* (toinen variaatio) of f, etc, Langhaar (1962). The formula in the box of (14) is an important general expression for the variation.

The first variation is seen to correspond the concept of total differential in the respect that a first order change is considered. The cause of the change only is different but the formulas obtained are analogous. In the following we only operate with the first variation and we then call it shortly variation.

When formula (14) is applied to the specific function $f = \phi'$, we obtain

$$\delta\phi' = 0 \cdot \varepsilon\eta + 1 \cdot \varepsilon\eta' = \varepsilon\eta' \tag{16a}$$

or

$$\delta\frac{\mathrm{d}\phi}{\mathrm{d}x} = \varepsilon\eta'. \tag{16b}$$

This is the variation of a derivative. On the other hand, the derivative of a variation

$$\frac{\mathrm{d}}{\mathrm{d}x}\delta\phi = \frac{\mathrm{d}}{\mathrm{d}x}(\varepsilon\eta) = \varepsilon\frac{\mathrm{d}\eta}{\mathrm{d}x} = \varepsilon\eta'. \tag{17}$$

Comparing (16) and (17), we have the important result

$$\boxed{\delta\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\delta\phi,} \qquad \delta\phi' = (\delta\phi)', \tag{18a, 18b}$$

or *the variation of a derivative is equal to the derivative of the variation*. In other words, variation and differentiation obey the commutative law. This result appears in Table D.1 as rule (5).

Using (5) and (16), expression (14) obtains the form

$$\delta f = \frac{\partial f}{\partial \phi}\delta\phi + \frac{\partial f}{\partial \phi'}\delta\phi'. \tag{19}$$

Making use of the total differential expression

$$\mathrm{d}f = \frac{\partial f}{\partial \phi}\mathrm{d}\phi + \frac{\partial f}{\partial \phi'}\mathrm{d}\phi' + \frac{\partial f}{\partial x}\mathrm{d}x \tag{20}$$

for a function $f(x, \phi, \phi')$, we would get analogously

$$\delta f = \frac{\partial f}{\partial \phi}\delta\phi + \frac{\partial f}{\partial \phi'}\delta\phi' + \frac{\partial f}{\partial x}\delta x. \tag{21}$$

However, as discussed in connection with Figure D.3, the independent variable experiences no variation, that is, we can put $\delta x = 0$ in (21) and we thus arrive at the correct formula (19).

Also the rules (1) to (4) of Table D.1 can be proved rather easily using the definition (14) of the variation. In fact, after some practise it is more convenient to drop the $\varepsilon\eta$-notation and to operate simply with the $\delta\phi$-notation and rely on familiar differentiation formulas.

The change of the value of a definite integral (2) is

$$\Delta\Pi = \int_a^b f(x, \phi^*, \phi^{*\prime})\mathrm{d}x - \int_a^b f(x, \phi, \phi')\mathrm{d}x = \int_a^b (f^* - f)\mathrm{d}x = \int_a^b \Delta f\,\mathrm{d}x$$

$$= \int_a^b \delta f\,\mathrm{d}x + \int_a^b \frac{1}{2}\delta^2 f\,\mathrm{d}x + \cdots \tag{22}$$

Form the definition of the variation, as the first term in the expansion contains ε in the first power,

$$\delta\Pi \equiv \boxed{\delta\int_a^b f\,\mathrm{d}x = \int_a^b \delta f\,\mathrm{d}x} \tag{23}$$

or *the variation of a definite integral is equal to the definite integral of the variation*. In other words, variation and integration (between fixed limits) obey the commutative law. This result is given in Table D.1 as rule (6).

The mathematical manipulations in the case of several independent and dependent variables are rather obvious from the presentation above. For instance, let x and y be the independent variables and let us have two dependent variables $u(x,y)$ and $v(x,y)$. The variations of u and v are represented in the form

$$\delta u = \varepsilon\xi(x,y), \qquad \delta v = \varepsilon\eta(x,y) \tag{24}$$

where $\xi(x,y)$ and $\eta(x,y)$ are arbitrary smooth enough functions and the parameter ε controls the magnitude of the change. The expressions containing u or v experience changes when u and v are replaced with u^* and v^*. For instance the function

$$f = f(x,y,u,v,u_x,u_y,v_x,v_y) \tag{25}$$

where $u_x = \partial u/\partial x$, etc, experiencies the change

$$\begin{aligned}\Delta f &= f(x,y,u^*,v^*,u_x^*,u_y^*,v_x^*,v_y^*) - f(x,y,u,v,u_x,u_y,v_x,v_y)\\ &= f^* - f.\end{aligned} \tag{26}$$

Again the change can be considered to depend on only one variable ε. There is obtained

$$\Delta f = \delta f + \frac{1}{2}\delta^2 f + \cdots \tag{27}$$

where the variation

$$\begin{aligned}\delta f &\equiv \left.\frac{\mathrm{d}f^*}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\varepsilon\\ &= \frac{\partial f}{\partial u}\varepsilon\xi + \frac{\partial f}{\partial v}\varepsilon\eta + \frac{\partial f}{\partial u_x}\varepsilon\xi_x + \frac{\partial f}{\partial u_y}\varepsilon\xi_y + \frac{\partial f}{\partial v_x}\varepsilon\eta_x + \frac{\partial f}{\partial v_y}\varepsilon\eta_y\\ &= \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial v}\delta v + \frac{\partial f}{\partial u_x}\delta u_x + \frac{\partial f}{\partial u_y}\delta u_y + \frac{\partial f}{\partial v_x}\delta v_x + \frac{\partial f}{\partial v_y}\delta v_y.\end{aligned} \tag{28}$$

Further, for example, the operators δ and $\partial/\partial x$ or $\partial/\partial y$ are commutative:

$$\begin{aligned}\delta\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\delta u, \qquad \delta u_x = (\delta u)_x,\\ \delta\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\delta u, \qquad \delta u_y = (\delta u)_y.\end{aligned} \tag{29}$$

These are generalizations of rule (5) of Table D.1. Obvious generalization is also valid for rule (6) with respect to the the integration domain.

D.2.3 One independent and one dependent variable

We now consider in detail the case of the heading. The change of a functional $\Pi(\phi)$ due to the variation $\delta\phi(x) = \varepsilon\eta(x)$ of the dependent variable or argument function $\phi(x)$ is

$$\Delta\Pi = \delta\Pi + \frac{1}{2}\delta^2\Pi + \cdots \tag{30}$$

As expressed in (D.2.4), the functional is said to have a stationary value when its (first) variation disappears with respect to all admissible variations of the argument function:

$$\delta\Pi = 0. \tag{31}$$

The stationarity is a necessary condition for an extremum. To find out the character of the stationary point, the behaviour of the second variation must be studied similarly as was done with the second differential in connection of functions in Section D.1.1. The treatment becomes complicated. In mechanics, however, the necessary equations are usually obtained already from the stationary condition and we therefore do not consider the second variation here. In addition, in practice we often immediately proceed with a discretized formulation, that is, we continue with the study of ordinary functions instead of functionals.

The expression for the variation from (14) is

$$\delta\Pi \equiv \left.\frac{\mathrm{d}\Pi^*}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\varepsilon = \left.\frac{\mathrm{d}\Pi(\phi+\varepsilon\eta)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\varepsilon. \tag{32}$$

Careful mathematical derivations usually treat the variation in the $\varepsilon\eta$-form and start from the condition

$$\left.\frac{\mathrm{d}\Pi^*}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = 0 \tag{33}$$

equivalent to (31). The formulas become however little tidier when the variation is denoted simply $\delta\phi$ and condition (31) is used.

We consider first the basic case

$$\Pi(\phi) = \int_a^b f(x,\phi,\phi')\,\mathrm{d}x. \tag{34}$$

Using formulas (23), (19) and (18) we have

$$\delta\Pi = \delta\int_a^b f\,\mathrm{d}x = \int_a^b \delta f\,\mathrm{d}x = \int_a^b (\frac{\partial f}{\partial \phi}\delta\phi + \frac{\partial f}{\partial \phi'}\delta\phi')\,\mathrm{d}x$$
$$= \int_a^b (\frac{\partial f}{\partial \phi}\delta\phi + \frac{\partial f}{\partial \phi'}\frac{\mathrm{d}\delta\phi}{\mathrm{d}x})\,\mathrm{d}x. \qquad (35)$$

The next step is based on the fact that the variation $\delta\phi$ is arbitrary. No conclusions can yet be drawn from expression (35) as it contains also the derivative of $\delta\phi$. The derivative is removed by applying integration by parts. Formula (B.1.1) with $g \mathrel{\hat{=}} \partial f / \partial \phi'$ and $h \mathrel{\hat{=}} \delta\phi$ gives

$$\int_a^b \frac{\partial f}{\partial \phi'}\frac{\mathrm{d}\delta\phi}{\mathrm{d}x}\,\mathrm{d}x = -\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial f}{\partial \phi'})\delta\phi\,\mathrm{d}x + \Big|_a^b \frac{\partial f}{\partial \phi'}\delta\phi \qquad (36)$$

and thus the stationarity condition obtains the form

$$\delta\Pi \equiv \int_a^b [\frac{\partial f}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial f}{\partial \phi'})]\delta\phi\,\mathrm{d}x + \Big|_a^b \frac{\partial f}{\partial \phi'}\delta\phi = 0. \qquad (37)$$

The expression (37) must vanish for an arbitrary $\delta\phi$. This means that the term inside the brackets in the integral must vanish:

$$\frac{\partial f}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial f}{\partial \phi'}) = 0, \qquad a < x < b. \qquad (38)$$

This is the Euler-Lagrange differential equation corresponding to functional (34). The partial derivatives $\partial f / \partial \phi$ and $\partial f / \partial \phi'$ are formed considering x, ϕ and ϕ' as independent variables in the expression of f. However, finally x is here the only actual independent variable and (38) is thus an ordinary differential equation. When the second term in (38) is developed by the chain rule, there is obtained

$$-\frac{\partial^2 f}{\partial \phi' \partial \phi'}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \frac{\partial^2 f}{\partial \phi \partial \phi'}\frac{\mathrm{d}\phi}{\mathrm{d}x} - \frac{\partial^2 f}{\partial x \partial \phi'} + \frac{\partial f}{\partial \phi} = 0 \qquad (39)$$

so the Euler-Lagrange equation is in general a second order differential equation.

Example D.7. We consider functional (3):

$$V(v) = \int_0^l [\frac{1}{2}S(v')^2 - qv]\mathrm{d}x \qquad (a)$$

representing the potential energy of a stretched string. Here

$$f(x, v, v') = \frac{1}{2}S(v')^2 - q(x)v \qquad (b)$$

and thus $\phi \mathrel{\hat{=}} v$. The Euler-Lagrange equation (38) is

$$\frac{\partial f}{\partial v} - \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial f}{\partial v'}) = 0 \qquad (c)$$

or

$$-q - \frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{2}S2v') = 0,$$
$$-q - Sv'' = 0 \qquad (d)$$

or finally

$$Sv'' = -q. \qquad (e)$$

In practice, it is usually more convenient to derive the Euler-Lagrange equation directly in each case without having recourse to some general formulas like (c). Thus here, making use of the rules of Table D.1 and of integration by parts, we obtain

$$\delta V = \int_0^l [\frac{1}{2}S2v'\delta v' - q\delta v]\mathrm{d}x = \int_0^l [Sv'(\delta v)' - q\delta v]\mathrm{d}x$$
$$= \int_0^l [-(Sv')'\delta v - q\delta v]\mathrm{d}x + \Big|_0^l Sv'\delta v$$
$$= \int_0^l [-Sv'' - q]\delta v\,\mathrm{d}x + \Big|_0^l Sv'\delta v. \qquad (f)$$

Demanding this to vanish gives again the differential equation (e).

We have not yet considered the boundary conditions which are classified in connection with functionals as explained in Section D.2.2 into essential and natural conditions. If, for instance, the essential boundary conditions consist of

$$\phi(a) = \alpha, \qquad \phi(b) = \beta, \qquad (40)$$

this means that all the competing admissible argument functions must satisfy in advance the conditions (40). Thus in addition to the assumed stationary function ϕ, also the varied function $\phi^* = \phi + \delta\phi$ must satisfy

$$\phi^*(a) \equiv \phi(a) + \delta\phi(a) = \alpha, \qquad \phi^*(b) \equiv \phi(b) + \delta\phi(b) = \beta. \qquad (41)$$

Subtracting (40) from (41) leads to the homogeneous conditions

$$\delta\phi(a) = 0, \qquad \delta\phi(b) = 0 \qquad (42)$$

for the admisible variations. In this case the boundary terms in (37) vanish automatically.

If, on the other hand, no essential conditions are presented, the variations $\delta\phi(a)$ and $\delta\phi(b)$ are arbitrary. Stationarity condition on expression (37) leads now to the natural boundary conditions

$$\left.\frac{\partial f}{\partial \phi'}\right|_{x=a} = 0, \quad \left.\frac{\partial f}{\partial \phi'}\right|_{x=b} = 0. \tag{43}$$

Further, if the essential boundary condition would be say $\phi(a) = \alpha$, the resulting natural boundary condition would be $(\partial f / \partial \phi')|_{x=b} = 0$.

Let us return in more detail on the logic of getting the Euler-Lagrange equation and the natural boundary conditions from the stationary condition (37). Use is made of the so called *fundamental lemma of variational calculus* (variaatiolaskennan peruslemma):

If the relation

$$\int_a^b f(x)g(x)\mathrm{d}x = 0 \tag{44}$$

where $f(x)$ is a continuous function, is valid for all continuous functions $g(x)$,

$$f(x) = 0 \quad \text{in } \Omega =]a,b[. \tag{45}$$

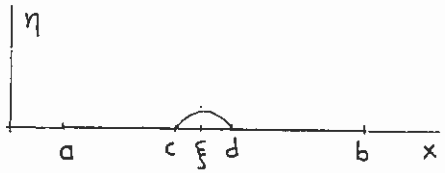


Figure D.5 Bubble function.

This is proved roughly as follows. Let us consider an arbitrary point $x = \xi$ in the open interval $]a,b[$ and assume that contrary to what has been stated, f is there non-zero and say positive. Because of the continuity of f there exists a neighbourhood $c < x < d$ of ξ where f is positive. We now select η as a "bubble" (Figure D.5) say of the type

$$g = \begin{cases} (x-c)(d-x), & c < x < d, \\ 0 \quad \text{elsewhere.} \end{cases} \tag{46}$$

The first expression in (46) is then positive and thus

$$\int_a^b f(x)g(x)\mathrm{d}x = \int_c^d f(x)g(x)\mathrm{d}x > 0 \tag{47}$$

which is against (44).

In our application the variation $\delta\phi$ has the role of g. In the weak forms treated earlier the weighting function w has had the role of g. The lemma can be extended in an obvious way to cases with several independent variables.

The possible natural boundary conditions are derived in detail as follows. The previous consideration showed that the integral in (37) vanishes and we are left with the equation

$$\left.\frac{\partial f}{\partial \phi'}\delta\phi\right|_{x=b} - \left.\frac{\partial f}{\partial \phi'}\delta\phi\right|_{x=a} = 0. \tag{48}$$

If we have for instance the case with no essential boundary conditions we can take first $\delta\phi(b) \neq 0$, $\delta\phi(a) = 0$ and then $\delta\phi(a) \neq 0$, $\delta\phi(b) = 0$ to deduce the natural boundary conditions (43). Other cases are dealt with similarly.

The generalization to formulations with higher order derivatives proceeds in a straightforward way; more applications of integration by parts are just needed. For instance, variation of the integral

$$\Pi(\phi) = \int_a^b f(x, \phi, \phi', \phi'')\mathrm{d}x \tag{49}$$

gives first

$$\begin{aligned} \delta\Pi &= \int_a^b \left(\frac{\partial f}{\partial \phi}\delta\phi + \frac{\partial f}{\partial \phi'}\delta\phi' + \frac{\partial f}{\partial \phi''}\delta\phi''\right)\mathrm{d}x \\ &= \int_a^b \left(\frac{\partial f}{\partial \phi}\delta\phi + \frac{\partial f}{\partial \phi'}\frac{\mathrm{d}\delta\phi}{\mathrm{d}x} + \frac{\partial f}{\partial \phi''}\frac{\mathrm{d}^2\delta\phi}{\mathrm{d}x^2}\right)\mathrm{d}x. \end{aligned} \tag{50}$$

After integrations by parts we find

$$\begin{aligned} \delta\Pi = &\int_a^b \left[\frac{\partial f}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \phi'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial \phi''}\right)\right]\delta\phi\,\mathrm{d}x + \\ &+ \left|_a^b \left[\frac{\partial f}{\partial \phi'} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \phi''}\right)\right]\delta\phi + \left|_a^b \frac{\partial f}{\partial \phi''}\delta\phi'\right. \end{aligned} \tag{51}$$

The stationary condition $\delta\Pi = 0$ gives the Euler-Lagrange equation

$$\frac{\partial f}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \phi'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial \phi''}\right) = 0. \tag{52}$$

The possible natural boundary conditions obtainable depend on the possible essential boundary conditions and can be deduced from the remaining boundary terms in (51).

Example D.8. The governing equilibrium equations for the elastic Bernoulli cantilever beam of Figure (a) are derived by the principle of stationary potential energy.

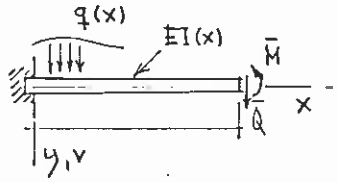


Figure (a)

The expression for the potential energy is

$$V(v) = \int_0^l [\frac{1}{2} EI(v'')^2 - qv]\mathrm{d}x - \overline{Q}v(l) + \overline{M}v'(l) \tag{a}$$

where the argument function $v(x)$ represents the lateral small deflection of the axis of the beam, $q(x)$ is the lateral loading per unit length and $EI(x)$ the bending stiffness. $\overline{Q}$ and $\overline{M}$ are the given shearing force and bending moment at the free end of the beam. The essential boundary conditions are

$$\begin{aligned} &v(0) = 0, \\ &v'(0) = 0 \end{aligned} \tag{b}$$

due to the kinematics of the fixed end.

The functional differs from (49) in the respect that terms from the boundary of the domain appear here. In the case of several independent variables the boundary terms are integrals. Because no boundary terms have been considered in connection of obtaining the Euler-Lagrange equation (52), we perform the calculations starting directly from expression (a).

Taking the variation of (a) gives first

$$\begin{aligned} \delta V &= \int_0^l [EIv''\delta v'' - q\delta v]\mathrm{d}x - \overline{Q}\delta v(l) + \overline{M}\delta v'(l) \\ &= \int_0^l [EIv''(\delta v)'' - q\delta v]\mathrm{d}x - \overline{Q}\delta v(l) + \overline{M}\delta v'(l). \end{aligned} \tag{c}$$

The integration by parts needed, goes in detail as follows:

$$\begin{aligned} \int_0^l EIv''(\delta v)''\,\mathrm{d}x &= -\int_0^l (EIv'')'(\delta v)'\,\mathrm{d}x + \Big|_0^l EIv''(\delta v)' \\ &= \int_0^l (EIv'')''\delta v\mathrm{d}x - \Big|_0^l (EIv'')'\delta v + \Big|_0^l EIv''(\delta v)'. \end{aligned} \tag{d}$$

Thus,

$$\begin{aligned} \delta V = &\int_0^l [(EIv'')'' - q]\delta v\mathrm{d}x - \Big|_0^l (EIv'')'\delta v + \Big|_0^l EIv''(\delta v)' + \\ &- \overline{Q}\delta v(l) + \overline{M}\delta v'(l). \end{aligned} \tag{e}$$

The stationary condition $\delta V = 0$ gives first the Euler-Lagrange equation

$$(EIv'')'' - q = 0 \tag{f}$$

and we are left with the equation

$$\begin{aligned} &[-(EIv'')' - \overline{Q}]\Big|_{x=l}\delta v(l) + (EIv'')'\Big|_{x=0}\delta v(0) + \\ &+ [EIv'' + \overline{M}]\Big|_{x=l}\delta v'(l) - EIv''\Big|_{x=0}\delta v'(0) = 0. \end{aligned} \tag{g}$$

Because of the essential boundary conditions (b),

$$\begin{aligned} &\delta v(0) = 0, \\ &\delta v'(0) = 0. \end{aligned} \tag{h}$$

On the other hand, $\delta v(l)$ and $\delta v'(l)$ are arbitrary, leading to the natural boundary conditions

$$\begin{aligned} &\overline{M} = -EIv''\Big|_{x=l}, \\ &\overline{Q} = -(EIv'')'\Big|_{x=l}. \end{aligned} \tag{i}$$

These mean physically that the external stress resultants are equal to the internally evaluated stress resultants at the free end.

The differential equation (f) together with the boundary conditions (b) and (i) represents the problem to solve the deflection $v(x)$ of the beam. When the functional (a) is used as the starting point for a numerical solution, considerable advantages are achieved. The derivative order appearing in the functional is two whereas in the differential equation it is four. The approximation needs only to satisfy the simple essential conditions (b), the more complicated natural conditions (i) are simulated automatically through the functional.

D.2.4 One independent and several dependent variables

The functional corresponding to the heading is now typically of the type

$$\Pi(\phi_1, \phi_2, \cdots, \phi_n) = \int_a^b f(x, \phi_1, \phi_2, \cdots, \phi_n, \phi_1', \phi_2', \cdots, \phi_n')\mathrm{d}x. \tag{53}$$

Boundary terms and higher order derivatives can additionally be present. By taking certain functions $\phi_1, \phi_2, \cdots, \phi_n$, the integral can be evaluated. The task is again to determine those functions which make the functional Π stationary (or to determine the conditions from which the functions can in principle be determined.)

Similarly as earlier, the task is approached by taking the varied functions

$$\phi_i^*(x) = \phi_i(x) + \varepsilon\eta_i(x), \qquad i = 1, 2, \cdots, n \tag{54}$$

and by considering the (first) variation

$$\delta\Pi \equiv \frac{\mathrm{d}\Pi^*}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \varepsilon = \frac{\mathrm{d}\Pi(\phi_1 + \varepsilon\eta_1, \phi_2 + \varepsilon\eta_2, \cdots, \phi_n + \varepsilon\eta_n)}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \varepsilon \tag{55}$$

which must vanish for arbitrary admissible functions $\eta_i(x)$. By taking the consecutive selections

$$\eta_1 \neq 0, \quad \eta_2 = 0, \quad \eta_3 = 0, \cdots \quad \text{or} \quad \delta\phi_1 \neq 0, \quad \delta\phi_2 = 0, \quad \delta\phi_3 = 0, \cdots$$
$$\eta_1 = 0, \quad \eta_2 \neq 0, \quad \eta_3 = 0, \cdots \quad \text{or} \quad \delta\phi_1 = 0, \quad \delta\phi_2 \neq 0, \quad \delta\phi_3 = 0, \cdots \tag{56}$$

etc, that is, by varying only one function at a time, n Euler -Lagrange equations

$$\frac{\partial f}{\partial \phi_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \phi_i'}\right) = 0, \qquad i = 1,2,\cdots,n \tag{57}$$

and the associated natural boundary conditions are arrived at. The Euler-Lagrange equations form thus a system of ordinary differential equations. It is seen that the structure of the equations is exactly the same as in (38).

Example D.9. We derive the equations of motion of the elastic spring system considered in Example D.4 using here Hamilton's principle.

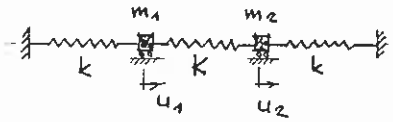


Figure (a)

The particles in Figure (a) have masses m_1 and m_2 and no external forces act on the system. According to Hamilton's principle the motion of a conservative system happens so that the functional

$$\int_{t_1}^{t_2} (T - V)\,\mathrm{d}t \tag{a}$$

obtains a stationary value. T is the kinetic energy and V the potential energy of the system.

Here

$$T = \frac{1}{2}m_1\dot{u}_1^2 + \frac{1}{2}m_2\dot{u}_2^2 \tag{b}$$

and

$$V = \frac{1}{2}ku_1^2 + \frac{1}{2}K(u_2 - u_1)^2 + \frac{1}{2}ku_2^2 \tag{c}$$

and thus we have the functional

$$\Pi(u_1,u_2) = \int_{t_1}^{t_2} [\frac{1}{2}m_1\dot{u}_1^2 + \frac{1}{2}m_2\dot{u}_2^2 - \frac{1}{2}ku_1^2 - \frac{1}{2}K(u_2 - u_1)^2 - \frac{1}{2}ku_2^2]\,\mathrm{d}t. \tag{d}$$

Comparison of (d) with (53) shows that here $n = 2$, $x \triangleq t$, $\phi_1 \triangleq u_1$, $\phi_2 \triangleq u_2$, $(\cdot)' \triangleq (\dot{\cdot})$. The admissible functions $u_1(t)$ and $u_2(t)$ have to correspond to the actual state at $t = t_1$ and $t = t_2$. These are essential conditions and thus the admissible variations δu_1 and δu_2 have to vanish at $t = t_1$ and $t = t_2$:

$$\delta u_1(t_1) = 0, \quad \delta u_1(t_2) = 0,$$
$$\delta u_2(t_1) = 0, \quad \delta u_2(t_2) = 0. \tag{e}$$

The Euler-Lagrange equations (57) are

$$-ku_1 + K(u_2 - u_1) - \frac{\mathrm{d}}{\mathrm{d}t}(m_1\dot{u}_1) = 0,$$
$$-K(u_2 - u_1) - ku_2 - \frac{\mathrm{d}}{\mathrm{d}t}(m_2\dot{u}_2) = 0 \tag{f}$$

or in matrix notation with a change of signs

$$\begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{g}$$

Alternatively, the same results can be derived without using the general formula (57). The variation of (d) is

$$\begin{aligned}
\delta\Pi &= \int_{t_1}^{t_2} [m_1\dot{u}_1\delta\dot{u}_1 + m_2\dot{u}_2\delta\dot{u}_2 - ku_1\delta u_1 - K(u_2 - u_1)(\delta u_2 - \delta u_1) - ku_2\delta u_2]\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} [m_1\dot{u}_1\frac{\mathrm{d}\delta u_1}{\mathrm{d}t} + m_2\dot{u}_2\frac{\mathrm{d}\delta u_2}{\mathrm{d}t}]\,\mathrm{d}t + \\
&\quad + \int_{t_1}^{t_2} [-ku_1\delta u_1 + K(u_2 - u_1)\delta u_1 - K(u_2 - u_1)\delta u_2 - ku_2\delta u_2]\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} [-\frac{\mathrm{d}(m_1\dot{u}_1)}{\mathrm{d}t}\delta u_1 - \frac{\mathrm{d}(m_2\dot{u}_2)}{\mathrm{d}t}\delta u_2]\,\mathrm{d}t + \Big|_{t_1}^{t_2} [m_1\dot{u}_1\delta u_1 + m_1\dot{u}_2\delta u_2] + \\
&\quad + \int_{t_1}^{t_2} [-ku_1\delta u_1 + K(u_2 - u_1)\delta u_1 - K(u_2 - u_1)\delta u_2 - ku_2\delta u_2]\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \{[-\frac{\mathrm{d}(m_1\dot{u}_1)}{\mathrm{d}t} - ku_1 + K(u_2 - u_1)]\delta u_1 + \\
&\quad + [-\frac{\mathrm{d}(m_2\dot{u}_2)}{\mathrm{d}t} - K(u_2 - u_1) - ku_2]\delta u_2\}\,\mathrm{d}t
\end{aligned} \tag{h}$$

The steps used are obvious. Demanding (h) to be stationary for arbitrary δu_1 and δu_2 gives again equations (f).

We do not discuss here the difficulties emerging in the case $K >> k$.

D.2.5 Several independent variables

Let us consider first the case of two independent variables x and y and one dependent (argument) function $\phi(x,y)$. The functional is typically of the type

$$\Pi(\phi) = \int_A f(x,y,\phi,\phi_x,\phi_y)\,\mathrm{d}A. \tag{58}$$

Boundary integrals and higher order derivatives can additionally be present. The stationary condition leads to Euler-Lagrange *partial* differential equation and possible to some natural boundary conditions. We do not present any general formulas.

Example D.10. We consider heat conduction in the domain A with boundary s (Figure (a)).

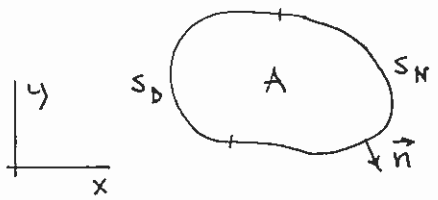


Figure (a)

The governing field equation for steady heat conduction in an isotropic medium is

$$\frac{\partial}{\partial x}(-k\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(-k\frac{\partial T}{\partial y})-Q=0 \quad \text{in } A \tag{a}$$

where $T(x,y)$ is the temperature to be determined, k the thermal conductivity (lämmönjohtavuus) of the medium and Q the heat source rate per unit volume (lämpölähteen antoisuus).The conventional boundary conditions are the Dirichlet type condition

$$T=\bar{T} \quad \text{on } s_D \tag{b}$$

where $\bar{T}(s)$ is the given temperature and the Neumann type condition

$$-k(n_x\frac{\partial T}{\partial x}+n_y\frac{\partial T}{\partial y})=\bar{q}_n \quad \text{on } s_N \tag{c}$$

where $\bar{q}_n(s)$ is the given heat flow rate density (lämpövirran tiheys). The two parts s_D and s_N form together the whole boundary s without gaps or overlaps; $s_D \cup s_N = s$, $s_D \cap s_N = \emptyset$.

There is a variational principle (without a well-established name) corresponding to the above problem. The functional is

$$\Pi(T)=\int_A\{\frac{1}{2}k[(\frac{\partial T}{\partial x})^2+(\frac{\partial T}{\partial y})^2]-QT\}dA+\int_{s_N}\bar{q}_nT\,ds \tag{d}$$

with the essential boundary condition (b). We show this in the following.

The variation of (d) is first

$$\delta\Pi=\int_A\{\frac{1}{2}k[2\frac{\partial T}{\partial x}\delta\frac{\partial T}{\partial x}+2\frac{\partial T}{\partial y}\delta\frac{\partial T}{\partial y}]-Q\delta T\}dA+\int_{s_N}\bar{q}_n\delta T\,ds$$

$$=\int_A[k\frac{\partial T}{\partial x}\frac{\partial \delta T}{\partial x}+k\frac{\partial T}{\partial y}\frac{\partial \delta T}{\partial y}-Q\delta T]dA+\int_{s_N}\bar{q}_n\delta T\,ds. \tag{e}$$

Integration by parts formulas (B.2.1a) give with $g \triangleq k\partial T/\partial x$ and $h \triangleq \delta T$ and with $g \triangleq k\partial T/\partial y$ and $h \triangleq \delta T$, respectively

$$\int_A k\frac{\partial T}{\partial x}\frac{\partial \delta T}{\partial x}dA=-\int_A\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})\delta T\,dA+\int_s k\frac{\partial T}{\partial x}\delta T n_x ds,$$
$$\int_A k\frac{\partial T}{\partial y}\frac{\partial \delta T}{\partial y}dA=-\int_A\frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})\delta T\,dA+\int_s k\frac{\partial T}{\partial y}\delta T n_y ds. \tag{f}$$

Further, because of the essential boundary condition (b), the admissible variation must satisfy

$$\delta T=0 \quad \text{on } s_D. \tag{g}$$

Taking (f) and (g) into account in (e) gives the expression

$$\delta\Pi=\int_A[\frac{\partial}{\partial x}(-k\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(-k\frac{\partial T}{\partial y})-Q]\delta T\,dA+$$
$$+\int_{s_N}[k(n_x\frac{\partial T}{\partial x}+n_y\frac{\partial T}{\partial y})+\bar{q}_n]\delta T\,ds. \tag{h}$$

As the variation δT is arbitrary in A and on s_N, the differential equation (a) and the boundary condition (c) result by demanding $\delta\Pi$ to vanish.

If the functional contains second or higher order partial derivatives, integration by parts has to be applied consecutively more than once.

How to proceed in the case of more than one independent (argument) functions should be obvious from the earlier text. The Euler-Lagrange equations form a system of partial differential equations.

Example D.11. We consider the application of the principle of stationary potential energy in connection of an isotropic homogeneous elastic body under small displacements in the plane strain case (Figure (a)).

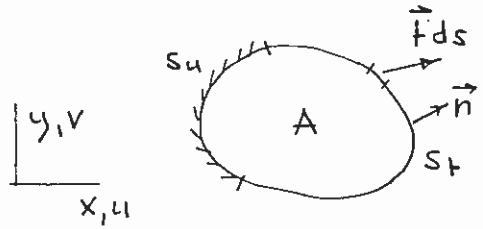


Figure (a)

The expression for the potential energy of the body (per unit length in the z - direction) is (to simplify the calculations we have taken the rather unrealistic case of Poisson's ratio $\nu=0$)

$$V(u,v)=\int_A\{\frac{1}{2}E[(\frac{\partial u}{\partial x})^2+(\frac{\partial v}{\partial y})^2+\frac{1}{2}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})^2]-f_x u-f_y v\}dA+$$
$$-\int_{s_t}(\bar{t}_x u+\bar{t}_y v)ds. \tag{a}$$

Here $u(x,y)$ and $v(x,y)$ are displacement components to be determined, E is the elastic modulus of the material (assumed constant in space), $f_x(x,y)$ and $f_y(x,y)$ are the body force components per volume and $\bar{t}_x(s)$ and $\bar{t}_y(s)$ are the given traction components on the boundary s_t. On the rest of the boundary s_u, the displacement components are given:

$$u = \bar{u}, \quad v = \bar{v} \quad \text{on } s_u. \tag{b}$$

We shall derive the governing differential equations and the traction boundary conditions by the principle of stationary potential energy.

The variation of (a) is

$$\delta V = \int_A \{E[\frac{\partial u}{\partial x}\delta\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\delta\frac{\partial v}{\partial y} + \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})(\delta\frac{\partial u}{\partial y} + \delta\frac{\partial v}{\partial x})] + \\ - f_x\delta u - f_y\delta v\}\,dA - \int_{s_t}(\bar{t}_x\delta u + \bar{t}_y\delta v)\,ds. \tag{c}$$

The kinematic boundary conditions (b) are essential ones in the principle of stationary potential energy. Thus the admissible variations δu and δv have to satisfy

$$\delta u = 0, \quad \delta v = 0 \quad \text{on } s_u. \tag{d}$$

Taking these into account and proceeding similarly as in Example D.10 gives finally the expression

$$\delta V = -\int_A \{[E\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}E(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\partial y}) + f_x]\delta u + \\ + [E\frac{\partial^2 v}{\partial y^2} + \frac{1}{2}E(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial x^2}) + f_y]\delta v\}\,dA + \\ + \int_{s_t}\{[n_x E\frac{\partial u}{\partial x} + n_y\frac{1}{2}E(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) - \bar{t}_x]\delta u + \\ + [n_y E\frac{\partial v}{\partial y} + n_x\frac{1}{2}E(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) - \bar{t}_y]\delta v\}\,ds. \tag{e}$$

The stationarity condition gives clearly the Euler-Lagrange equations

$$E\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}E(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\partial y}) + f_x = 0, \\ E\frac{\partial^2 v}{\partial y^2} + \frac{1}{2}E(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial x^2}) + f_y = 0 \tag{f}$$

in A and the natural (traction) boundary conditions

$$n_x E\frac{\partial u}{\partial x} + n_y\frac{1}{2}E(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = \bar{t}_x, \\ n_y E\frac{\partial v}{\partial y} + n_x\frac{1}{2}E(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = \bar{t}_y \tag{g}$$

on s_t. If the medium is not homogeneous (E is not constant in space), the equilibrium equations (f) become very complicated but expression (a) remains the same this indicating again the usefulness of the variational formulation.

D.2.6 Constrained stationary problems

Constrained stationary problems appear also often in connection of functionals. Thus (using matrix notation) we have to determine the argument functions

$\{\phi\} = [\phi_1, \phi_2, \cdots]^T$ giving a functional $\Pi(\{\phi\})$ a stationary value and additionally constraints

$$\{C(\{\phi\})\} = \{0\} \tag{59}$$

in the form of differential (or algebraic) equations have to be satisfied.

As an example we may consider the functional associated with the slow or so called "creeping flow" of a viscous incompressible Newtonian fluid in two dimensions:

$$\Pi(u,v) = \int_A \{\mu[(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + \frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})^2] + \\ - \rho b_x u - \rho b_y v\}\,dA - \int_{s_t}(\bar{t}_x u + \bar{t}_y v)\,ds. \tag{60}$$

Here $u(x,y)$ and $v(x,y)$ are velocity components to be determined and μ is the viscosity of the medium. Some of the other notations can be understood on the basis of Example D.11. The equations of motion are obtained from the stationarity of (60) but the admissible argument functions must satisfy in advance in addition to the essential (kinematic) boundary conditions

$$u = \bar{u}, \\ v = \bar{v} \tag{61}$$

on s_u, the incompressibility condition

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{in } A. \tag{62}$$

The constraints (61) and (62) are examples of (59). We have considered essential boundary conditions this far not "seriously" as constraints as they can usually be taken into account rather easily directly both in an analytical or numerical solution. This way to treat them could be called the elimination method by the analogy to the discussion in Section D.1.2. However, they can be considered strictly as constraints similarly as (62) which is already difficult to take care of by the elimination method.

The Lagrange multiplier method is the classical way to treat constraints also in connection with functionals. Thus, we can form a modified functional

$$\Pi_L(u,v;\lambda) = \Pi(u,v) + \int_A \lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})\,dA \tag{63}$$

to take care of (62). Here $\lambda(x,y)$ is a new unknown *function*; *Lagrange multiplier function* (Lagrangen kertojafunktio). The admissible u and v need no more to satisfy in advance (62). Further, we can extend the functional to the form

$$\Pi_L(u,v;\lambda,\lambda_u,\lambda_v) = \Pi(u,v) + \int_A \lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})\mathrm{d}A + \\ + \int_{s_u} [\lambda_u(u-\bar{u}) + \lambda_v(v-\bar{v})]\mathrm{d}s \tag{64}$$

to take care additionally even of (61). Two new unknown Lagrange multiplier functions $\lambda_u(s)$ and $\lambda_v(s)$ emerge. There is now no constraints on the admissible u and v (except reasonable smoothness).

We do not consider the theoretical justification of the use of the Lagrange multiplier function method here. On the the basis of Section D.1.2 it can however be roughly understood as follows. Let us consider first the problem discretize in such a form where the functions u and v are represented as a set of pointwise values $u_1, u_2, \cdots$. $v_1, v_2, \cdots$. The equivalent of the constraint equation (62) is a set of several algebraic equations

$$g_k(u_1,u_2,\cdots,v_1,v_2,\cdots) = 0, \qquad k = 1,2,\cdots \tag{65}$$

These can be produced say by approximating (62) at certain points with the finite difference method. In Section D.1.2 the function under study was modified in connection of constraints by adding a term of the type

$$\sum_k \lambda_k g_k(u_1,u_2,\cdots,v_1,v_2,\cdots). \tag{66}$$

In the limit this may be consider to transform into an integral

$$\int_A \lambda(x,y)\, C(u,v)\mathrm{d}A \tag{67}$$

where in the case at hand

$$C(u,v) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \tag{68}$$

In the general case the constraints (59) can be taken into account by introducing a modified functional

$$\Pi_L(\{\phi\};\{\lambda\}) = \Pi(\{\phi\}) + \int_\Omega (\lambda_1 C_1 + \lambda_2 C_2 + \cdots)\mathrm{d}\Omega \tag{69}$$

where $\{\lambda\}$ contains as many Lagrange multiplier functions as there are constraints. Here we have assumed that the constraints are valid in the domain Ω. If there are constraints valid on the boundary Γ, naturally corresponding integrals over Γ will appear.

The Euler-Lagrange equations are derived as earlier considering now in addition the contributions from the variations $\delta\lambda$. For instance, the variation of (63) is first

$$\delta\Pi_L = \delta\Pi(u,v) + \int_A \delta\lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})\mathrm{d}A + \int_A \lambda(\delta\frac{\partial u}{\partial x} + \delta\frac{\partial v}{\partial y})\mathrm{d}A. \tag{70}$$

Without continuing, it is immediately seen that one Euler-Lagrange equation will be the constraint (62) as $\delta\lambda$ is arbitrary.

The constraints can in some problems be in the form containing a definite integral over the domain (or over the boundary), say

$$\int_\Omega C(\{\phi\})\mathrm{d}\Omega - c = 0 \tag{71}$$

where c is a given constant. These kind of conditions are called *isoperimetric constraints* (isoperimetrinen rajoite) "since the first historically recorded extremum problem, that of finding the maximum area bounded by a perimeter of given length (Dido's problem) prescribes a condition of this nature", Lanczos (1974, p. 66).

Constraint (71) can be taken into account by the Lagrange multplier method by augmenting the original functional with the term

$$\lambda(\int_\Omega C(\{\phi\})\mathrm{d}\Omega - c) \tag{72}$$

where λ is now an unknown multiplier and *not a function*. This can be understood again by considering the problem first as discretized. The equivalent of (71) is then always *one algebraic equation* be the discrete model however large. If there are several isoperimetric constraints, the augmented term obviosly consists of a sum of the terms like (72).

The penalty method can be used similarly as explained in Section D.1.3 also in connection of functionals. Same advantages and disadvantages are present. For instance, the penalty method versions for (63) and (64) would be

$$\Pi_p(u,v) = \Pi(u,v) + \frac{1}{2}\int_A \alpha(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})^2\,\mathrm{d}A \tag{73}$$

and

and the virtual work equation (8") obtains the specific form

$$\boxed{\int_\Omega ([S]\delta\{u\})^{\mathrm{T}}[D][S]\{u\}\mathrm{d}\Omega - \int_\Omega \delta\{u\}^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \int_\Gamma \delta\{u\}^{\mathrm{T}}\{t\}\mathrm{d}\Gamma = 0.} \tag{28}$$

12.1.4 Sensitized principle of virtual work

Equilibrium equations (1) are in detail in the two-dimensional case (substitute expressions (11.2.2') into (1))

$$\begin{aligned} &\frac{\partial\sigma_a}{\partial a} + \frac{\partial\tau_{ba}}{\partial b} + \rho b_a = 0, \\ &\frac{\partial\tau_{ab}}{\partial a} + \frac{\partial\sigma_b}{\partial b} + \rho b_b = 0, \end{aligned} \tag{29}$$

or

$$\begin{bmatrix} \partial/\partial a & 0 & \partial/\partial b \\ 0 & \partial/\partial b & \partial/\partial a \end{bmatrix}\begin{Bmatrix} \sigma_a \\ \sigma_b \\ \tau_{ab} \end{Bmatrix} + \rho\begin{Bmatrix} b_a \\ b_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{30}$$

or

$$\{R(\{\sigma\})\} \equiv \{L(\{\sigma\})\} + \rho\{b\} \equiv [E]\{\sigma\} + \rho\{b\} = \{0\}, \tag{31}$$

where the *equilibrium* operator matrix

$$[E] = \begin{bmatrix} \partial/\partial a & 0 & \partial/\partial b \\ 0 & \partial/\partial b & \partial/\partial a \end{bmatrix}. \tag{32}$$

It is seen that here $[E] = [S]^{\mathrm{T}}$. The least-squares functional corresponding to equations (31) is (see Section D.2.8)

$$\Pi(\{\sigma\}) = \frac{1}{2}\int_\Omega \{R\}^{\mathrm{T}}[\tau]\{R\}\mathrm{d}\Omega. \tag{33}$$

The symmetric matrix $[\tau]$ is a sensitizing parameter matrix discussed in Section D.2.9. We consider here just the case with one sensitizing integral. Demanding (33) to have a stationary value gives the equation (see Section D.2.11)

$$\int_\Omega \{L(\{\delta\sigma\})\}^{\mathrm{T}}[\tau]\{R(\{\sigma\})\}\mathrm{d}\Omega = 0. \tag{34}$$

As discussed in Section D.2.11, a sensitized weak form — here a sensitized principle of virtual work — is obtained as a linear combination of (8") and (34):

$$\boxed{\begin{aligned} &\int_\Omega ([S]\delta\{u\})^{\mathrm{T}}\{\sigma\}\mathrm{d}\Omega - \int_\Omega \delta\{u\}^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \int_\Gamma \delta\{u\}^{\mathrm{T}}\{t\}\mathrm{d}\Gamma + \\ &+\int_\Omega ([E]\{\delta\sigma\})^{\mathrm{T}}[\tau]([E]\{\sigma\} + \rho\{b\})\mathrm{d}\Omega = 0. \end{aligned}} \tag{35}$$

The expression $\{\delta\sigma\}$ is to interpreted here finally as the virtual change of stresses due to a virtual displacement $\delta\{u\}$.

12.1.5 Sensitized principle of virtual work for an elastic body

For an elastic material we have the expression (27):

$$\{\sigma\} = [D][S]\{u\} \tag{36}$$

and thus

$$\delta\{\sigma\} = [D][S]\delta\{u\}. \tag{37}$$

The virtual work equation (35) obtains the specific form

$$\boxed{\begin{aligned} &\int_\Omega ([S]\delta\{u\})^{\mathrm{T}}[D][S]\{u\}\mathrm{d}\Omega - \int_\Omega \delta\{u\}^{\mathrm{T}}\rho\{b\}\mathrm{d}\Omega - \int_\Gamma \delta\{u\}^{\mathrm{T}}\{t\}\mathrm{d}\Gamma + \\ &+\int_\Omega ([E][D][S]\delta\{u\})^{\mathrm{T}}[\tau]([E][D][S]\{u\} + \rho\{b\})\mathrm{d}\Omega = 0. \end{aligned}} \tag{38}$$

Again it should be noticed that the operator matrices are to be applied on the (total) quantities on the right-hand sides of them as far as indicated by parentheses.

12.2 LARGE DISPLACEMENTS

12.2.1 Principle of virtual work

The governing equilibrium equations are in the body

$${}^0\rho\boldsymbol{b} + \frac{\partial \boldsymbol{T}^j}{\partial a_j} = \boldsymbol{0} \qquad \text{in } {}^0V \tag{1}$$

and on the body surface

$$\boldsymbol{T} = {}^0n_j\boldsymbol{T}^j \qquad \text{on } {}^0S. \tag{2}$$

These are exactly of the same form as equations (1) and (2) in Section 12.1, only the lower case traction symbols replaced here with the capital ones. Repeating the manipulation with the weighting function

$$\boldsymbol{w}(\boldsymbol{a}) = w_i(\boldsymbol{a})\boldsymbol{i}_i \tag{3}$$

produces thus the counterpart of equation (6) in Section 12.1:

$$\int_{{}^0V} {}^0\rho \boldsymbol{b} \cdot \boldsymbol{w}\, \mathrm{d}^0V + \int_{{}^0S} \boldsymbol{T} \cdot \boldsymbol{w}\, \mathrm{d}^0S - \int_{{}^0V} \boldsymbol{T}^j \cdot \frac{\partial \boldsymbol{w}}{\partial a_j}\, \mathrm{d}^0V = 0. \tag{4}$$

The interpretation

$$\begin{aligned} \boldsymbol{w} = \delta\boldsymbol{r} = \delta\boldsymbol{u} &= \delta u_k \boldsymbol{i}_k = \delta u_1 \boldsymbol{i}_1 + \delta u_2 \boldsymbol{i}_2 + \delta u_3 \boldsymbol{i}_3 \\ &= \delta u \boldsymbol{i} + \delta v \boldsymbol{j} + \delta w \boldsymbol{k} \end{aligned} \tag{5}$$

leads to the *principle of virtual work* (statics) for a continuum in the case of large displacements:

$$\boxed{\int_{{}^0V} {}^0\rho \boldsymbol{b} \cdot \delta\boldsymbol{u}\, \mathrm{d}^0V + \int_{{}^0S} \boldsymbol{T} \cdot \delta\boldsymbol{u}\, \mathrm{d}^0S - \int_{{}^0V} \boldsymbol{S} : \delta\boldsymbol{E}\, \mathrm{d}^0V = 0.} \tag{6}$$

Similarly as in Section 12.2, this can be written

$$\delta' W \equiv \delta' W_{\text{ext}} + \delta' W_{\text{int}} = 0 \tag{7}$$

with

$$\begin{aligned} \delta' W_{\text{ext}} &= \int_{{}^0V} {}^0\rho \boldsymbol{b} \cdot \delta\boldsymbol{u}\, \mathrm{d}^0V + \int_{{}^0S} \boldsymbol{T} \cdot \delta\boldsymbol{u}\, \mathrm{d}^0S, \\ \delta' W_{\text{int}} &= -\int_{{}^0V} \boldsymbol{S} : \delta\boldsymbol{E}\, \mathrm{d}^0V. \end{aligned} \tag{8}$$

The double-dot product is in rectangular cartesian coordinates

$$\begin{aligned} \boldsymbol{S} : \delta\boldsymbol{E} = S_{ij}\delta E_{ij} = {} & S_{11}\delta E_{11} + S_{12}\delta E_{12} + S_{13}\delta E_{13} + \\ & + S_{21}\delta E_{21} + S_{22}\delta E_{22} + S_{23}\delta E_{23} + \\ & + S_{31}\delta E_{31} + S_{32}\delta E_{32} + S_{33}\delta E_{33}. \end{aligned} \tag{9}$$

The terms

$$\begin{aligned} \delta E_{ij} &= \delta \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right) \\ &= \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_j}{\partial a_i} + \frac{\partial \delta u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j} + \frac{\partial u_k}{\partial a_i}\frac{\partial \delta u_k}{\partial a_j}\right) \end{aligned} \tag{10}$$

are virtual Green strains. The detailed derivation concerning internal virtual work is performed in Example 12.2.

Example 12.2. We consider the term

$$\boldsymbol{T}^j \cdot \frac{\partial \boldsymbol{w}}{\partial a_j} = \boldsymbol{T}^j \cdot \frac{\partial \delta\boldsymbol{u}}{\partial a_j} \tag{a}$$

in (4). We have

$$\boldsymbol{T}^j = S_{ji}\boldsymbol{G}_i, \quad \boldsymbol{G}_i = \boldsymbol{i}_i + \frac{\partial u_l}{\partial a_i}\boldsymbol{i}_l, \quad \delta\boldsymbol{u} = \delta u_k \boldsymbol{i}_k. \tag{b}$$

The first and second forms in (b) are given in (11.2.7') and (10.2.23'). Thus

$$\begin{aligned} \boldsymbol{T}^j \cdot \frac{\partial \delta\boldsymbol{u}}{\partial a_j} &= S_{ji}\boldsymbol{G}_i \cdot \frac{\partial \delta u_k}{\partial a_j}\boldsymbol{i}_k = S_{ji}\frac{\partial \delta u_k}{\partial a_j}\boldsymbol{G}_i \cdot \boldsymbol{i}_k = S_{ji}\frac{\partial \delta u_k}{\partial a_j}\left(\boldsymbol{i}_i \cdot \boldsymbol{i}_k + \frac{\partial u_l}{\partial a_i}\boldsymbol{i}_l \cdot \boldsymbol{i}_k\right) \\ &= S_{ji}\frac{\partial \delta u_k}{\partial a_j}\left(\delta_{ik} + \frac{\partial u_l}{\partial a_i}\delta_{lk}\right) = S_{ji}\frac{\partial \delta u_k}{\partial a_j}\left(\delta_{ik} + \frac{\partial u_k}{\partial a_i}\right) \\ &= S_{ji}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_k}{\partial a_j}\frac{\partial u_k}{\partial a_i}\right) \\ &= \frac{1}{2}S_{ij}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_k}{\partial a_j}\frac{\partial u_k}{\partial a_i}\right) + \frac{1}{2}S_{ji}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_k}{\partial a_j}\frac{\partial u_k}{\partial a_i}\right) \\ &= \frac{1}{2}S_{ij}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_k}{\partial a_j}\frac{\partial u_k}{\partial a_i}\right) + \frac{1}{2}S_{ij}\left(\frac{\partial \delta u_j}{\partial a_i} + \frac{\partial \delta u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right) \\ &= S_{ij}\frac{1}{2}\left(\frac{\partial \delta u_i}{\partial a_j} + \frac{\partial \delta u_j}{\partial a_i} + \frac{\partial \delta u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j} + \frac{\partial u_k}{\partial a_i}\frac{\partial \delta u_k}{\partial a_j}\right). \end{aligned} \tag{c}$$

The steps used should be rather obvious. Symmetry of the Kirchhoff stress has been made use of. The expression for the Green strain from (10.2.36) is

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right). \tag{d}$$

Its variation δE_{ij} is seen to appear in (c) and thus

$$\boldsymbol{T}^j \cdot \frac{\partial \delta\boldsymbol{u}}{\partial a_j} = S_{ij}\delta E_{ij}. \tag{e}$$

Remark 12.5. We repeat here the discussion of Remark 12.1. If we let the weighting function $\boldsymbol{w}$ remain finite and take no specific interpretation, we have from formula (c) of Example 12.2 the result

$$\boldsymbol{T}^j \cdot \frac{\partial \boldsymbol{w}}{\partial a_j} = S_{ij}\frac{1}{2}\left(\frac{\partial w_i}{\partial a_j} + \frac{\partial w_j}{\partial a_i} + \frac{\partial w_k}{\partial a_i}\frac{\partial u_k}{\partial a_j} + \frac{\partial u_k}{\partial a_i}\frac{\partial w_k}{\partial a_j}\right). \tag{11}$$

The large displacement theory is seen to be present via the current displacement state $\boldsymbol{u}$. We again see that there is no absolute need to consider $\boldsymbol{w}$ as infinitesimal and as the variation of $\boldsymbol{u}$. If we consider $\boldsymbol{w}$ to be the current finite displacement $\boldsymbol{u}$, we obtain

$$\Pi_P(u,v) = \Pi(u,v) + \frac{1}{2}\int_A \alpha\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 \, dA + \\ + \frac{1}{2}\int_{S_u} [\alpha_u(u-\bar{u})^2 + \alpha_v(v-\bar{v})^2] ds, \qquad (74)$$

respectively. Here $\alpha(x,y)$, $\alpha_u(s)$, $\alpha_v(s)$ are given functions (normally constants).

D.2.7 Finding a variational principle

We have considered the Euler-Lagrange equations and natural boundary conditions above as the consequency of the stationarity condition. These are always arrived at in principle easily by performing certain mathematical manipulations. To proceed in the opposite direction is more difficult: Is there a variational principle equivalent (and what is the functional and what are the essential boundary conditions) to a given set of differential equations and boundary conditions? This problem is dealt with for instance in Finlayson and Sriven (1967). We do not consider it here. It is usually, however, well documented in the literature, which kind of problems are amenable to the variational treatment. If a variational principle is not available, the numerical solution must be based on a suitable weak form.

Even in the case where a variational principle is available it seems that when numerical methods are used, it is convenient to perform first the variation in analytical form to obtain in fact a weak form and only then apply the discretization. So the most convenient and consistent starting point is always a weak form and we do not actually care if there is a variational principle somewhere in the background. This theme is treated more in Section 2.10.

D.2.8 Least-squares functional

Let us consider the set of differential equations

$$\underset{n\times 1}{\{R(\{\phi\})\}} = \underset{n\times 1}{\{0\}} \qquad (75)$$

in a domain Ω concerning certain functions ϕ with certain boundary conditions on the boundary Γ. We write a functional

$$\Pi(\{\phi\}) = \frac{1}{2}\int_\Omega \{R\}^T[\alpha]\{R\} d\Omega \qquad (76)$$

where $[\alpha]$ is a positive definite $n \times n$ matrix which may be called the *weight factor matrix* (painotekijämatriisi). First, the multiplier $1/2$ in front of the integral is included just for aesthetic reasons. Second, the weight factor matrix can be taken symmetric without loss of generality (See Example D.2). Third, the

elements in the weight factor matrix must naturally have such physical dimensions that the whole expression remains physically dimensionally homogeneous. The admissible ϕ are assumed to satisfy in advance the boundary conditions.

It is obvious why (76) is called the *least-squares functional* (pienimmän neliön funktionaali). It is also obvious that when the functional obtains a stationary value — which is actually a minimum (when $[\alpha]$ is taken to be positive definite) and has the value zero — the admissible functions are the solution of the differential equations (75).

A least-squares functional with its stationary principle can apparently thus be formed without difficulty for any set of differential equations — linear or nonlinear — and the approach seems at the outset to be very promising. However, in practice there are serious disadvantages. The admissible functions must have higher continuity than in the corresponding weak forms. All the boundary conditions must be satisfied in advance. It is difficult to find a logic how to select the elements of the weight factor matrix. If the boundary conditions are not satisfied in advance (in strong sense) they can obviously be appended to the functional (76) as an additional least-squares integral over the boundary but then again the question of suitable weights arises.

Here we have introduced the concept of the least-squares functional mainly because it can be used advantageously as a "sensitizing" appended term to a conventional functional to enhance the discrete solution behaviour. This is considered in Section D.2.9.

Remark D.6. The fact that the stationarity of the least-squares functional is not really a "proper" variational principle is seen by looking at the corresponding Euler-Lagrange equations. It is found that they are not directly the original differential equations of the problem but some differentiated forms of them (See Example D.12). □

Remark D.7. In the applications of the least-squares method in the literature, often new unknowns are introduced into the formulation so that the order of derivatives appearing in the equations get lower and less continuity is needed from the approximations in discrete solutions (See example D.12). The problem with the selection of the proper values for the weight factors, however, remains and in fact gets more difficult. □

Example D.12. We consider the simple stretched string problem studied in Example D.7 starting now from the differential equation

$$R(v) \equiv -Sv'' - q = 0, \qquad 0 < x < l \qquad (a)$$

with the boundary conditions

$$v(0)=0, \quad v(l)=0. \tag{b}$$

(The left-hand side of the differential equation can naturally be written with an arbitrary sign change without any effect on the final results; the minus sign used here is selected for aesthetic reasons appearing later.) The least-squares functional corresponding to (a) is

$$\Pi(v)=\frac{1}{2}\int_0^l R^2\,dx=\frac{1}{2}\int_0^l(-Sv''-q)^2\,dx \tag{c}$$

where the admissible $v(x)$ has to satisfy conditions (b). No weight factor is needed as there is only one differential equation. The variation is

$$\begin{aligned}\delta\Pi &= \int_0^l(-Sv''-q)(-S\delta v'')\,dx=\int_0^l S(Sv''+q)\delta v''\,dx\\ &=-\int_0^l[S(Sv''+q)]'\delta v'\,dx+\Big|_0^l S(Sv''+q)\delta v'\\ &=\int_0^l[S(Sv''+q)]''\delta v\,dx-\Big|_0^l[S(Sv''+q)]'\delta v+\Big|_0^l S(Sv''+q)\delta v'\\ &=\int_0^l[S(Sv''+q)]''\delta v\,dx+\Big|_0^l S(Sv''+q)\delta v'.\end{aligned} \tag{d}$$

Due to the essential boundary conditions (b), $\delta v(0)=0$, $\delta v(l)=0$, and the corresponding terms above vanish. From (d), the Euler-Lagrange differential equation is (S is constant)

$$(Sv''+q)''=0, \quad 0<x<l \tag{e}$$

and the natural boundary conditions are

$$(Sv''+q)_{x=0}=0, \quad (Sv''+q)_{x=l}=0. \tag{f}$$

Thus, in fact, the boundary conditions are the original differential equation evaluated at the ends and the Euler-Lagrange equation is a fourth order differential equation obtained by differentation from the original second order equation.

The second derivative of v appears in expression (c). To use (c) as the starting point for a discrete solution would demand a C^1 -continuous approximation. This can be avoided at the price of introducing a new unknown function $u \equiv v'$. Now the problem reads

$$\begin{aligned}R_1 &\equiv -Su'-q=0, \quad 0<x<l\\ R_2 &\equiv u-v'=0, \quad 0<x<l\end{aligned} \tag{g}$$

with the boundary conditions

$$v(0)=0, \quad v(l)=0. \tag{h}$$

The least-squares functional would be

$$\Pi(v,u)=\frac{1}{2}\int_0^l\begin{Bmatrix}R_1\\R_2\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\alpha_{11}&\alpha_{12}\\\alpha_{21}&\alpha_{22}\end{bmatrix}\begin{Bmatrix}R_1\\R_2\end{Bmatrix}dx \tag{i}$$

where the admissible $v(x)$ has to satisfy (h). We face the problem of selecting the elements of the weight factor matrix. Actually, only the ratios between the elements are essential. Usually the matrix is taken for lack of better logic as diagonal and if we further put arbitrarily $\alpha_{11}=1$, we end with the expression

$$\Pi(v,u)=\frac{1}{2}\int_0^l\begin{Bmatrix}R_1\\R_2\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}1&0\\0&\alpha_{22}\end{bmatrix}\begin{Bmatrix}R_1\\R_2\end{Bmatrix}dx=\frac{1}{2}\int_0^l[(-Su'-q)^2+\alpha_{22}(u-v')^2]dx. \tag{j}$$

Now it would be enough to have a C^0 - continuous approximation for v and u. Some numerical experimentation could hopefully give a clue to select the proper value for α_{22} for a certain discrete formulation.

D.2.9 Sensitized functional

Courant has presented in two articles from years 1923 and 1943 a formulation where a conventional variational principle is "sensitized" by appending the variational expression with terms of higher order which vanish for the actual solution. This powerful formulation has found practical use with finite elements only quite recently.

We quote from Courant (1943): "These facts which are intimately related to more profound questions in the general theory of the variational calculus have suggested the following method for obtaining better convergence in the Rayleigh-Ritz method. Instead of considering the simple variational problem for the corresponding boundary value problem, we modify the former problem without changing the solution of the latter. This is accomplished by adding to the original variational expression terms of higher order which vanish for the actual solution u. For example, we may formulate the equilibrium problem for a membrane under the external pressure f as follows:

$$I(v)=\iint_B(v_x^2+v_y^2+vf)dxdy+\iint_B k(\Delta v-f)^2\,dxdy=\min.,$$

where k is an arbitrary positive constant or function. Such additional terms make $I(v)$ more *sensitive* to the variations of v without changing the solution. In other words, minimizing sequences attached to such a "sensitized" functional will by force behave better as regards convergence [7].

The practical value of the method of sensitizing the integral by the addition of terms of higher order has not yet been sufficiently explored. Certainly the sensitizing terms will lead to a more complicated system of equations for the c_i. This means that a compromise must be made for a suitable choice of the arbitrary positive function k so that good convergence is assured while the necessary labor is kept within bounds."

We further quote from Courant (1923): "Zur Erläuterung behandeln wir die Randwertaufgabe der Potentialtheorie für einen Bereich G in der xy-Ebene. Die vorgegebenen Randwerte mögen identisch sein mit den Werten, die ein Polynom $p(x,y)$ auf dem Rand annimmt. Der Rand von G möge abgesehen von endlich vielen Ecken eine sich stetig drehende Tangente besitzen. Wir betrachen das Integral

(1) $D[\varphi]=\int\int_G\{\varphi_x^2+\varphi_y^2+(\Delta\varphi)^2+(\Delta\varphi_x)^2+(\Delta\varphi_y)^2+(\Delta\varphi_{xx})^2+\cdots\}dxdy,$

wobei rechts über alle in Frage kommenden Ableitungen zu summieren ist, und fordern, das Integral $D[\varphi]$ zum Minimum zu machen, wenn zum Vergleiche alle in G mit ihren Ableitungen stetigen Funktionen φ zugelassen werden, welche die vorgeschriebenen Randwerte besitzen."

Remark D.8. In the formula of the quotation from Courant (1943) there is obviously a slight misprint and there should probably read

$$I(v)=\iint_B(v_x^2+v_y^2+2vf)dxdy+\iint_B k(\Delta v-f)^2dxdy=\min., \tag{77}$$

Remark D.9. From the formula of the quotation from Courant (1923) it is apparent that the presentation there is assumed to be in a dimensionless form as otherwise the total expression would not be dimensionally homogeneous. □

Let us assume that corresponding to the set of differential equations (75) with some boundary conditions we have a functional $\Pi(\{\phi\})$ with a variational principle $\delta\Pi=0$. In other words, the governing differential equations are the Euler-Lagrange differential equations of the variational principle. We proceed the way suggested by Courant and write a *sensitized functional* (sensitoitu funktionaali)

$$\begin{aligned}\Pi_s(\{\phi\})=\Pi(\{\phi\})&+\frac{1}{2}\int_\Omega\{R\}^T[\tau]^{(0)}\{R\}d\Omega+\\&+\frac{1}{2}\int_\Omega\frac{d}{dx}\{R\}^T[\tau]^{(1)}\frac{d}{dx}\{R\}d\Omega+\\&+\cdots\end{aligned} \tag{78}$$

where the $n\times n$ matrices $[\tau]^{(0)}$, $[\tau]^{(1)}$, ⋯ can be called *sensitizing parameter matrices* (sensitointiparametrimatriisi). In addition to an appended least-squares functional a so called *gradient least-squares functional* (gradientti pienimmän neliön funktionaali) has been appended etc. It is realized that by differentiating the governing differential equations with respect to the independent variables we can produce new differential equations which can again be included by the least squares method into the formulation. This is apparent from the quotation above from Courant (1923) In (78) we have assumed a one-dimensional case for simplicity of presentation. Also, as remarked in the quotation from Courant (1943), the appended terms vanish for the actual solution. Of course, we must have some logic to determine suitable values for the sensitizing parameters. This is considered in the connection of the finite element method in Section 13.3. In any case, we are no more just "at the mercy" of the pure conventional variational principle. We have now available the possibility to try to steer the

discrete solution in the direction we want by suitable selection of the values of the sensitizing parameters.

Example D.13. We consider the stretched string problem studied in Example D.7 and D.12 and formulate a sensitized variational principle.

The governing differential equation is

$$R(v)\equiv -Sv''-q=0,\qquad 0<x<l. \tag{a}$$

Differing from the earlier cases, we now take for demonstration purposes the non-homogeneous boundary conditions

$$v(0)=v_0, \tag{b}$$

$$Sv'(l)=V_l \tag{c}$$

where v_0 and V_l are given quantities. Thus, at the left-hand end the Dirichlet type boundary condition fixes the vertical displacement of the support. The physical content of the Neumann type boundary condition at the right-hand end tells that a vertical force of magnitude V_l acts there (positive when acting in the positive direction of v) in addition to the horizontal string force S. (Small displacement theory is assumed.) The potential energy functional corresponding to the problem is

$$V(v)=\int_0^l[\frac{1}{2}S(v')^2-qv]dx-V_l\,v(l). \tag{d}$$

The admissible $v(x)$ has to satisfy the essential boundary condition (b). The natural boundary condition (c) is found to be a consequence of the variational principle. The reader is suggested to find this by performing the variation.

The sensitized variational principle is thus (only the least-squares sensitizing term is included here)

$$\begin{aligned}V_s(v)&=V(v)+\frac{1}{2}\int_0^l R\tau\,Rdx\\&=\int_0^l[\frac{1}{2}S(v')^2-qv]dx-V_l\,v(l)+\frac{1}{2}\int_0^l\tau(-Sv''-q)^2dx.\end{aligned} \tag{e}$$

Only one sensitizing parameter τ appears. It should be noticed that the parameter must again have such a physical dimension that the whole expression (e) remains dimensionally homogeneous.

D.2.10 Obtaining a weak form

Section 12.1 contains some discussion of the concept of a weak form in connection with the principle of virtual work. We repeat here some themes discussed earlier but now in a more general context so the terminology is not necessarily the same as earlier with the principle of virtual work. We here first explain how a *weak form can always be generated from a variational principle.* The steps needed are just: write the stationarity condition, call the variations of the argument functions weight functions and use new notation for them. This is explained in Example D.14.

Example D.14. We consider the stretched string problem studied in Example D.13.

The governing differential equation is

$$R(v) \equiv -Sv'' - q = 0, \qquad 0 < x < l \tag{a}$$

and the boundary conditions are

$$v(0) = v_0, \tag{b}$$

$$Sv'(l) = V_l. \tag{c}$$

The corresponding functional is

$$V(v) = \int_0^l [\frac{1}{2} S(v')^2 - qv]\,\mathrm{d}x - V_l\, v(l) \tag{d}$$

where the admissible $v(x)$ must satisfy the essential boundary condition (b).

The variational principle gives the equation

$$\begin{aligned} \delta V &\equiv \int_0^l [Sv'\delta v' - q\delta v]\,\mathrm{d}x - V_l\,\delta v(l) \\ &= \int_0^l [(\delta v)' Sv' - \delta v q]\,\mathrm{d}x - \delta v(l) V_l = 0. \end{aligned} \tag{e}$$

We now denote

$$\delta v \equiv w, \tag{f}$$

call w weighting function and write (e) again:

$$\int_0^l (w'Sv' - wq)\,\mathrm{d}x - w(l)V_l = 0. \tag{g}$$

From the essential boundary condition follows that $\delta v(0) = 0$ so the weighting function has also to satisfy

$$w(0) = 0. \tag{h}$$

Equation like (g) is called a weak form and it is a ready starting point for a discrete solution, in fact, a more convenient starting point than the corresponding functional. For instance, it is seen that here in (g) the unknown function is present only linearly but in (d) also quadratic terms appear.

Finally, if we had started from the differential equation (a) with the signs changed, obviously the weak form (g) would have emerged also with different signs. If we would have preferred for aesthetic reason just the present outlook (g), we could have naturally changed the signs at the end.

If a variational principle is not available, a weak form is produced typically as follows. Let us consider the set (75) or written in more detail the set

$$\begin{aligned} &R_1(\phi_1, \phi_2, \cdots, \phi_n) = 0, \\ &R_2(\phi_1, \phi_2, \cdots, \phi_n) = 0, \\ &\cdots \qquad\qquad \text{in } \Omega \\ &R_n(\phi_1, \phi_2, \cdots, \phi_n) = 0 \end{aligned} \tag{79}$$

with some boundary conditions on Γ. Functions $\phi_1(x)$, $\phi_2(x)$, $\cdots$, $\phi_n(x)$ are the unknowns to be determined and x refers to the generic point in $\overline{\Omega} = \Omega \cup \Gamma$. The left-hand sides of the differential equations are called often *residuals* (residuaali, jäännös) in the literature. When the differential equations are satisfied the residuals vanish. We multiply (both sides of) the differential equations by *arbitrary* functions $w_1(x)$, $w_2(x)$, $\cdots$, $w_n(x)$ — called *weighting functions* (painofunktio) — integrate (both sides of) the resulting equations and add (both sides of) the resulting equations to obtain finally *one scalar equation*

$$\boxed{\int_\Omega \{w\}^{\mathrm{T}} \{R\}\,\mathrm{d}\Omega = 0} \tag{80}$$

or in more detail

$$\int_\Omega (w_1 R_1 + w_2 R_2 + \cdots + w_n R_n)\,\mathrm{d}\Omega = 0. \tag{81}$$

This is called a *weak form* (heikko muoto) of the problem. (The differential equations (79) are said in this connection to present the problem in a strong form.) As the weighting functions in the weak form are arbitrary, we see by applying the fundamental lemma of variational calculus (Section D.2.3) that the *satisfaction of the weak form for any arbitrary set of weighting functions gives as a consequence the satisfaction of the differential equations* (See remark D.10). It is thus understandable that we can start the discrete solutions instead of directly from the differential equations also from the weak form and indeed this latter approach has many advantages.

We have not yet discussed the boundary conditions. To apply form (80) would demand that the functions to be determined would have to satisfy in advance (in a strong sense) all the boundary conditions. However, *in practice nearly always integrations by parts are performed in the preliminary weak form* so that some higher derivatives in the unknown functions vanish and some derivatives appear in the weighting functions. Boundary terms emerge in the integrations by parts operations and very often some information from the boundary conditions can be fed into the weak form through them so that finally only part of the boundary conditions have to be satisfied in advance and the rest of the boundary conditions are satisfied implicitly (in a weak sense). Also here it is usual to speak about essential and natural boundary conditions with an obvious meaning. We do not attempt to write down a general form of the weak form manipulated by integration by parts etc. Also this resulting form is still called a weak form. The particular examples treated later give indication of the steps used in the manipulations.

Remark D.10. In mathematical texts, careful description of the properties and smoothness of the unknown functions and the weighting functions have to be discussed to be able to draw conclusions about the equivalence of the strong and

weak formulation of a problem. This is outside the scope of this text. However, in the discrete methods we are mainly interested in, some rather simple rules are usually enough to give us converging solutions. □

Remark D.11. The weighting functions must naturally have such physical dimensions between them that the total left hand side of a weak form is dimensionally homogeneous. □

Example D.15. The string problem of Example D.14 is used again. We assume now that we are not aware of the existence of a variational principle and start from the strong form

$$R(v) \equiv -Sv'' - q = 0, \quad 0 < x < l \tag{a}$$

with the boundary conditions

$$v(0) = v_0, \tag{b}$$

$$Sv'(l) = V_l. \tag{c}$$

The preliminary weak form is

$$\begin{aligned} &\int_0^l w(-Sv'' - q)\,dx = 0, \\ &\int_0^l (-wSv'' - wq)\,dx = 0. \end{aligned} \tag{d}$$

As there is only one differential equation, the weighting function $w(x)$ can have here any physical dimension. Integration by parts of the first term in the integral gives the form

$$\begin{aligned} &\int_0^l [(wS)'v' - wq]\,dx - \Big|_0^l wSv' = 0, \\ &\int_0^l [(w'Sv' - wq)\,dx - w(l)Sv'(l) + w(0)Sv'(0) = 0. \end{aligned} \tag{e}$$

It has been taken into account that that S is a constant. At $x = 0$, v' is not given. We can avoid its appearance in the weak form by restricting the weighting function to disappear at $x = 0$, i.e. we take $w(0) = 0$. At $x = l$, v' (or directly Sv') is given. We substitute this information from boundary condition (c) into the weak form. The final weak form is thus

$$\int_0^l (w'Sv' - wq)\,dx - w(l)V_l = 0. \tag{f}$$

It is identical to the form obtained via another route in Example D.14 (formula (g)). As boundary condition (b) has not yet been taken into account, it must be satisfied in advance in the the weak form: it is an essential boundary condition. Information about boundary condition (c) was introduced into the weak form so this boundary condition is natural. Finally, the weighting function must satisfy the condition

$$w(0) = 0. \tag{g}$$

D.2.11 Sensitized weak form

Courant presented his sensitizing idea in connection with variational principles. By starting from this point of view and performing the variation, the basis to

obtain a sensitized weak form in connection with any standard weak form can be easily detected.

Let us write equations (79) now in the form

$$\begin{aligned} &R_1(\phi_1, \phi_2, \cdots, \phi_n) \equiv L_1(\phi_1, \phi_2, \cdots, \phi_n) - f_1 = 0, \\ &R_2(\phi_1, \phi_2, \cdots, \phi_n) \equiv L_2(\phi_1, \phi_2, \cdots, \phi_n) - f_2 = 0, \\ &\cdots \qquad\qquad \text{in } \Omega \\ &R_n(\phi_1, \phi_2, \cdots, \phi_n) \equiv L_n(\phi_1, \phi_2, \cdots, \phi_n) - f_n = 0 \end{aligned} \tag{82}$$

where the symbols L_1, L_2, $\cdots$, L_n refer to *linear differential operators* and the given quantities f_1, f_2, $\cdots$, f_n are often called *source terms* (lähdetermi). For instance, the differential equation (a) in Example D.15 can be written as

$$R(v) \equiv L(v) - f \equiv -S\frac{d^2v}{dx^2} - q = 0 \tag{83}$$

so here

$$L(v) = -S\frac{d^2v}{dx^2}, \qquad f = q. \tag{84}$$

By the notation $L_1(\phi_1, \phi_2, \cdots, \phi_n)$, etc. we just mean that we have some given rules by which the functions ϕ_1, ϕ_2, $\cdots$, ϕ_n are operated on by the operators.

We have assumed that the differential equations are here linear. The sensitizing terms can be appended clearly as such also in non-linear cases. To proceed in more detail we, however, assume linear (or linearized) behaviour in the following.

The sensitized functional considered in Section D.2.9 was written as

$$\begin{aligned} \Pi_s(\{\phi\}) = \Pi(\{\phi\}) &+ \frac{1}{2}\int_\Omega \{R\}^T[\tau]^{(0)}\{R\}d\Omega + \\ &+ \frac{1}{2}\int_\Omega \frac{d}{dx}\{R\}^T[\tau]^{(1)}\frac{d}{dx}\{R\}d\Omega + \\ &+ \cdots \end{aligned} \tag{85}$$

The variational equation obtained from the stationarity of (85) is

$$\begin{aligned} \delta\Pi_s \equiv \delta\Pi &+ \int_\Omega \{L(\{\delta\phi\})\}^T[\tau]^{(0)}\{R(\{\phi\})\}d\Omega + \\ &+ \int_\Omega \frac{d}{dx}\{L(\{\delta\phi\})\}^T[\tau]^{(1)}\frac{d}{dx}\{R(\{\phi\})\}d\Omega + \\ &+ \cdots = 0. \end{aligned} \tag{86}$$

We show in detail how the first varied sensitized term is arrived at:

$$\delta\left(\frac{1}{2}\int_\Omega \{R\}^{\mathrm{T}}[\tau]^{(0)}\{R\}\mathrm{d}\Omega\right)=\frac{1}{2}\int_\Omega \delta\{R\}^{\mathrm{T}}[\tau]^{(0)}\{R\}\mathrm{d}\Omega+$$
$$+\frac{1}{2}\int_\Omega \{R\}^{\mathrm{T}}[\tau]^{(0)}\delta\{R\}\mathrm{d}\Omega$$
$$=\int_\Omega \delta\{R\}^{\mathrm{T}}[\tau]^{(0)}\{R\}\mathrm{d}\Omega=\int_\Omega \{L(\{\delta\phi\})\}^{\mathrm{T}}[\tau]^{(0)}\{R(\{\phi\})\}\mathrm{d}\Omega. \tag{87}$$

Rules of variational calculus — such as those shown in Table D.1 — have been used. Further, the symmetry of the sensitizing parameter matrix has been taken into account. Finally, it is realized that again by the rules of variational calculus

$$\delta\{R(\{\phi\})\}=\{L(\{\delta\phi\})\}. \tag{88}$$

As a simple example from (83),

$$\delta R(v)=\delta\left(-S\frac{\mathrm{d}^2v}{\mathrm{d}x^2}-q\right)=-S\frac{\mathrm{d}^2\delta v}{\mathrm{d}x^2}=L(\delta v). \tag{89}$$

The second varied sensitized term can be explained similarly. Now we make the interpretation and notational change

$$\{\delta\phi\}=\{w\} \tag{90}$$

and write (86) in the form

$$\boxed{F+F^{(0)}+F^{(1)}+\cdots=0} \tag{91}$$

where

$$F\equiv\delta\Pi,\quad\text{(more generally, the left – hand side of the standard weak form)} \tag{92}$$

$$F^{(0)}\equiv\int_\Omega \{L(\{w\})\}^{\mathrm{T}}[\tau]^{(0)}\{R(\{\phi\})\}\mathrm{d}\Omega, \tag{93}$$

$$F^{(1)}\equiv\int_\Omega \frac{\mathrm{d}}{\mathrm{d}x}\{L(\{w\})\}^{\mathrm{T}}[\tau]^{(1)}\frac{\mathrm{d}}{\mathrm{d}x}\{R(\{\phi\})\}\mathrm{d}\Omega, \tag{94}$$

...

Let us now forget the variational principle and consider that a standard weak form $F=0$ has been arrived at from the governing differential equations (82) by multiplying them with the weighting functions, integrating over the domain, integrating by parts in the usual way etc. A "least-squares weak form" $F^{(0)}=0$

is seen to be arrived at directly from the corresponding least-squares functional. Similarly "the gradient least-squares weak form" $F^{(1)}=0$ is seen to follow from the corresponding gradient least-squares functional etc. The *sensitized weak form* (sensitoitu heikko muoto) (91) can thus be interpreted as a *linear combination* (lineaarikombinaatio) of several weak forms. It contains free parameters by which we can again try to steer the discrete solution in the direction we want.

Remark D.12. When applying a sensitized formulation in connection with a variational principle, the sensitizing parameter matrices could be taken symmetric without loss of generality. In a sensitized weak form we can, on the contrary, finally alter the sensitizing matrices non-symmetric in the hope of achieving more modelling possibilities. It is seen, for instance, that in (93) the integrand consists just of a linear combination of differential equation residuals and it remains naturally zero for the exact solutions even in the case of a non-symmetric matrix. We at the same time now drop the interpretation that the weak form $F^{(0)}=0$ was arrived at starting from a least-squares functional. □

Remark D.13. Sensitized weak forms have gained lately much use in fluid mechanics with finite elements where earlier difficulties to cope with convection terms have disappeared by sensitizing. However, in fluid mechanics literature the term " stabilized" formulation is employed usually in this context. □

Remark D.14. The sensitizing terms are of the least-squares type in connection with a variational principle and consist in connection with a weak form in any case of the equation residuals. In Section D.2.8 the drawbacks (for instance the higher demand on continuity of the approximation with discrete methods) of the least-squares method were discussed. Fortunately, it will be found that at least when applying the finite element method these drawbacks to a great deal vanish. This is shortly because the standard formulation takes care of convergence when the mesh gets denser and the sensitizing terms are actually just needed to give accurate results already with coarse meshes. A kind of *patch test* (tilkkutesti) can be used to determine suitable values for the sensitizing parameters and it will be found that the parameter values tend to zero with the element sizes. This means among other things that some "crimes" against conventional rules given in the literature can be performed when evaluating the sensitizing terms. □

Example D.16. We employ the setting of Example D.15 with the only change that the string now rests on an elastic foundation having foundation modulus k.

The differential equation is

$$R(v)\equiv L(v)-q\equiv -Sv''+kv-q=0,\qquad 0<x<l \tag{a}$$

with the boundary conditions

$$v(0)=v_0. \tag{b}$$

$$Sv'(l) = V_l. \tag{c}$$

The elastic foundation means that the reactive force per unit length from the foundation on the string is of the form kv. This can be taken conveniently into account by performing the substitution

$$q := q - kv \tag{d}$$

in the differential equation (a) of Example D.15 where the elastic foundation was missing. (In (a), the positive direction of q has been taken to be the same as the positive direction of v.)

The potential energy functional is available also in this case and it is

$$V(v) = \int_0^l [\frac{1}{2}S(v')^2 + \frac{1}{2}kv^2 - qv]\mathrm{d}x - V_l v(l) \tag{e}$$

with the essential boundary condition (b). However, we pretend that we do not know this and we produce a weak form starting from (a), (b) and (c). The manipulations proceed similarly as in Example D.15 and we obtain

$$\int_0^l (w'Sv' + wkv - wq)\mathrm{d}x - w(l)V_l = 0 \tag{f}$$

with the essential boundary condition (b) and with the restriction $w(0) = 0$. As a check we can perform the variation in (e) to see readily the correctness of (f).

The sensitized weak form (91) with two sensitizing terms is

$$F + F^{(0)} + F^{(1)} = 0 \tag{g}$$

where here

$$F = \int_0^l (w'Sv' + wkv - wq)\mathrm{d}x - w(l)V_l \tag{h}$$

$$F^{(0)} = \int_0^l L(w)\tau^{(0)}R(v)\mathrm{d}x = \int_0^l (-Sw'' + kw)\tau^{(0)}(-Sv'' + kv - q)\mathrm{d}x, \tag{i}$$

$$F^{(1)} = \int_0^l \frac{\mathrm{d}}{\mathrm{d}x}L(w)\tau^{(1)}\frac{\mathrm{d}}{\mathrm{d}x}R(v)\mathrm{d}x = \int_0^l [-Sw''' + (kw)']\tau^{(1)}[-Sv''' + (kv)' - q']\mathrm{d}x. \tag{j}$$

D.3 DISCRETIZATION

As has been emphasized frequently earlier, we can base our calculations on a variational principle or more generally on a weak form and in fact a variational principle can be always reformulated as a weak form. A discrete version based on a weak form is called often as a *residual method* or a *weighted residual method* or a *residual formulation* (jäännösmenetelmä, painotettujen jäännösten menetelmä, jäännösformulaatio) in the literature. Same type of approximations is used both in variational formulations and in residual formulations. The only difference comes from the way of generating the system equations to determine the unknown parameters of the approximation.

D.3.1 Ritz method

The most common discretization procedure in connection with variational principles is the *Rayleigh-Ritz method* or shortly the *Ritz method* (Ritzin menetelmä), Ritz (1909). Let us consider first the case of only one independent variable x and only one dependent variable $\phi(x)$. The starting point is to assume an approximation of the type — called *trial solution* (yriteratkaisu, yrite)

$$\boxed{\tilde{\phi}(x) = a_j\varphi_j(x)} = a_1\varphi_1(x) + a_2\varphi_2(x) + \cdots + a_n\varphi_n(x). \tag{1}$$

where $\varphi_j(x)$ are *given* functions, called *coordinate functions, (trial) basis functions, trial functions* (koordinaattifunktio, kantafunktio, yritefunktio) and the multipliers a_j — called *indetermined parameters* (määräämätön parametri) — are unknown constants. Approximation (1) is mathematically a *linear combination* (lineaarikombinaatio) of the trial functions. It is often said that the trial solution is *spanned* (virittää) by the trial basis functions. This type of representation abounds in mathematics. For instance, Fourier series is one important example, the basis functions consisting of sines and cosines. After agreeing on a suitable form of (1), the only task left is to select "good" values for the unknown parameters a_j.

When a variational principle is available, approximation (1) is substituted into the corresponding functional $\Pi(\phi)$. After performing in principle the indicated integrations over Ω and Γ (it is not necessary and not efficient to actually do the integrations at this phase), *the functional becomes in fact an ordinary function*

$$\tilde{\Pi}(a_1, a_2, \cdots, a_n) = \Pi(\tilde{\phi}) \tag{2}$$

in the undetermined parameters. This relationship is indicated here shortly with the notation $\tilde{\Pi}(\{a\})$. The variational principle in the form of the stationarity condition $\delta\Pi = 0$ is transformed into the stationarity condition of the ordinary function $\tilde{\Pi}(\{a\})$:

$$\boxed{\frac{\partial \tilde{\Pi}}{\partial a_i} = 0,} \qquad i = 1,2,\cdots,n \tag{3a}$$

or shortly,

$$\boxed{\frac{\partial \tilde{\Pi}}{\partial \{a\}} = \{0\}.} \tag{3b}$$

These are the discrete equations, called in the following the *system equations* (systeemiyhtälö), from which the parameters a are determined. The principle of the Ritz method is contained in formulas (1) to (3).

Remark D.15. It is often useful to represent the unknown function in the form

$$\boxed{\phi(x) = \bar{\phi}(x) + \Delta\phi(x)} \tag{4}$$

where $\bar{\phi}(x)$ is a *given* function in $\bar{\Omega}$, a "smooth extension" of the boundary data from the essential boundary conditions and $\Delta\phi(x)$ is a *new unknown* function to be determined. First, this formulation has the advantage that as the non-homogeneous essential (linear) boundary conditions are satisfied by $\bar{\phi}$, the function $\Delta\phi$ has only to satisfy the essential conditions in the homogeneous form (= with zero right hand side). Second, in non-linear problems $\bar{\phi}$ may represent conveniently the initial solution guess or the current updated solution in an iterative procedure. We will call formulation (4) as the *deltaform* (deltamuoto) in the following. Using approximation (1), the essential boundary conditions must be satisfied; at least approximately. This means in fact that some of the parameters a are used for this purpose and are not included any more as free variables in equations (3). A modified version of approximation (1), corresponding to the deltaform (4), may be written as

$$\boxed{\tilde{\phi}(x) = \varphi_0(x) + a_j\varphi_j(x)} = \varphi_0(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \cdots + a_n\varphi_n(x). \tag{5}$$

Here $\varphi_0(x)$ is a given function satisfying the essential boundary conditions, in fact, we can have $\varphi_0 = \bar{\phi}$. This form of approximation will be called similarly the *deltaform approximation* in the following. □

In the general case, the functional will be of the type

$$\Pi(\{\phi\}) = \int_\Omega f(x,\{\phi\},\cdots)\,\mathrm{d}\Omega + \int_\Gamma g(x,\{\phi\},\cdots)\,\mathrm{d}\Gamma \tag{6}$$

where the meaning of the notations should be rather obvious, the dots referring to some lists of derivatives of the argument functions $\{\phi\}$. Let the approximations of the argument functions be of the types of (1) or (5) giving

approximatios $\{\tilde{\phi}\}$ and $\tilde{f} = f(x,\{\tilde{\phi}\},\cdots)$, $\tilde{g} = g(x,\{\tilde{\phi}\},\cdots)$. The system equations are, using more detail than in (3a),

$$\frac{\partial \tilde{\Pi}}{\partial a_i} \equiv \int_\Omega \frac{\partial \tilde{f}}{\partial a_i}\,\mathrm{d}\Omega + \int_\Gamma \frac{\partial \tilde{g}}{\partial a_i}\,\mathrm{d}\Gamma = 0, \qquad i = 1,2,\cdots \tag{7}$$

As the undetermined quantities a are parameters in the definite integrals, it is permissible to take the partial derivatives with respect to a inside the integrals. The detailed system equation form (7) is usually more convenient in practice than form (3a) where the integrals are supposed to be evaluated before the differentiations.

Remark D.16. A complicated question in connection of all numerical work is the possible convergence of the approximate solution $\{\tilde{\phi}\}$ to the exact solution $\{\phi\}$ measured in some norm as trial basis is enlarged without limit. In practice only a finite dimensional basis can be used and the resulting error is difficult to estimate and some "engineering judgment" is normally necessary. In this course we are satisfied to introduce the main approximate procedures and leave the detailed error analyses to more advanced presentations. In equilibrium problems some qualitative knowledge of the error can be obtained by simply comparing the fictitious loading corresponding to the approximate solution to the actual loading. This is demonstated in Example D.17. □

Example **D.17.** Figure (a) represent a stretched string on an elastic foundation under a point load F at the midpoint. We determine the deflection of the string by the Ritz method using the principle of stationary potential energy.

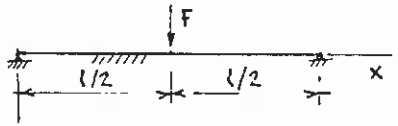


Figure (a)

The expression for the potential energy of the system is

$$V(v) = \int_0^l [\frac{1}{2}S(v')^2 + \frac{1}{2}kv^2]\mathrm{d}x - Fv(\frac{l}{2}). \tag{a}$$

where the main of the notations are clear from Example D.16. The deflections at the supports at the ends are given:

$$v(0) = \bar{v} \equiv \frac{1}{10}\frac{F}{kl}, \qquad v(l) = \bar{v} \equiv \frac{1}{10}\frac{F}{kl}. \tag{b}$$

These are the essential boundary conditions.

We take the approximation (5):

$$\tilde{v}(x) = \varphi_0(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + \cdots \tag{c}$$

Perhaps the simplest choice for function φ_0 is here the constant value

$$\varphi_0(x) = \bar{v} = \frac{1}{10}\frac{F}{kl}. \tag{d}$$

There is an infinite number of smooth extensions available and a very clever guess could in principle in the best case give directly the exact solution. The "delta part"

$$a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + \cdots \tag{e}$$

should now satisfy the homogeneous boundary conditions

$$v(0) = 0, \quad v(l) = 0. \tag{f}$$

This is achieved if each of the basis functions $\varphi_1, \varphi_2, \cdots$ separately satisfies these conditions. The sine functions

$$\varphi_1 = \sin\frac{\pi x}{l}, \quad \varphi_2 = \sin\frac{3\pi x}{l}, \quad \varphi_3 = \sin\frac{5\pi x}{l}, \cdots, \quad \varphi_j = \sin\frac{(2j-1)\pi x}{l}, \cdots \tag{g}$$

for example, depicted in Figure (b), are here one suitable possibility. The odd factors in the expansion have been selected in advance on the basis of the obvious symmetry of the problem.

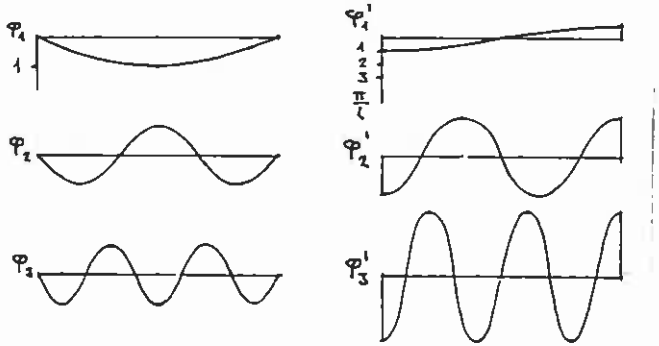


Figure (b)

The approximation is written shortly

$$\tilde{v}(x) = \varphi_0(x) + a_j\varphi_j(x) \tag{h}$$

and thus

$$\tilde{v}'(x) = \varphi_0'(x) + a_j\varphi_j'(x) \tag{i}$$

Functional (a) transforms into

$$\tilde{V}(\{a\}) = \int_0^l [\frac{1}{2}S(\varphi_0' + a_j\varphi_j')^2 + \frac{1}{2}k(\varphi_0 + a_j\varphi_j)^2]\mathrm{d}x + \\ - F[\varphi_0(\frac{l}{2}) + a_j\varphi_j(\frac{l}{2})]. \tag{j}$$

Equations (7) are

$$\frac{\partial\tilde{V}}{\partial a_i} \equiv \int_0^l [S(\varphi_0' + a_j\varphi_j')\varphi_i' + k(\varphi_0 + a_j\varphi_j)\varphi_i]\mathrm{d}x - F\varphi_i(\frac{l}{2}) \\ = \int_0^l (S\varphi_i'\varphi_j' + k\varphi_i\varphi_j)a_j\,\mathrm{d}x + \int_0^l (S\varphi_i'\varphi_0' + k\varphi_i\varphi_0)\mathrm{d}x - F\varphi_i(\frac{l}{2}) \\ = \int_0^l (S\varphi_i'\varphi_j' + k\varphi_i\varphi_j)\mathrm{d}x a_j + \int_0^l (S\varphi_i'\varphi_0' + k\varphi_i\varphi_0)\mathrm{d}x - F\varphi_i(\frac{l}{2}) \\ = K_{ij}a_j - b_i = 0, \tag{k}$$

that is, a system of linear equations

$$K_{ij}a_j = b_i \tag{l}$$

where

$$K_{ij} = \int_0^l (S\varphi_i'\varphi_j' + k\varphi_i\varphi_j)\mathrm{d}x, \\ b_i = F\varphi_i(\frac{l}{2}) - \int_0^l (S\varphi_i'\varphi_0' + k\varphi_i\varphi_0)\mathrm{d}x. \tag{m}$$

The steps used to obtain (k) should be obvious. Chain derivation is applied and it is again realized that the quantities a are parametrs with respect to integration and can be taken outside the integral sign.

It may be mentioned that (a) is an example of a *quadratic functional* (kvadraattinen funktionaali) — meaning that the argument functions and its derivatives appear at most quadratically in it. The Ritz method applied to a quadratic functional *always produces a linear system of equations* for the undetermined parameters. Further, the coefficient matrix — which is called here the *stiffness matrix* (jäykkyysmatriisi) — is then always *symmetric*. This is seen to be valid in (m). In the general case this is seen as follows. After discretization, a quadratic functional $\Pi(\{\phi\})$ contains the undetermined parameters a at most quadratically. Clearly, the term $K_{ij} = \partial(\partial\tilde{\Pi}(\{a\})/\partial a_i)\partial a_j$ and the term $K_{ji} = \partial(\partial\tilde{\Pi}(\{a\})/\partial a_j)\partial a_i$. As the differentiation order does not change the value of the derivative, $K_{ij} = K_{ij}$.

In (m), S is a constant and k is here assumed to be a constant. Continuing in more detail,

$$K_{ij} = S\frac{(2i-1)\pi}{l}\frac{(2j-1)\pi}{l}\int_0^l \cos\frac{(2i-1)\pi x}{l}\cos\frac{(2j-1)\pi x}{l})\mathrm{d}x + \\ + k\int_0^l \sin\frac{(2i-1)\pi x}{l}\sin\frac{(2j-1)\pi x}{l})\mathrm{d}x, \tag{n}$$

$$b_i = F\sin\frac{(2i-1)\pi}{2} - k\bar{v}\int_0^l \sin\frac{(2i-1)\pi x}{l}\mathrm{d}x.$$

Performing the integrations gives

$$K_{ij} = 0, \quad i \neq j, \\ K_{(i)(i)} = \frac{(2i-1)^2\pi^2}{2l}S + \frac{1}{2}kl, \\ b_i = F - \frac{2}{(2i-1)\pi}kl\bar{v}, \quad i \text{ is odd}, \tag{o} \\ b_i = -F - \frac{2}{(2i-1)\pi}kl\bar{v}, \quad i \text{ is even}.$$

The basis functions happen to be here orthogonal and the coefficient matrix is thus diagonal. For example in the case $\sqrt{k/S}\,l = 4$ and taking nine basis functions, the system of equations (l) are

$$kl\begin{bmatrix} 0.8084 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.276 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8.211 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 15.61 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 25.48 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 37.82 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 52.62 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 69.90 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 89.63 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{Bmatrix} = F\begin{Bmatrix} 0.9363 \\ -1.0212 \\ 0.9873 \\ -1.0091 \\ 0.9929 \\ -1.0056 \\ 0.9951 \\ -1.0042 \\ 0.9963 \end{Bmatrix}. \quad \text{(p)}$$

It is thus trivial to solve this for the parameters. After substituting them in (c) we have the deflection. If more terms are taken into the approximation this happens here conveniently as the earlier parameters a keep their old values unchanged.

The exact solution is available in this simple case. The differential equation is (See Example D.16)

$$-Sv'' + kv - q = 0 \qquad \text{(q)}$$

where here $q = 0$. At the midpoint the differential equation is not valid but must be replaced by the jump condition

$$(v')^- - (v')^+ = \frac{F}{S} \qquad \text{(r)}$$

concerning the left and right hand side limit values of the derivative. The exact solution is found to be

$$v = \{2\frac{\sinh(4x/l)}{\cosh 2} + \frac{1}{10}[\cosh(4x/l) - \tanh 2 \cdot \sinh(4x/l)]\}\frac{F}{kl}, \quad 0 \le x \le \frac{l}{2}. \qquad \text{(s)}$$

Due to symmetry, the solution on $l/2 \le x \le l$ is obtained by reflection about the point $x = l/2$. The exact and the nine parameter approximate solution are shown in Figure (c). The graphs are nearly identical except in the neighbourhood of the point load. It is understandable that here the smooth basis functions have difficulties in simulating the kink in the exact solution.

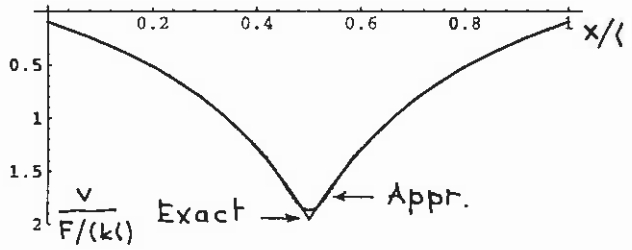


Figure (c)

At the midpoint $v(l/2) = 1.955\,F/(kl)$ and $\bar{v}(l/2) = 1.865\,F/(kl)$ so the error of the approximate solution is there about 4.6%. Figure (d) shows the approximate midpoint

deflection $\bar{v}(l/2)$ as a function of the number n of parameters in the approximation. The exact value is seen to be approached here from below. This behaviour under a single point load is in fact not accidental (strictly only in the case where the displacements of the supports would be zero) but is based on the fact that the principle of stationary potential energy is here a minimum principle.

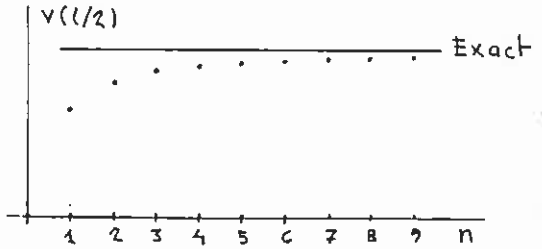


Figure (d)

Considering the differential equation (q) we can feed in the approximate $\bar{v}(x)$ and determine the fictitious lateral loading intensity $\bar{q}(x)$, necessary to keep the system in equilibrium:

$$\bar{q} = -S\bar{v}'' + k\bar{v}. \qquad \text{(t)}$$

This way of thinking has been commented on in Remark D.16. *For this loading the approximate solution is the exact one.* Figure (e) shows the loading in the case of nine basis functions considered above. This gives some idea of the nature of the solution. The value of integral

$$\int_0^l \bar{q}\,dx = 1.038F \qquad \text{(u)}$$

is not very far from the exact value F.

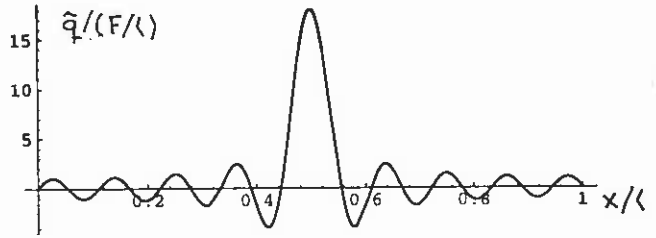


Figure (e)

It is obvious how the Ritz method can be applied in the case of several independent and dependent variables. For instance, say in the case treated in Example D.11, we could have the type of approximations

$$\begin{aligned} \bar{u}(x,y) &= \varphi_0^{(u)}(x,y) + a_j^{(u)}\varphi_j^{(u)}(x,y), \\ \bar{v}(x,y) &= \varphi_0^{(v)}(x,y) + a_j^{(v)}\varphi_j^{(v)}(x,y). \end{aligned} \qquad (8)$$

Supercripts (u) and (v) have been employed to discern the relevant quantities. The ranges for the summation index j can be in principle different for $\tilde{u}$ and $\tilde{v}$. Usually the same basis functions are used for all the dependent variables in which case we can write

$$\begin{aligned} \tilde{u}(x,y) &= \varphi_0^{(u)}(x,y) + a_j^{(u)}\varphi_j(x,y), \\ \tilde{v}(x,y) &= \varphi_0^{(v)}(x,y) + a_j^{(v)}\varphi_j(x,y). \end{aligned} \tag{9}$$

Remark D.17. If a sensitized variational principle is taken as the starting point for the discetization — we may then speak about a *sensitized Ritz method* (sensitoitu Ritzin menetelmä) — the system equations can be produced in principle exactly in the same way as explained above; evaluation of the least-squares type contributions just mean some extra calculations. Of course, some criterion must be available to select proper values for the sensitizing parameters. □

D.3.2. Galerkin method

In this section we consider shortly the discretization of weak forms, i.e. residual formulations. Again we explain the ideas for simplicity first in the case of only one independent variable x and only one dependent variable $\phi(x)$. The basic idea is extremely simple and general. The trial solution $\tilde{\phi}(x)$ (expression (1) or in case of the deltaform, expression (5)) is substituted for $\phi(x)$. It is realized that the field equation and the boundary conditions cannot in general be satisfied exactly for any selection of the values of the undetermined parameters. *The parameters can be selected, however, so that the equations are satisfied in some average, integral sense through satisfying the weak form with respect to some suitable weighting functions.* This procedure produces the system equations from which the parameters can be determined.

Different versions of the residual formulation are obtained according to type of finite dimensional weighting functions used. The most common versions are the *Galerkin method* (Galerkinin keino), the *subdomain method* or subdomain collocation (osa-aluekeino), *collocation* or point collocation (kollokaatio). These are explained very clearly in Crandall (1956). The *least-squares method* as a discrete form of the least-squares functional stationarity principle can be also be interpreted as a residual method.

In the Galerkin method the weighting functions are taken from the set of trial basis functions. (What is meant by this in a general case of several unknown functions with different type of approximations is not necessarily quite obvious.) The Galerkin method is in practice by far the most important version. Sometimes the Galerkin method is called the *Bubnov-Galerkin method* and as its opposite, if the weighting functions are not from the set of trial functions, the

name *Petrov-Galerkin method* is used. In fluid mechanics finite element applications, for example, some efforts to overcome problems with dominant convection have been based on using so called *upwinding* (ylävirtapainotus) which means a kind of Petrov-Galerkin method.

Remark D.18. When using the variational formulation as the basis for the discretization — and thus the Ritz method — after the applier has decided on the approximation, there is nothing else to decide on (of course in a sensitized principle the sensitizing parameter values); the system equations follow by turning the handle in a prescribed way. In a residual formulation, on the contrary, there exists in addition the decision on the type of weighting functions to be used. This clearly increases the discrete modelling capabilities but it also makes things more complicated for the applier. In practice the Galerkin method is a good compromise and when it is applied it in connection with a sensitized weak form — then it may be called a *sensitized Galerkin method* (sensitoitu Galerkinin keino) — increased modelling capabilities come through the sensitizing parameter values. In the following, we sometimes use the terminology *standard Galerkin method* when the Galerkin method is used in its pure classical form as opposed to the sensitized form. □

How one proceeds in detail in the Galerkin method and the generalization to other cases should become clear from the following example.

Example D.18. We look at Example D.17 now from the corresponding weak form point of view using the Galerkin method.

The weak form is

$$\int_0^l (w'Sv' + wkv)\,dx - w(\tfrac{l}{2})F = 0 \tag{a}$$

with

$$v(0) = \bar{v} = \frac{1}{10}\frac{F}{kl}, \qquad v(l) = \bar{v} = \frac{1}{10}\frac{F}{kl} \tag{b}$$

as essential boundary conditions and with the restrictions $w(0) = 0$ and $w(l) = 0$. These can be obtained immediately from the variational presentation of Example D.17 in the way explained in Example D.14. If we start alternatively from the differential equation, when employing integration by parts, care must be exercised to take into account the discontinuity in the first derivative of v due to the point load.

We take the same approximation

$$\tilde{v}(x) = \varphi_0(x) + a_j\varphi_j(x) \tag{c}$$

as in Example D.17. When this is substituted in (a), the left-hand side is transformed into

$$\int_0^l [w'S(\varphi_0' + a_j\varphi_j') + wk(\varphi_0 + a_j\varphi_j)]\,dx - w(\tfrac{l}{2})F. \tag{d}$$

The system equations

$$F_i = 0, \quad i = 1,2,\cdots \tag{e}$$

using the Galerkin method, are obtained by taking consecutively $w = \varphi_1$, $w = \varphi_2$, $\cdots$ and by demanding the discrete weak form to be valid:

$$F_i \equiv \int_0^l [\varphi_i' S(\varphi_0' + a_j \varphi_j') + \varphi_i k(\varphi_0 + a_j \varphi_j)]\mathrm{d}x - \varphi_i(\tfrac{l}{2})F = 0, \quad i = 1,2,\cdots \tag{f}$$

Further manipulation produces exactly the same linear system of equations

$$K_{ij} a_j = b_i \tag{g}$$

as in Example D.17. It is in fact a general result that application of the Ritz method with a variational principle is equivalent to the application of the Galerkin method with the corresponding weak form. However, as stressed earlier, the Galerkin method can be applied in connection with weak forms even in those cases where there exist no corresponding variational principles.

D.3.3 Kantorovitch method

In some applications the independent variables are of distinctly different character. In a bridge, the coordinates in a cross-sectional direction and the coordinate in the spanwise direction have clearly a different role. The most obvious example is the difference in character between the space coordinates and the time coordinate in a dynamics problem. It is then quite natural to let this kind of situation to affect the type of approximation used. As an example, let us consider a time dependent displacement field in two space dimensions. A consistent generalization of (8) would be

$$\begin{aligned} \bar{u}(x,y,t) &= \varphi_0^{(u)}(x,y,t) + a_j^{(u)} \varphi_j^{(u)}(x,y,t), \\ \bar{v}(x,y,t) &= \varphi_0^{(v)}(x,y,t) + a_j^{(v)} \varphi_j^{(v)}(x,y,t), \end{aligned} \tag{10}$$

that is, the basis functions are defined in the xyt-space and the undetermined parameters a are again *unknown constants*. An alternative, giving the time a different role would be

$$\begin{aligned} \bar{u}(x,y,t) &= \varphi_0^{(u)}(x,y,t) + a_j^{(u)}(t) \varphi_j^{(u)}(x,y), \\ \bar{v}(x,y,t) &= \varphi_0^{(v)}(x,y,t) + a_j^{(v)}(t) \varphi_j^{(v)}(x,y). \end{aligned} \tag{11}$$

Here the undetermined parameters a are *unknown functions* of time. This kind of representation applied in connection with variational principles is called the *Kantorovitch method* (Kantorovitchin menetelmä). More generally, a representation like (11) is called for obvious reasons as *semidiscretization* (semidiskretointi, osittainen diskretointi).

When a semidiscrete form like (11) is substituted into a corresponding functional and integration over (in this case) the space coordinates is performed there remains a transformed approximate functional in the form of a definite

integral with t (here) as the independent variable containing several arguments functions $a(t)$. This is the case considered already in Section D.2.4. The stationarity condition gives several Euler-Lagrange ordinary differential equations and possible natural boundary conditions. This problem must usually be solved in practice again by some numerical method.

Example D.19. We consider the problem of vibration of a stretched string on the x-axis (Figure (a)).

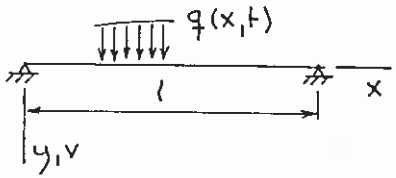


Figure (a)

Some of the relevant notations have appeared earlier. The solution domain in the xt-plane is shown in Figure (b). The differential equation formulation of the problem consists of the partial differential equation

$$\rho \frac{\partial^2 v}{\partial t^2} - S \frac{\partial^2 v}{\partial x^2} - q = 0 \tag{a}$$

with the boundary conditions

$$v(0,t) = \bar{v}_0(t), \qquad v(l,t) = \bar{v}_l(t) \tag{b}$$

and the initial conditions

$$v(x,0) = g(x), \qquad \frac{\partial v}{\partial t}(x,0) = h(x). \tag{c}$$

Here $\rho(x)$ is the mass density (per unit length) of the string. The dynamic differential equation is obtained from the corresponding static one just by the substitution $q := q - \rho \partial^2 v / \partial t^2$ and by some obvious change in notation. The quantities $\bar{v}_0(t)$, $\bar{v}_l(t)$, $g(x)$, $h(x)$ in the boundary and initial conditions are given functions.

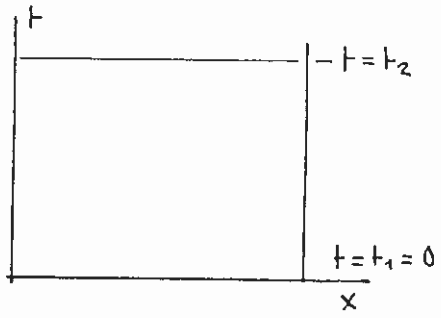


Figure (b)

An alternative is to start from the Hamilton's principle applied already in Example D.9. The functional is

$$\Pi(v)=\int_{t_1}^{t_2}\{\int_0^l[\frac{1}{2}\rho(\frac{\partial v}{\partial t})^2-\frac{1}{2}S(\frac{\partial v}{\partial x})^2+qv]\mathrm{d}x\}\mathrm{d}t. \tag{d}$$

If q depends on time, it is no more conservative but however monogenic (See Section 4.1.3) and Hamilton's principle is still valid. The essential boundary conditions for the argument function $v(x,t)$ consist first of all of the boundary conditions (b) similarly as would be the case in a static problem using the principle of stationary potential energy. Further, the argument function must satisfy the first of (c) at $t=t_1=0$ and a similar condition at $t=t_2$ where the condition is is fact not available but nothing prevents us for the time being to imagine that we know it. At the end we find that we do not actually need this condition to solve the problem.

We take a representation like (11):

$$\tilde{v}(x,t)=\varphi_0(x,t)+\varphi_j(x)a_j(t). \tag{e}$$

Function φ_0 could be for instance a linear interpolation in the x-direction between the values $\bar{v}_0$ and $\bar{v}_l$:

$$\varphi_0(x,t)=(1-x/l)\cdot\bar{v}_0(t)+x/l\cdot\bar{v}_l(t). \tag{f}$$

One convenient selection for the basis functions φ_j is

$$\varphi_j(x)=\sin\frac{j\pi x}{l}. \tag{g}$$

This choice makes the representation (e) to satisfy the boundary conditions (b).

From (e),

$$\frac{\partial\tilde{v}}{\partial x}=\frac{\partial\varphi_0}{\partial x}+\frac{\mathrm{d}\varphi_j(x)}{\mathrm{d}x}a_j(t)=\frac{\partial\varphi_0}{\partial x}+\varphi_j'a_j \tag{h}$$

and

$$\frac{\partial\tilde{v}}{\partial t}=\frac{\partial\varphi_0}{\partial t}+\varphi_j(x)\frac{\mathrm{d}a_j(t)}{\mathrm{d}t}=\frac{\partial\varphi_0}{\partial t}+\varphi_j\dot{a}_j. \tag{i}$$

Substituting these into the functional produces first

$$\Pi(\{a\})=\int_{t_1}^{t_2}\{\int_0^l[\frac{1}{2}\rho(\frac{\partial\varphi_0}{\partial t}+\varphi_j\dot{a}_j)(\frac{\partial\varphi_0}{\partial t}+\varphi_k\dot{a}_k)+$$
$$-\frac{1}{2}S(\frac{\partial\varphi_0}{\partial x}+\varphi_j'a_j)(\frac{\partial\varphi_0}{\partial x}+\varphi_k'a_k)+q(\varphi_0+\varphi_ja_j)]\mathrm{d}x\}\mathrm{d}t. \tag{j}$$

It should be noticed that for instance in the expression $(\partial\tilde{v}/\partial t)^2=(\partial\tilde{v}/\partial t)(\partial\tilde{v}/\partial t)$ we must use different letters for the summation indices to avoid errors. Performing the integrations with respect to x and taking into account that a_j and $\dot{a}_j$ do not depend on x we arrive at

$$\Pi(\{a\})=\int_{t_1}^{t_2}[\frac{1}{2}(\int_0^l\rho\varphi_j\varphi_k\mathrm{d}x)\dot{a}_j\dot{a}_k-\frac{1}{2}(\int_0^lS\varphi_j'\varphi_k'\mathrm{d}x)a_ja_k+$$

$$+(\int_0^lq\varphi_j\mathrm{d}x)a_j+(\int_0^l\rho\frac{\partial\varphi_0}{\partial t}\varphi_j\mathrm{d}x)\dot{a}_j-(\int_0^lS\frac{\partial\varphi_0}{\partial x}\varphi_j'\mathrm{d}x)a_j+c]\mathrm{d}t. \tag{k}$$

Shorthand notation c refers to those terms not containing functions a and $\dot{a}$. After performing the integration with respect to x, there remains at most the dependence on t. Functional is of the type (D.2.53). If orthogonal basis functions such as (g) are used, (k) simplifies. We do not continue into the details. Constant (c with respect to the argument functions) has no effect on the Euler-Lagrange equations.

After deriving the Euler-Lagrange differential equations we can forget the imaginary boundary condition at the future time $t=t_2$ and start to make use of both of the initial conditions (c). With a finite number of basis functions we cannot in general satisfy these conditions exactly. A residual formulation, for instance, can be used here. As an example, a least squares formulation is described. We form the integral

$$I(\{a(0)\})=\frac{1}{2}\int_0^l[\tilde{v}(x,0)-g(x)]^2\mathrm{d}x$$
$$=\frac{1}{2}\int_0^l[\varphi_0(x,0)+\varphi_j(x)a_j(0)-g(x)]^2\mathrm{d}x. \tag{l}$$

Demanding this to be stationary with respect to $a_j(0)$ gives a linear set

$$K_{ij}a_j(0)=b_i \tag{m}$$

with

$$K_{ij}=\int_0^l\varphi_i\varphi_j\mathrm{d}x,$$
$$b_i=\int_0^l[\varphi_ig-\varphi_i\varphi_0(x,0)]\mathrm{d}x \tag{n}$$

from which the $a_j(0)$ can be determined. A similar formulation can be used with respect to the second boundary condition starting from

$$I(\{\dot{a}(0)\})=\frac{1}{2}\int_0^l[\frac{\partial\tilde{v}}{\partial t}(x,0)-h(x)]^2\mathrm{d}x$$
$$=\frac{1}{2}\int_0^l[\frac{\partial\varphi_0}{\partial t}(x,0)+\varphi_j(x)\dot{a}_j(0)-h(x)]^2\mathrm{d}x. \tag{o}$$

This gives the $\dot{a}_j(0)$. Thereafter the pure initial value problem with the given initial values $a_j(0)$ and $\dot{a}_j(0)$ can be solved in one or the other way.

Semidiscretization can be applied naturally also in connection of a weak form. The following example gives an illustration.

Example D.20. A weak form corresponding to case of vibration of a stretched string considered in the previous example is

$$\int_0^l(\frac{\partial w}{\partial x}S\frac{\partial v}{\partial x}+w\rho\frac{\partial^2v}{\partial t^2}-wq)\mathrm{d}x=0 \tag{a}$$

with the essential boundary conditions

$$v(0,t)=\bar{v}_0(t),\qquad v(l,t)=\bar{v}_l(t) \tag{b}$$

and with the restrictions

$$w(0,t)=0, \quad w(l,t)=0. \tag{c}$$

Equation (a) is arrived at in the same way as explained for instance in Example D.15. This is in accordance with the semidiscretization to follow: the solution domain is in space and time but we here concentrate first on a fixed moment of time and integrate just over the space coordinate to obtain the weak form. Similarly, we do not care about the initial conditions at all yet.

Approximation

$$\tilde{v}(x,t)=\varphi_0(x,t)+\varphi_j(x)a_j(t) \tag{d}$$

is substituted into (a) and the left-hand side transformes into

$$\int_0^l [\frac{\partial w}{\partial x} S(\frac{\partial \varphi_0}{\partial x}+\varphi_j' a_j)+w\rho(\frac{\partial^2 \varphi_0}{\partial t^2}+\varphi_j \ddot{a}_j)-wq]\mathrm{d}x. \tag{e}$$

The system equations

$$F_i(t)=0, \quad i=1,2,\cdots \tag{f}$$

using the Galerkin method, are obtained by taking consecutively $w=\varphi_1$, $w=\varphi_2$, $\cdots$ and by demanding the discrete weak form to be valid:

$$F_i(t)\equiv \int_0^l [\varphi_i' S(\frac{\partial \varphi_0}{\partial x}+\varphi_j' a_j)+\varphi_i \rho(\frac{\partial^2 \varphi_0}{\partial t^2}+\varphi_j \ddot{a}_j)-\varphi_i q]\mathrm{d}x=0, \quad i=1,2,\cdots \tag{g}$$

When these are manipulated further in detail, we find that the have obtained a linear system of ordinary differential equations with time as the independent variable:

$$M_{ij}\ddot{a}_j + K_{ij}a_j = b_i \tag{h}$$

where

$$\begin{aligned} M_{ij} &= \int_0^l \rho\varphi_i \varphi_j \mathrm{d}x, \\ K_{ij} &= \int_0^l S\varphi_i' \varphi_j' \mathrm{d}x, \\ b_i(t) &= \int_0^l \varphi_i q\,\mathrm{d}x - \int_0^l S\varphi_i' \frac{\partial \varphi_0}{\partial x}\mathrm{d}x - \int_0^l \rho\varphi_i \frac{\partial^2 \varphi_0}{\partial t^2}\mathrm{d}x. \end{aligned} \tag{i}$$

It is seen that the right-hand side of (h) can depend on time through q and φ_0. Matrix with the elements M_{ij} is called the *mass matrix* (massamatriisi). It is seen to be symmetric. This is clearly due to the Galerkin way of selecting the discrete weighting functions. The initial conditions on $a_j(t)$ can be obtained the way explained in Example D.19.

It is obvious that by starting from the weak form we can arrive at the discrete equations with less effort than by using Hamilton's principle.

Remark D.19. In dynamics, a procedure familiar from particle mechanics to generate the equations of motion is the application of Lagrange's equations (Section 5.6). We can alternatively rely directly on this method when faced with a continuum problem. After applying semidiscretization and by then

interpreting the undetermined parameters, say, $a_j^{(u)}(t)$ and $a_j^{(v)}(t)$, in equations like (11) to be generalized coordinates $q(t)$ we are at the position to apply Lagrange's equations of motion. We just first form the discretized expressions for the kinetic energy of the system and for the generalized forces. There is then actually no need even to see the corresponding continuum equations. Naturally, system equations for equilibrium problems are obtained also as special cases of Lagrange's equations with zero kinetic energy. □

Example D.21. We consider the problem treated in Examples D.19 and D.20 once more now making use Lagrange's equations of motion.

The kinetic energy of the string is

$$T=\frac{1}{2}\int_0^l \rho(\frac{\partial v}{\partial t})^2\mathrm{d}x \tag{a}$$

and strain energy (potential energy of the internal forces) is

$$V_{\mathrm{int}}=\frac{1}{2}\int_0^l S(\frac{\partial v}{\partial x})^2\mathrm{d}x \tag{b}$$

We deal with the distributed loading here by considering its virtual work

$$\delta' W_{\mathrm{ext}}=\int_0^l q\delta v\,\mathrm{d}x. \tag{c}$$

The approximation

$$\tilde{v}(x,t)=\varphi_0(x,t)+\varphi_j(x)a_j(t) \tag{d}$$

is introduced. Expressions (a), (b) and (c) are transformed into

$$\begin{aligned} \tilde{T} &= \frac{1}{2}\int_0^l \rho(\frac{\partial \varphi_0}{\partial t}+\varphi_j \dot{a}_j)(\frac{\partial \varphi_0}{\partial t}+\varphi_k \dot{a}_k)\mathrm{d}x \\ &= \frac{1}{2}(\int_0^l \rho\varphi_j\varphi_k \mathrm{d}x)\dot{a}_j\dot{a}_k + (\int_0^l \rho\frac{\partial \varphi_0}{\partial t}\varphi_j \mathrm{d}x)\dot{a}_j + \frac{1}{2}\int_0^l \rho(\frac{\partial \varphi_0}{\partial t})^2\mathrm{d}x, \end{aligned} \tag{e}$$

$$\begin{aligned} \tilde{V}_{\mathrm{int}} &= \frac{1}{2}\int_0^l S(\frac{\partial \varphi_0}{\partial x}+\varphi_j' a_j)(\frac{\partial \varphi_0}{\partial x}+\varphi_k' a_k)\mathrm{d}x \\ &= \frac{1}{2}(\int_0^l S\varphi_j'\varphi_k' \mathrm{d}x)a_j a_k + (\int_0^l S\frac{\partial \varphi_0}{\partial x}\varphi_j' \mathrm{d}x)a_j + \frac{1}{2}\int_0^l S(\frac{\partial \varphi_0}{\partial x})^2\mathrm{d}x, \end{aligned} \tag{f}$$

$$\delta' \tilde{W}_{\mathrm{ext}}=\int_0^l q(\varphi_j \delta a_j)\mathrm{d}x = (\int_0^l q\varphi_j\,\mathrm{d}x)\delta a_j. \tag{g}$$

It is seen that some terms appearing in Hamilton's principle in Example D.19 naturally emerge also here. It is realized that the variation δv in its discretized form $\delta\tilde{v}$ is restricted to the variations δa_j of the undetermined parameters.

We write further

$$\tilde{T} = \frac{1}{2} M_{jk} \dot{a}_j \dot{a}_k + (\int_0^l \rho \frac{\partial \varphi_0}{\partial t} \varphi_j \, dx) \dot{a}_j + \frac{1}{2} \int_0^l \rho (\frac{\partial \varphi_0}{\partial t})^2 dx,$$

$$Q_i^{int} \equiv -\frac{\partial \tilde{V}_{int}}{\partial a_i} = -K_{ij} a_j - \int_0^l S \frac{\partial \varphi_0}{\partial x} \varphi_i' \, dx, \qquad \text{(h)}$$

$$\delta \tilde{W}_{ext} = Q_j^{ext} \delta a_j$$

with the mass matrix and stiffness matrix elements defined earlier and with

$$Q_j^{ext} = \int_0^l q \varphi_j \, dx. \qquad \text{(i)}$$

It can now be said that "the system can not see the difference if it is the result of a discretization or if an original finite degree of freedom problem is represented." We obtain by applying formulas of Section 5.6 the system equations as the Lagrange's equations

$$Q_i^{ext} + Q_i^{int} = \frac{d}{dt} \frac{\partial \tilde{T}}{\partial \dot{a}_i}, \quad i = 1,2,\cdots \qquad \text{(j)}$$

Writing these in detail, it is found that equations (h) of Example D.20 are reproduced.

D.3.4 Finite element method

Introduction. In the immediately following presentation we assume some basic knowledge of the finite element method from the reader. We later elaborate in more detail on the subject. In the *finite element approximation* (elementti-approksimaatio) the basis functions are defined in a *piecewise manner*. Let us consider Figure D.6 showing schematically two typical basis functions of the classical Ritz method and of the finite element method in a simple two-dimensional rectangular domain with one dependent function $\phi(x,y)$.

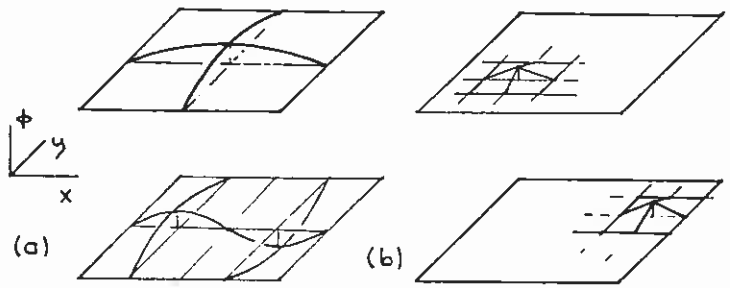


Figure D.6 (**a**) Two basis functions of the classical Ritz method. (**b**) Two typical basis functions of the finite element method.

In both methods the approximation is mathematically still a linear combination of the basis functions (we leave here out the possible $\varphi_0(x,y)$ part in the approximation; see Remark D.15):

$$\tilde{\phi}(x,y) = a_j \varphi_j(x,y), \qquad (12)$$

the difference is in the character of the basis functions used. In the Ritz method — or to emphasize the old original form of the Ritz method we might call it the *classical Ritz method* — the basis functions are *smooth functions in general non-zero in the whole domain* Ω. In contrast, in the finite element method, the basis functions are due to their piecewise definition *non-smooth and they are non-zero only in a small subdomain of* Ω. Further, an additional difference is in the interpretation of the parameters a. *In the classical Ritz method they usually have no clear physical meaning; in the finite element method they are the values of the function to be determined at certain points*, called nodal points or nodes. Using the conventional notation established in the finite element literature, we can write (12) in a the form

$$\boxed{\tilde{\phi}(x,y) = N_j(x,y)\phi_j} \qquad (13)$$

where ϕ_j are the nodal values and $N_j(x,y)$ are the basis functions, called usually shape functions.

In the classical Ritz method it is often very difficult to find suitable basis functions in complicated geometries satisfying the essential boundary conditions. By defining the basis functions in a piecewise — elementwise — manner, it is easy to satisfy the boundary conditions at least in an accurate approximate way.

The final system equations after discretization are in a linear problem of the linear type

$$K_{ij} a_j = b_i. \qquad (14)$$

both in the classical Ritz method and in the finite element method. If the basis functions have no special properties the system coefficient matrix becomes fully populated and the solution is expensive. It would be advantageous to have *orthogonal* basis functions satisfying

$$\int_\Omega \varphi_i f \varphi_j \, d\Omega = 0, \qquad i \neq j. \qquad (15)$$

If these type of relations are valid in all terms forming K_{ij}, we end up with a diagonal coefficient matrix which would be ideal from the computational point of view. (In Example D.17 this happened to be the case.). In two or more

dimensions it is usually not possible two achieve this in practice completely. However, the *finite element method automatically almost achieves this goal* . It is namely realized that

$$\int_\Omega N_i f N_j \, d\Omega = 0, \qquad \begin{array}{l}\text{when nodes } i \text{ and } j \text{ do not belong} \\ \text{together to any element in the mesh.}\end{array} \tag{16}$$

This is obvious from a look at Figure D.6(b). In a mesh of say hundreds or thousands of elements, two shape functions N_i an N_j are each non-zero in small subdomains consisting of just those elements having node i or j common, respectively. The non-zero parts thus seldom happen to overlap leading to most of the coefficients K_{ij} to have the value zero. We could thus say that the finite element shape functions are "*almost orthogonal*" basis functions. This is computationally extremly advantageous and one main reason for the practical success of the finite element method.

Finite element approximation. Due to the practical importance of the finite element method we describe concisely in some detail the main concepts and terminology of it.

Essential in the finite element method is its systematic way to approximate continuous functions by a *discrete model* (diskreetti malli). The model is generated by dividing the domain of the function under consideration in sub-domains or so called *finite elements* (elementti) (total number n_e). On the boundaries of the elements and often also inside them certain points, so called *nodal points* or shortly *nodes* (solmupiste, solmu) (total number n_n), are further selected. The resulting configuration is called the *element mesh* (elementti-verkko).

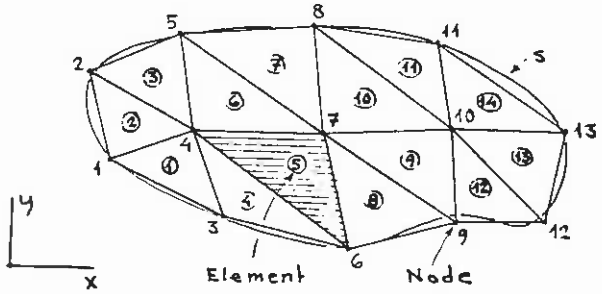


Figure D.7 Division of a two-dimensional domain in triangular elements ($n_e = 14$, $n_n = 13$).

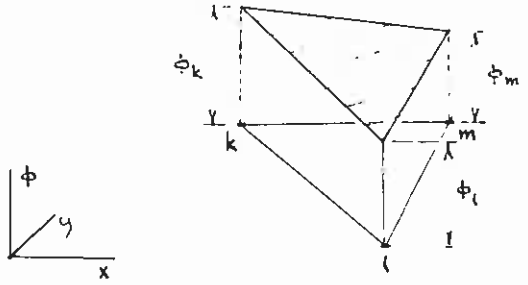
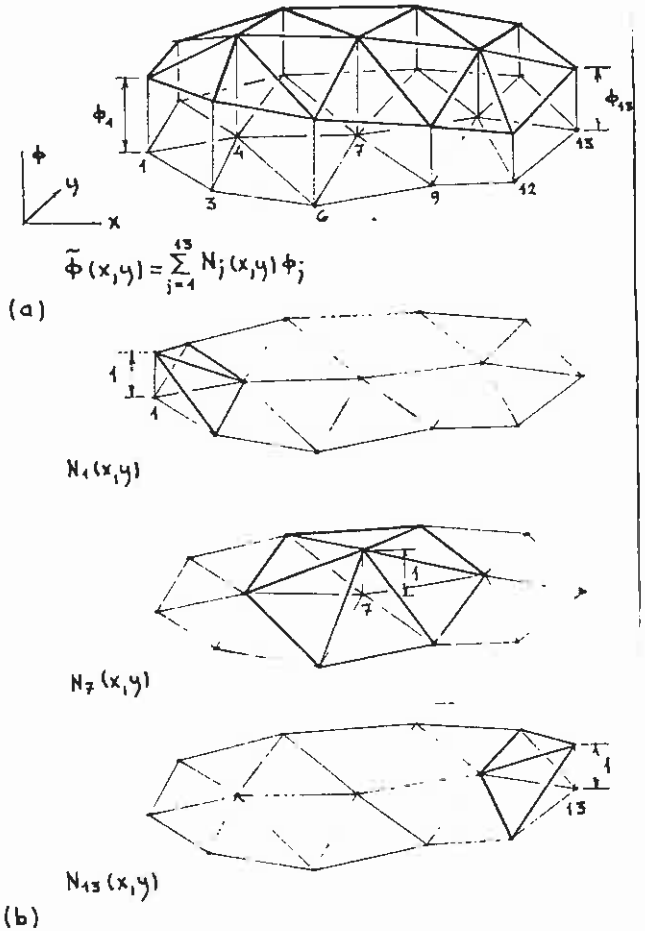


Figure D.8 Linear approximation of $\phi(x, y)$ in an element having nodes k, l, m.



$$\tilde{\phi}(x,y) = \sum_{j=1}^{13} N_j(x,y)\,\phi_j$$

Figure D.9 (a) Approximation $\tilde{\phi}$ of ϕ. **(b)** Three global shape functions.

The function is approximated in each element with simple functions — usually polynomials — by which it is interpolated inside the element employing its values at the nodes, the so called *nodal values* (solmuarvo).This procedure is illuminated in Figures D.7, D.8 and D.9 for a function $\phi(x,y)$ of two independent variables x and y.

We can realize with the help of Figure D.9 that it is possible to define interpolation functions or in the finite element terminology so called *shape functions* (muotofunktio) $N_i(x,y)$, $i=1,2,\cdots n_n$ so that the approximation in the whole domain can be expressed in the linear form (with respect to the nodal values)

$$\boxed{\bar{\phi}(x,y)=\sum_{i=1}^{n_n} N_j(x,y)\phi_j} = N_1(x,y)\phi_1 + N_2(x,y)\phi_2 + \cdots \qquad (17)$$

A shape function *obtains the value one at the node corresponding to its index and the value zero at all other nodes* and differs from zero at most in the elements connected to the node in question.

After a certain element mesh with its corresponding shape functions has been selected, according (1) *the approximation is wholly determined by fixing the discrete nodal values* ϕ_i.

The way of presentation (1) is suitable for theoretical considerations but in practical calculations these so called *global shape functions* (gobaali muotofunktio) N_j are not used. Namely in the domain of a certain element e, approximation (17) can be clearly given simply as

$$\boxed{\phi^e(x,y)=\sum_{i=1}^{n_n^e} N_j^e(x,y)\phi_j^e} = N_1^e(x,y)\phi_1^e + N_2^e(x,y)\phi_2^e + \cdots \qquad (18)$$

where the quantities N_j^e are so called *local* or *element shape functions* (lokaalinen muotofunktio, paikallinen muotofunktio, elementtimuotofunktio) which have been defined only in the domain of element e. (They coincide with the global shape functions in the element domain because the global shape functions are obtained in a piecewise manner from the local ones.) n_n^e is the total number of nodes of element e. The values $1,2,\cdots,n_n^e$ of index i refer to the so called *local node numbers* or local indices or node identifiers (sisäinen, paikallinen, lokaali solmunumero). At a local node r

$$\phi_r^e = \phi_i \qquad (19)$$

where i is the so called *global node number* (ulkoinen, globaali solmunumero) corresponding to the local node number r. In (19), we can similarly also speak about *local and global nodal values*. The global and local numbering of the nodes and also the numbering of the elements is performed normally starting from number 1 without "gaps". Figure D.10 describes the local shape functions in our example case in a generic element e.

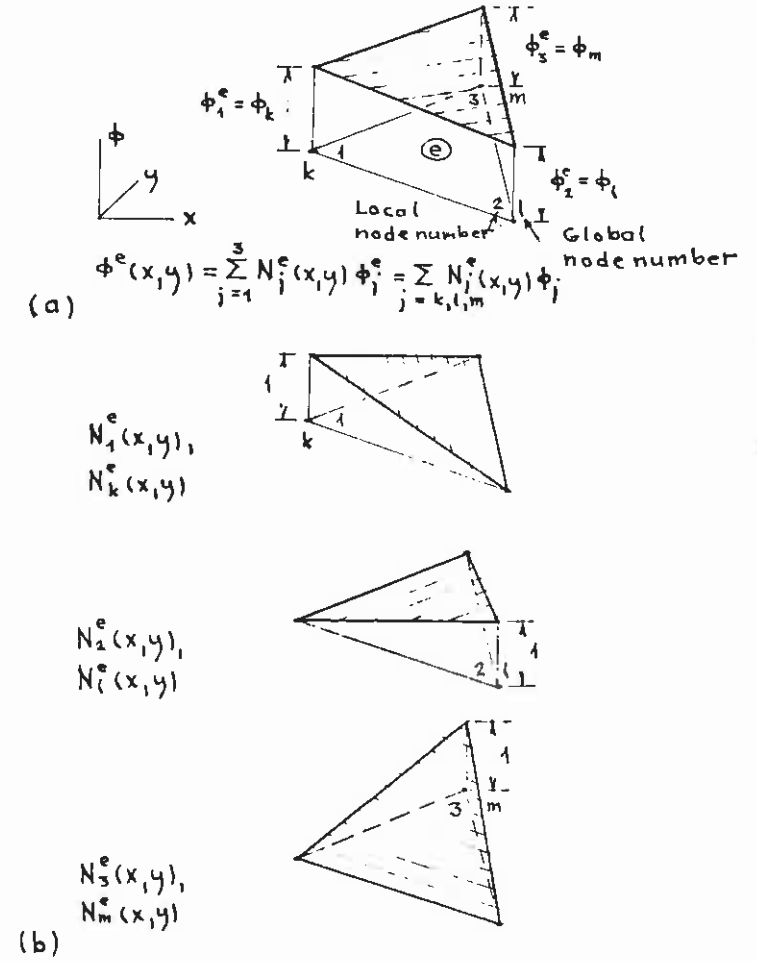


Figure D.10 (a) Approximation of ϕ in element e. **(b)** Element shape functions.

Let us consider as a specific example the element 5 of Figure. D.7. Let the local node numbering for it be that shown in Figure. D.11. Thus according to (18)

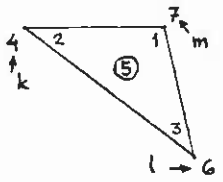


Figure D.11 Element 5.

$$\phi^5(x,y) = N_1^5(x,y)\phi_1^5 + N_2^5(x,y)\phi_2^5 + N_3^5(x,y)\phi_3^5 \tag{20}$$

and as additional information there exists the correspondence given in the following table

Local number r		Global number i
1	≙	7
2	≙	4
3	≙	6

(21)

This type of data for each element is enough to describe the connection between the global and local shape functions and between the global and local nodal values. For instance, we now know based on the table and equation (19) that

$$\phi_1^5 = \phi_7, \quad \phi_2^5 = \phi_4, \quad \phi_3^5 = \phi_6. \tag{22}$$

The element used in this demonstration example is called *three-noded* or *linear triangular element* (kolmisolmunen tai lineaarinen kolmioelementti). There are many kinds of elements well documented in the literature. We do not attempt to describe them at any length, some are considered in the applications.

System equations and the assembly process. It should again be emphasized that the finite element method applied in connection of a variational principle is just a version of the general Ritz method. Thus the system equations are still obtained from (3):

$$\frac{\partial \tilde{\Pi}}{\partial a_i} = 0. \tag{23}$$

Remark D.20. This far we have called the discrete unknowns in the finite element method as nodal values (solmuarvo). This is appropriate if we have only one unknown quantity per node, say the nodal value of the temperature. In more general situations we may have several unknowns per node, say two displacement component components, the temperature, etc. The discrete unknowns are numbered starting from number one (usually in some fashion

following the nodal numbering both for the mesh and for an element) and we shall call them here in general as *nodal parameters* (solmuparametri). A much used synonym in the literature is *degree of freedom* (vapausaste) but this is not very pertinent as the proper term from classical mechanics would be *generalized coordinate*; see Section 4.1.2. We often still use the general notation a for the nodal parameters as in (23). □

One basic property of the finite element method — not present in the classical Ritz method — is the following formula

$$\boxed{\int_\Omega (\cdot)\,\mathrm{d}\Omega = \sum_{e=1}^{n_e} \int_{\Omega^e} (\cdot)\,\mathrm{d}\Omega} \tag{24}$$

based clearly on the properties of the definite integral.This simple fact is fundamental in the finite element method: *the integral over Ω can be evaluated by evaluating separate integrals over the subdomains Ω^e of Ω and by summing the contributions.* (A similar statement can be given with respect to Γ.)

In the variational finite element method this means first that

$$\Pi = \sum_{e=1}^{n_e} \Pi^e \tag{25}$$

and second that

$$\tilde{\Pi} = \sum_{e=1}^{n_e} \tilde{\Pi}^e \tag{26}$$

and finally that

$$\frac{\partial \tilde{\Pi}}{\partial a_i} = \sum_{e=1}^{n_e} \frac{\partial \tilde{\Pi}^e}{\partial a_i}. \tag{27}$$

The meaning of the notations should be obvious. The last result means that *the left-hand sides of the system equations can be assembled by summation from separate element contributions*. Literature contains detailed bookkeeping rules for the *assembly process* (kokoamisprosessi) based on the connections between local and global nodal parameter numbers. In practice this phase is important for efficient calculations. However, in the small simple demonstration applications of this course we will generate the system equations without using any detailed assembly processes.

Example D.22. The problem of a stretched string on an elastic foundation of Example D.17 and D.18 is treated here by the finite element method.

Two-noded or *linear line elements* (kaksisolmuinen tai lineaarinen janaelementti) are used. The element shape functions are (Figure (a))

$$N_1^e = 1-\xi, \quad N_1^e = \xi \tag{a}$$

where $\xi = x/h$ $(0 \le \xi \le 1)$ is a dimensionless elementwise coordinate, x is a local elementwise coordinate measured from the left-hand end of the element (this double usage of the notation x should not cause confusion) and h is the length of the element.

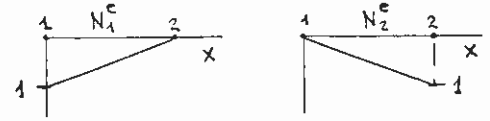


Figure (a)

The interval $[0,l]$ is discretized by eight equal length $h = l/8$ elements. The global numbering of the nodes is consecutively from left to right. Figure (b) shows the graphs of the first four global shape functions and their derivatives.

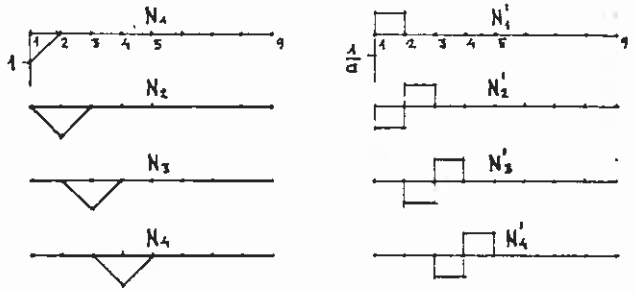


Figure (b)

The finite element approximation is

$$\bar{v}(x) \approx \sum_{j=1}^{9} N_j(x) v_j \,. \tag{b}$$

The nodal parameters v_j are here approximate nodal values of the deflection v. The essential boundary conditions $v(0) = v(l) = \bar{v}$ can be satisfied by taking

$$v_1 = v_9 = \bar{v} \approx \frac{1}{10}\frac{F}{kl}. \tag{c}$$

This is an example of the simplicity in the finite element method to deal with the essential boundary conditions. There are two ways to proceed in detail with respect to the boundary conditions. This is connected to Remark D.15. First, we can introduce a φ_0 - function (or using the current notation, a N_0 -function) as

$$\varphi_0(x) = N_1(x) v_1 + N_9(x) v_9 \,. \tag{d}$$

This is clearly the simplest possibility using the finite element representation. The corresponding graph is shown in Figure (c).

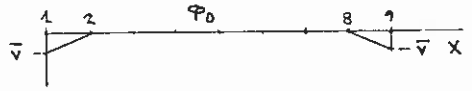


Figure (c)

The parameters v_1 and v_9 have been made use of for taking care of the boundary conditions and only the rest of the parametrs are free to be used in connection of the stationary condition. An alternative way to proceed, often employed in the literature and practice, is to "forget" the boundary conditions at this phase and to take them into account later. We proceed in this second fashion here.

The potential energy functional is

$$V(v) = \int_0^l [\frac{1}{2} S(v')^2 + \frac{1}{2} k v^2] \mathrm{d}x - F v(\frac{l}{2}). \tag{e}$$

We will derive the relevant expressions without the assembly details. The Ritz method in Example D.17 gave the system equations

$$K_{ij} a_j = b_i \tag{f}$$

with the expressions

$$\begin{aligned} K_{ij} &= \int_0^l (S\varphi_i' \varphi_j' + k\varphi_i \varphi_j)\mathrm{d}x, \\ b_i &= F\varphi_i(\frac{l}{2}) - \int_0^l (S\varphi_i' \varphi_0' + k\varphi_i \varphi_0)\mathrm{d}x. \end{aligned} \tag{g}$$

There is no need to repeat the steps as we would naturally get the same result only with a little bit different notation. Thus we have the system equations

$$\sum_{j=1}^{9} K_{ij} v_j = b_i, \quad i = 1,2,\cdots,9 \tag{h}$$

with (as we have ignored the boundary conditions at this phase, we put here $\varphi_0 = 0$)

$$\begin{aligned} K_{ij} &= \int_0^l (S N_i' N_j' + k N_i N_j)\mathrm{d}x, \\ b_i &= F N_i(\frac{l}{2}). \end{aligned} \tag{i}$$

The terms can be produced by Mathematica or as S and k are constants, they can be rather easily evaluated just by the help of Figure (b). The final outcome is (again the case $\sqrt{k/S}\, l = 4$ is taken)

$$\frac{kl}{48}\begin{bmatrix} 26 & -23 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -23 & 52 & -23 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -23 & 52 & -23 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -23 & 52 & -23 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -23 & 52 & -23 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -23 & 52 & -23 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -23 & 52 & -23 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -23 & 52 & -23 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -23 & 26 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{Bmatrix} = F\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{j}$$

The coefficient matrix is no more diagonal but it is however rather sparsely populated for reasons discussed above. Now it is remembered that variables v_1 and v_9 are not free and thus the first and last equations $\partial\bar{V}/\partial v_1 = 0$ and $\partial\bar{V}/\partial v_9 = 0$ are in fact not correct. They are discarded from the set and in the rest of the equations the given values $v_1 = \bar{v}$ and $v_9 = \bar{v}$ are introduced and the corresponding terms (only two terms) are transferred to the right hand side. This gives the final set

$$\frac{kl}{48}\begin{bmatrix} 52 & -23 & 0 & 0 & 0 & 0 & 0 \\ -23 & 52 & -23 & 0 & 0 & 0 & 0 \\ 0 & -23 & 52 & -23 & 0 & 0 & 0 \\ 0 & 0 & -23 & 52 & -23 & 0 & 0 \\ 0 & 0 & 0 & -23 & 52 & -23 & 0 \\ 0 & 0 & 0 & 0 & -23 & 52 & -23 \\ 0 & 0 & 0 & 0 & 0 & -23 & 52 \end{bmatrix}\begin{Bmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{Bmatrix} = \frac{F}{480}\begin{Bmatrix} 23 \\ 0 \\ 0 \\ 480 \\ 0 \\ 0 \\ 23 \end{Bmatrix}. \tag{k}$$

The solution is

$$\begin{aligned} &v_2 = v_8 = 0.334\,F/(kl), \quad v_3 = v_7 = 0.655\,F/(kl), \\ &v_4 = v_6 = 1.146\,F/(kl), \quad v_5 = 1.937\,F/(kl). \end{aligned} \tag{l}$$

At the midpoint the deflection $\bar{v}(l/2) = v_5 = 1.937F/(kl)$ whose error compared with the exact value $v(l/2) = 1.955\,F/(kl)$ is now only 0.9%.

The solution by the Ritz method formulation of Example D.17 using five undetermined parameters and the present finite element solution with four elements are shown in Figure (d). To see the differencies clearly, only this small number of parameters have been used. Though the finite element solution generates jumps in the derivative at points where the exact solution is smooth, it can on the other hand automatically simulate the kink under the point force where the classical Ritz method has poor approximation properties.

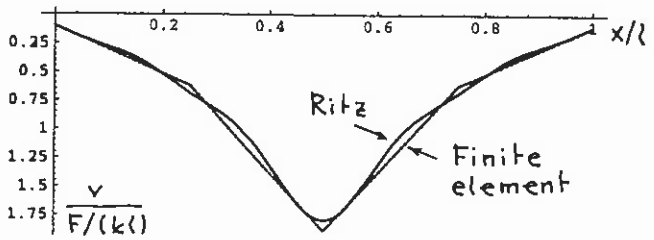


Figure (d)

It should be mentioned that the order of the numbering of the nodes of a mesh influencies the distribution of the non-zero terms in the system coefficient matrix because normally the equations are listed in the order indicated by the numbering. Thus, for instance, if the numbering here would be the one shown in Figure (e), say the second system equation would contain the nodal values v_2, v_3, v_9 which would be quite different from the corresponding equation in (j) which contains the values v_1, v_2, v_3. If a so called banded solver is used, the numbering of Figure (e) would be disadvantageous.

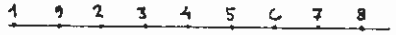


Figure (e)

We have described the application of the finite element method and especially the assembly process this far from the variational principle point of view. If a weak form is taken as the starting point, it is readily seen that the assembly process remains the same in the sense that that the left-hand sides of the system equations can be generated from summations from the elements. As an example, let us consider the third system equation in Example D.22 using the Galerkin method. It is obtained by taking in the discretized weak form

$$\int_0^l (w'S\bar{v}' + wk\bar{v})\mathrm{d}x - w(\frac{l}{2})F = 0 \tag{28}$$

the weighting function w to be the global shape function N_3. We see from Figure (b) of Example D.22 that the contributions to the left-hand side of the system equation come separately from element 2 and element 3. The use of the finite element method in connection with a weak form is explained further in the next section.

D.3.5 Sensitized finite element method

The sensitizing idea of Courant has advanced combined with the finite element method recently so far that specific recipes can be given (at least in one dimensional cases) for the appropriate sensitizing parameter values. We will describe this in connection with the setting of Example D.16.

The governing differential equation is

$$R(v) \equiv L(v) - q \equiv -Sv'' + kv - q = 0, \qquad 0 < x < l. \tag{29}$$

We employ here the weak formulation and the Galerkin method. The standard weak form is

$$\int_0^l (w'Sv' + wkv - wq)\mathrm{d}x + \mathrm{bt} = 0 \tag{30}$$

with some essential boundary conditions. The notation bt refers to some possible boundary terms due to the natural boundary conditions.

An ideal finite element solution would be such that its nodal values are exact, in other words, it would be the interpolant to the exact solution. A nodally exact solution cannot be unstable and it is a very good starting point for adaptive procedures and for error estimation.

Let us consider (29) in a specific case with zero loading, with some given boundary displacements $v(0) = v(l) = \bar{v}$ and the case of a very high value of the representative foundation modulus k. In more detail, the dimensionless number

$$C \equiv \frac{kl^2}{S} \tag{31}$$

is large. Figure D.12 shows the solution obtained with four equal length two noded (linear) elements in the case of constant k and with $C = 1600$ using the standard Galerkin method.

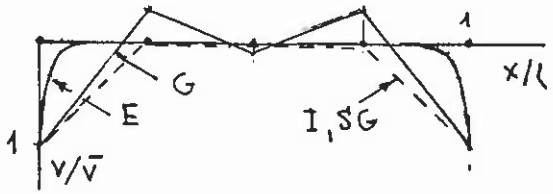


Figure D.12 Exact solution (E), standard Galerkin finite element solution (G), finite element interpolant to the exact solution (I), sensitized Galerkin finite element solution (SG).

The exact solution shows *boundary layer* (rajakerros) behaviour. The higher the value of C, the shorter the length of the interval near the ends where the solution is essentially non-zero. The standard Galerkin method solution is in fact the best solution in the so called energy norm

$$\| v \|_a = [\int_0^l (v'Sv' + vkv)\,\mathrm{d}x]^{1/2}. \tag{32}$$

This result is, however, not at all satisfactory in practice. Although the desired interpolant to the exact solution, indicated also in Figure D.12, is similarly very far from the exact solution, it is clearly a better starting point for refining the mesh adaptively as it is relatively easy to evaluate the errors in the elements (errors are clearly large in the first and in the last element) as we have exact boundary conditions available for each element.

Some consideration shows that the oscillations in connection with the boundary layer type behaviour can be removed here in particular by the gradient least squares sensitizing term. Thus we write the sensitized weak form

$$\int_\Omega (w'Sv' + wkv - wq)\,\mathrm{d}\Omega + \mathrm{bt} + \int_\Omega \frac{\mathrm{d}}{\mathrm{d}x} L(w)\tau^{(1)} \frac{\mathrm{d}}{\mathrm{d}x} R(v)\mathrm{d}\Omega = 0 \tag{33}$$

where

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} R(v) &= -Sv''' + (kv)' - q', \\ \frac{\mathrm{d}}{\mathrm{d}x} L(w) &= -Sw''' + (kw)'. \end{aligned} \tag{34}$$

We have used general domain notations to simplify the subsequent expressions. The discretized version of the weak form (33) looks (two noded linear elements are used)

$$\sum_e \int_{\Omega^e} (\bar{w}'S\bar{v}' + \bar{w}k\bar{v} - \bar{w}q)\,\mathrm{d}\Omega + \tilde{\mathrm{b}}\tilde{\mathrm{t}} + \sum_e \int_{\Omega^e} \frac{\mathrm{d}}{\mathrm{d}x} L(\bar{w})\tau^{(1)} \frac{\mathrm{d}}{\mathrm{d}x} R(\bar{v})\mathrm{d}\Omega = 0 \tag{35}$$

where the summation is over the elements of the mesh used. Here, in an element,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} R(\bar{v}) &\approx -S\bar{v}''' + k\bar{v}' - q' = k\bar{v}' - q', \\ \frac{\mathrm{d}}{\mathrm{d}x} L(\bar{w}) &\approx -S\bar{w}''' + k\bar{w}' = k\bar{w}'. \end{aligned} \tag{36}$$

The latter expressions are due to the first degree approximation used. It is seen that the expressions have been also simplified by assuming some constant representative value for k even in the case where it might depend on position. Based on a similar argument as that explained in Remark D.21, no violation with respect to convergence is introduced. It should also be noticed that by the notation $\bar{w}$ is meant here that the weighting function is taken from a finite dimensional set (here in fact from the set of trial functions); the tilde above the symbol w does not mean approximation as it means above the symbol v; *we cannot approximate a weighting function, we just select a weighting function to be employed in a weak form.*

Remark D.21. At a first glance, an objection about the formulation could be easily raised with respect to the low continuity of the approximation. For instance, the derivative $\bar{v}'''$ appears in the gradient residual $\mathrm{d}R(\bar{v})/\mathrm{d}x$. According to well-known conventional rules of the finite element literature, this would demand a C^2-continuous approximation for convergence. However, it

must be remembered that when studying convergence, the element sizes tend by definition towards zero. Thus roughly speaking, *with finite sizes of the elements any kind of extra terms producing beneficial behaviour can be used if these terms vanish fast enough when the mesh gets denser.* The sensitizing term is in fact later seen to vanish when the element size gets to zero. We need no sensitizing in the theoretical limit never reached in reality. It is good to know that a formulation is a converging one, but in practice, after all, the main thing we are interesting in is to have accurate results already with reasonable meshes. □

We can now see why the sensitizing term may produce the behaviour we want. The term is in detail

$$\int_{\Omega^e} \frac{\mathrm{d}}{\mathrm{d}x} L(\tilde{w}) \tau^{(1)} \frac{\mathrm{d}}{\mathrm{d}x} R(\tilde{v}) \mathrm{d}\Omega = \int_{\Omega^e} k\tilde{w}' \tau^{(1)} (k\tilde{v}' - q') \mathrm{d}\Omega$$
$$= \int_{\Omega^e} (\tilde{w}' \tau^{(1)} k^2 \tilde{v}' - \tilde{w}' \tau^{(1)} k q') \mathrm{d}\Omega. \tag{37}$$

Comparison with the standard part in (35) shows that we can interpret the multiplier $\tau^{(1)}k^2$ as an additional tension in the string (if the sensitizing parameter $\tau^{(1)}$ is positive) and correspondingly, an additional source term is introduced. If q is constant, this source term vanishes. Now the elementwise non-dimensional measure

$$c \equiv \frac{kh^2}{S} \tag{38}$$

where h is the element length, gets effectively smaller meaning that the effective tension is larger and the oscillations near the boundary layers are damped.

For demonstration purposes, the sensitizing parameter is taken to be here originally a second degree polynomial in x. Using the elementwise non-dimensional coordinate ξ, introduced in Example D.22, we take in particular

$$\tau^{(1)} = \alpha + \beta(\xi - \frac{1}{2}) + \gamma\xi(1 - \xi). \tag{39}$$

This form has been selected so that parameters α, β, γ to be determined have rather obvious geometrical interpretations.

For the goal of achieving the nodally exact solution a version of the *patch test* (tilkkutesti), Zienkiewicz and Taylor (1989), can be used. The main point is to have a set of solutions, called *reference solutions* (referenssiratkaisu), injecting important information of the real local solution behaviour into the discrete formulation. These reference solutions are employed via a patch test to produce a system of equations from which the sensitizing parameter values can be

determined. This procedure has been described to our knowledge for the first time in Salonen and Freund (1994).

First the data in the operators of the differential equations under study are replaced by some constant local representative values so that we arrive at a constant coefficient system which is simpler to deal with but still contains the essential features of the problem. In (29) this means that also k is considered to be a representative constant in the neighbourhood of the position under study. For convenience of presentation we assume a local origin $x = 0$ to this position. The continuous loading is assumed to be represented as a power series (Taylor series) in the form

$$q(x) = q_0 + q_0' \cdot x + \frac{1}{2} q_0'' \cdot x^2 + \cdots \tag{40}$$

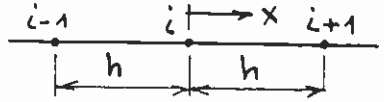
where the meaning of the notations is obvious. According to the theory for linear differential equations the solution consist of the general solution of the homogeneous system (29) plus of the particular solutions for the full system. There is found

$$v(x) = Ae^{\sqrt{k/S}\cdot x} + Be^{-\sqrt{k/S}\cdot x} + q_0 \frac{1}{k} + q_0' \frac{1}{k} x + q_0''(\frac{S}{k^2} + \frac{1}{2k} x^2) + \cdots \tag{41}$$

This can be written also as

$$\begin{Bmatrix} v(x) \\ q(x) \end{Bmatrix} = A \begin{Bmatrix} e^{\sqrt{k/S}\cdot x} \\ 0 \end{Bmatrix} + B \begin{Bmatrix} e^{-\sqrt{k/S}\cdot x} \\ 0 \end{Bmatrix} +$$
$$+ q_0 \begin{Bmatrix} 1/k \\ 1 \end{Bmatrix} + q_0' \begin{Bmatrix} 1/k \cdot x \\ x \end{Bmatrix} + q_0'' \begin{Bmatrix} S/k^2 + 1/(2k) \cdot x^2 \\ 1/2 \cdot x^2 \end{Bmatrix} + \cdots \tag{42}$$

In more detail we mean by this that here five different reference solution-source pairs (v,q) are available with the consecutive selections $A \neq 0$, $B \neq 0$, $q_0 \neq 0$, $q_0' \neq 0$, $q_0'' \neq 0$.



D.13 Two element patch.

The next step is to "clone" a generic (here two noded) element to form a uniform mesh and to consider a typical two element patch (Figure D.13). Finite

element system equation corresponding to node i is formed using the discretized weak form (35) with expression (39) for $\tau^{(1)}$. Now the nodal values are taken from the reference solutions by choosing consecutively each of the factors A, B, etc. non-zero. Correspondingly, there are obtained the following five system equations (some preliminary details are given in Example (D.23)):

$$
\begin{aligned}
&(-\frac{S}{h}+\frac{kh}{6}-\frac{\alpha k^2}{h}-\frac{\gamma k^2}{h})e^{-\sqrt{k/S}\cdot h}+\\
&+(2\frac{S}{h}+4\frac{kh}{6}+2\frac{\alpha k^2}{h}+2\frac{\gamma k^2}{h})1+\\
&+(-\frac{S}{h}+\frac{kh}{6}-\frac{\alpha k^2}{h}-\frac{\gamma k^2}{h})e^{\sqrt{k/S}\cdot h}=0,
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
&(-\frac{S}{h}+\frac{kh}{6}-\frac{\alpha k^2}{h}-\frac{\gamma k^2}{h})e^{\sqrt{k/S}\cdot h}+\\
&+(2\frac{S}{h}+4\frac{kh}{6}+2\frac{\alpha k^2}{h}+2\frac{\gamma k^2}{h})1+\\
&+(-\frac{S}{h}+\frac{kh}{6}-\frac{\alpha k^2}{h}-\frac{\gamma k^2}{h})e^{-\sqrt{k/S}\cdot h}=0,
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
&(-\frac{S}{h}+\frac{kh}{6}-\frac{\alpha k^2}{h}-\frac{\gamma k^2}{h})\frac{1}{k}+\\
&+(2\frac{S}{h}+4\frac{kh}{6}+2\frac{\alpha k^2}{h}+2\frac{\gamma k^2}{h})\frac{1}{k}+\\
&+(-\frac{S}{h}+\frac{kh}{6}-\frac{\alpha k^2}{h}-\frac{\gamma k^2}{h})\frac{1}{k}=h,
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&(-\frac{S}{h}+\frac{kh}{6}-\frac{\alpha k^2}{h}-\frac{\gamma k^2}{h})(-\frac{h}{k})+\\
&+(2\frac{S}{h}+4\frac{kh}{6}+2\frac{\alpha k^2}{h}+2\frac{\gamma k^2}{h})(0)+\\
&+(-\frac{S}{h}+\frac{kh}{6}-\frac{\alpha k^2}{h}-\frac{\gamma k^2}{h})(\frac{h}{k})=0,
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
&(-\frac{S}{h}+\frac{kh}{6}-\frac{\alpha k^2}{h}-\frac{\gamma k^2}{h})(\frac{S}{k^2}+\frac{h^2}{2k})+\\
&+(2\frac{S}{h}+4\frac{kh}{6}+2\frac{\alpha k^2}{h}+2\frac{\gamma k^2}{h})(\frac{S}{k^2})+\\
&+(-\frac{S}{h}+\frac{kh}{6}-\frac{\alpha k^2}{h}-\frac{\gamma k^2}{h})(\frac{S}{k^2}+\frac{h^2}{2k})=\frac{h^3}{12}-\alpha kh+\frac{\beta kh}{6}-\frac{\gamma kh}{6}.
\end{aligned}
\tag{47}
$$

It is readily seen that equations (45) and (46) are satisfied irrespective of the values of the sensitizing parameters. The three equations (43), (44), (47) contain the three parameters α, β, γ linearly and they should hopefully be determined from these equations. We, however, consider from this on only the case of a constant $\tau^{(1)}=\alpha$ so that β and γ are put to zero in (39) and thus also in equations (43) to (47). Either of equations (43) or (44) gives the value

$$
\tau^{(1)}=\frac{h^2}{k}(\frac{1}{6}\frac{\cosh\sqrt{c}+2}{\cosh\sqrt{c}-1}-\frac{1}{c})
\tag{48}
$$

where c is defined by formula (38). As the finite element system equations are linear with respect to the nodal values and with respect to the source term, the patch test is now passed for the full expression (42) with arbitrary values of A, B, etc. up to the linear (in x) source term. Thus, in the case of constant properties, uniform mesh and constant or linear source term, the finite element solution will be nodally exact. Further, even though the sensitizing parameter expression has been determined using a regular mesh, it is reasonable to expect — and the numerical results confirm this — that this generic expression can be employed with good accuracy for individual elements in an irregular mesh.

For instance, the sensitized formulation in the problem discussed in connection with Figure D.12 is found to produce the nodally exact solution. However, if the loading consists of a point load, the numerical results are found not to be very accurate with the parameter value (48). This is understandable as the point load can be interpreted as a Dirac delta function loading which is far from the case considered when expression (48) was derived. The point load case thus needs further study.

The sensitized formulation will be applied in a more practical problem setting in Section 13.3.

Example D.23. We develop the system equation for the midpoint of the patch in Figure D.13. Taking expression (37) into acccount, the discretized weak form (35) is

$$
\sum_e\int_{\Omega^e}(\tilde{w}'S\tilde{v}'+\tilde{w}k\tilde{v}-\tilde{w}q)\,\mathrm{d}\Omega+\tilde{b}\tilde{t}+\sum_e\int_{\Omega^e}(\tilde{w}'\tau^{(1)}k^2\tilde{v}'-\tilde{w}'\tau^{(1)}kq')\,\mathrm{d}\Omega=0.
\tag{a}
$$

Using the Galerkin method, a typical system equation is thus

$$
K_{ij}v_j=b_i
\tag{b}
$$

with

$$
\begin{aligned}
K_{ij}&=\sum_e\int_{\Omega^e}(N_i'SN_j'+N_ikN_j)\,\mathrm{d}\Omega+\sum_e\int_{\Omega^e}N_i'\tau^{(1)}k^2N_j'\,\mathrm{d}\Omega,\\
b_i&=\sum_e\int_{\Omega^e}N_iq\,\mathrm{d}\Omega+\sum_e\int_{\Omega^e}N_i'\tau^{(1)}kq'\,\mathrm{d}\Omega.
\end{aligned}
\tag{c}
$$

We have assumed that node i is inside the domain so that no contributions come from the boundary terms. With constant S and k, the element contributions are in more detail

$$K_{ij}^e = \int_{\Omega^e}(N_i' S N_j' + N_i k N_j)\,d\Omega + \int_{\Omega^e} N_i' \tau^{(1)} k^2 N_j'\,d\Omega$$
$$= S\int_{\Omega^e} N_i' N_j'\,d\Omega + k\int_{\Omega^e} N_i N_j\,d\Omega +$$
$$+ \alpha k^2 \int_{\Omega^e} N_i' N_j'\,d\Omega + \beta k^2 \int_{\Omega^e}(\xi - 1/2) N_i' N_j'\,d\Omega + \gamma k^2 \int_{\Omega^e} \xi(1-\xi) N_i' N_j'\,d\Omega, \quad \text{(d)}$$

$$b_i = \int_{\Omega^e} N_i q\,d\Omega + \int_{\Omega^e} N_i' \tau^{(1)} k q'\,d\Omega = \int_{\Omega^e} N_i q\,d\Omega + \frac{k}{h}\int_{\Omega^e} N_i' \tau^{(1)}[\bar\beta + \bar\gamma(1-2\xi)]\,d\Omega$$
$$= \bar\alpha \int_{\Omega^e} N_i\,d\Omega + \bar\beta \int_{\Omega^e}(\xi - 1/2) N_i\,d\Omega + \bar\gamma \int_{\Omega^e} \xi(1-\xi) N_i\,d\Omega +$$
$$+ \alpha[\bar\beta \frac{k}{h}\int_{\Omega^e} N_i'\,d\Omega + \bar\gamma \frac{k}{h}\int_{\Omega^e}(1-2\xi) N_i'\,d\Omega] +$$
$$+ \beta[\bar\beta \frac{k}{h}\int_{\Omega^e}(\xi - 1/2) N_i'\,d\Omega + \bar\gamma \frac{k}{h}\int_{\Omega^e}(\xi - 1/2)(1-2\xi) N_i'\,d\Omega] +$$
$$+ \gamma[\bar\beta \frac{k}{h}\int_{\Omega^e} \xi(1-\xi) N_i'\,d\Omega + \bar\gamma \frac{k}{h}\int_{\Omega^e} \xi(1-\xi)(1-2\xi) N_i'\,d\Omega]. \quad \text{(e)}$$

To simplify the presentation, the element indices have been left out from the shape function symbols. The second degree loading (40) has been expressed here in an element as

$$q = \bar\alpha + \bar\beta(\xi - \frac{1}{2}) + \bar\gamma\xi(1-\xi). \quad \text{(f)}$$

where the values of the coefficients $\bar\alpha$, $\bar\beta$, $\bar\gamma$ are easily found from (40) when the position of the element is given. Similarly, the derivative

$$q' = \frac{1}{h}[\bar\beta + \bar\gamma(1-2\xi)]. \quad \text{(g)}$$

Performing the (by hand) tedious integrations gives the following element contributions

$$[K]^e = \frac{S}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{kh}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{\alpha k^2}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\gamma k^2}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \text{(h)}$$

$$\{b\}^e = \frac{\bar\alpha h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \frac{\bar\beta h}{12}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} + \frac{\bar\gamma h}{12}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \frac{\alpha\bar\beta k}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} + \frac{\beta\bar\gamma k}{6h}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \frac{\gamma\bar\beta k}{6h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}. \quad \text{(i)}$$

We number the nodes and the elements of the two element patch of Figure D. 13 here for simplicity as shown in Figure (a).

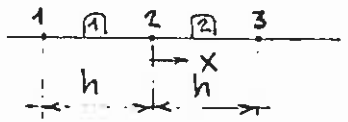


Figure (a)

The system equation for node 2 is

$$K_{21}v_1 + K_{22}v_2 + K_{23}v_3 = b_2 \quad \text{(j)}$$

and the assembly process gives the formulas

$$K_{21} = K_{21}^1, \quad K_{22} = K_{22}^1 + K_{11}^2, \quad K_{23} = K_{12}^2$$
$$b_2 = b_2^1 + b_1^2 \quad \text{(k)}$$

The coefficients in (j) are thus

$$K_{21} = -\frac{S}{h} + \frac{kh}{6} - \frac{\alpha k^2}{h} - \frac{\gamma k^2}{h},$$
$$K_{22} = 2\frac{S}{h} + 4\frac{kh}{6} + 2\frac{\alpha k^2}{h} + 2\frac{\gamma k^2}{h}, \quad \text{(l)}$$
$$K_{23} = -\frac{S}{h} + \frac{kh}{6} - \frac{\alpha k^2}{h} - \frac{\gamma k^2}{h}.$$

It is seen that the parameter β does not appear in them. The right hand side of (j) depends on the loading and is evaluated separately for the reference cases.

In the first case the nodal values are

$$v_1 = e^{-\sqrt{k/S}h}, \quad v_2 = 1, \quad v_3 = e^{\sqrt{k/S}h} \quad \text{(m)}$$

and there is no loading so

$$b_2 = 0. \quad \text{(n)}$$

(The coefficient A would appear as a multiplier in all the terms in the system equation for node 2 and we cancel it by formally taking $A = 1$.)

In the second case the nodal values are

$$v_1 = e^{\sqrt{k/S}h}, \quad v_2 = 1, \quad v_3 = e^{-\sqrt{k/S}h} \quad \text{(o)}$$

and there is no loading so

$$b_2 = 0. \quad \text{(p)}$$

In the third case,

$$v_1 = \frac{1}{k}, \quad v_2 = \frac{1}{k}, \quad v_3 = \frac{1}{k} \quad \text{(q)}$$

and the loading is $q = 1$. In elements 1 and 2, $\bar\alpha = 1$, $\bar\beta = 0$, $\bar\gamma = 0$. There is obtained

$$\{b\}^1 = \frac{h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \{b\}^2 = \frac{h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \text{(r)}$$
$$b_2 = h.$$

In the fourth case,

$$v_1 = -\frac{h}{k}, \quad v_2 = 0, \quad v_3 = \frac{h}{k} \quad \text{(s)}$$

and the loading is $q = x$. In element 1, $\bar\alpha = -h/2$, $\bar\beta = h$, $\bar\gamma = 0$. In element 2, $\bar\alpha = h/2$, $\bar\beta = h$, $\bar\gamma = 0$. There is obtained

$$\{b\}^1 = \frac{h^2}{4}\begin{Bmatrix}-1\\-1\end{Bmatrix} + \frac{h^2}{12}\begin{Bmatrix}-1\\1\end{Bmatrix} + \alpha k\begin{Bmatrix}-1\\1\end{Bmatrix} + \frac{\gamma k}{6}\begin{Bmatrix}-1\\1\end{Bmatrix},$$

$$\{b\}^2 = \frac{h^2}{4}\begin{Bmatrix}1\\1\end{Bmatrix} + \frac{h^2}{12}\begin{Bmatrix}-1\\1\end{Bmatrix} + \alpha k\begin{Bmatrix}-1\\1\end{Bmatrix} + \frac{\gamma k}{6}\begin{Bmatrix}-1\\1\end{Bmatrix}, \qquad (t)$$

$$b^2 = 0.$$

In the fifth case

$$\nu_1 = \frac{S}{k^2} + \frac{h^2}{2k}, \quad \nu_2 = \frac{S}{k^2}, \quad \nu_3 = \frac{S}{k^2} + \frac{h^2}{2k} \qquad (u)$$

and the loading is $q = 1/2 \cdot x^2$. In element 1, $\bar{\alpha} = 1/4 \cdot h^2$, $\bar{\beta} = -1/2 \cdot h^2$, $\bar{\gamma} = -1/2 \cdot h^2$. In element 2, $\bar{\alpha} = 1/4 \cdot h^2$, $\bar{\beta} = 1/2 \cdot h^2$, $\bar{\gamma} = -1/2 \cdot h^2$. There is obtained

$$\{b\}^1 = \frac{h^3}{8}\begin{Bmatrix}1\\1\end{Bmatrix} + \frac{h^3}{24}\begin{Bmatrix}1\\-1\end{Bmatrix} + \frac{h^3}{24}\begin{Bmatrix}-1\\-1\end{Bmatrix} + \frac{\alpha hk}{2}\begin{Bmatrix}-1\\1\end{Bmatrix} + \frac{\beta hk}{12}\begin{Bmatrix}1\\-1\end{Bmatrix} + \frac{\gamma hk}{2}\begin{Bmatrix}1\\-1\end{Bmatrix},$$

$$\{b\}^2 = \frac{h^3}{8}\begin{Bmatrix}1\\1\end{Bmatrix} + \frac{h^3}{24}\begin{Bmatrix}-1\\1\end{Bmatrix} + \frac{h^3}{24}\begin{Bmatrix}-1\\-1\end{Bmatrix} + \frac{\alpha hk}{2}\begin{Bmatrix}-1\\1\end{Bmatrix} + \frac{\beta hk}{12}\begin{Bmatrix}1\\-1\end{Bmatrix} + \frac{\gamma hk}{2}\begin{Bmatrix}-1\\1\end{Bmatrix}, \qquad (v)$$

$$b_2 = \frac{h^3}{12} - \alpha hk + \frac{\beta hk}{6} - \frac{\gamma hk}{6}.$$

The resulting five system equations are written down in the main text as equations (43) to (47).

D.3.6 Global-local method

The *global-local method* (globaali-lokaalimenettely) is a mixture of the classical Ritz method and the finite element method, Mote (1971). Approximation

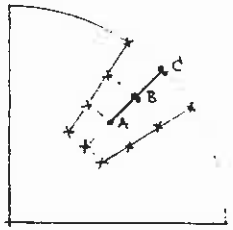
$$\bar{\phi}(x) = \varphi_0(x) + a_j \varphi_j(x) \qquad (49)$$

is written in the form

$$\boxed{\bar{\phi}(x) = \varphi_0(x) + a_k^* \varphi_k^*(x) + a_l^{**} \varphi_l^{**}(x).} \qquad (50)$$

Here the $\varphi_k^*(x)$ are classical (global) Ritz basis functions and the $\varphi_l^{**}(x)$ are finite element shape functions. (The term "global" means here that the Ritz basis functions are defined to be generally non-zero in the whole domain. Thus it has nothing to do with term "global" shape function used in connection of the finite element method.) The ranges of the summation indices k and l usually differ. The parameters a_l^{**} no more simply have the meaning of nodal function values as they now give the change or "correction" with respect to the global approximation.

With a careful selection of the type of representation (50) one can achieve in some cases with a relatively small number of parameters good accuracy.



• Node
: Double node
× Node with the associated nodal value zero

Figure D.14 Quadrant of a circular plate.

Figure D.14 shows a case modified from Mote (1971). The case concern the vibration of a circular plate. The plate has on line ABC a cut which is assumed to be infinitely narrow. It is easy to generate suitable global basis functions for a circular plate without the cut. The neighbourhood of the cut is patched in order by six four-noded quadrilateral elements. At points B and C the necessary discontinuity in deflection is achieved by associating two nodes at a geometrically same point which belong to different sides of the cut. At the nodes denoted with × the corresponding finite element nodal parameter value is fixed to zero to achieve a continuous deflection approximation.

It is possible to apply the global local method also in such a form that the global form is a suitable given function.

The drawbacks of the method are the complicated form of the system equations: The basis functions φ_k^* and φ_l^{**} are generally not orthogonal and the system equations thus contain corresponding couplings.

Example D.24. The stretched string on an elastic foundation treated in Examples D.17, D.18 and D.22 is considered again now using the global-local method.

Approximation according to (50) is here

$$\begin{aligned}\bar{v}(x) &= \varphi_0(x) + a_k^* \varphi_k^*(x) + a_k^{**} \varphi_k^{**}(x) \\ &= \varphi_0(x) + a_1^* \varphi_1^*(x) + a_2^* \varphi_2^*(x) + a_3^* \varphi_3^*(x) + a_1^{**} \varphi_1^{**}(x).\end{aligned} \qquad (a)$$

We take as in Example D.17

$$\varphi_0(x) = \bar{v} = \frac{1}{10}\frac{F}{kl}. \qquad (b)$$

The basis functions used are shown in Figure (a). The global basis functions are the same as used in Example D.17. The only local basis function is the shape function N_5 of Example D.22.

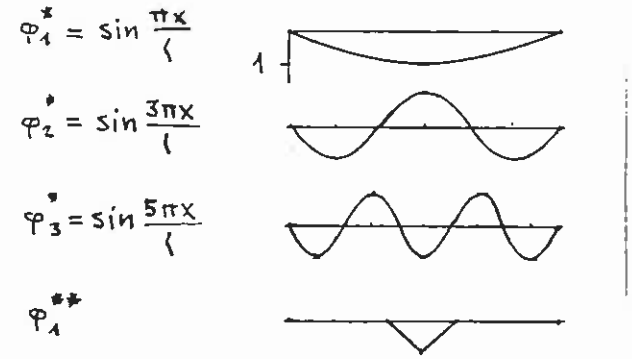


Figure (a)

The system equations are found again as equations (l) and (m) from Example D.17. Here the Mathematica program is especially convenient for performing the integrations. There is obtained ($\sqrt{k/S}\,l = 4$)

$$kl\begin{bmatrix} 0.8084 & 0 & 0 & 0.1995 \\ 0 & 3.2758 & 0 & -0.7285 \\ 0 & 0 & 8.2106 & 1.4723 \\ 0.1995 & -0.7285 & 1.4723 & 1.0833 \end{bmatrix}\begin{Bmatrix} a_1^* \\ a_2^* \\ a_3^* \\ a_1^{**} \end{Bmatrix} = F\begin{Bmatrix} 0.9363 \\ -1.0212 \\ 0.9873 \\ 0.9875 \end{Bmatrix}. \tag{c}$$

The solution is

$$a_1^* \approx 1.0152\,F/(kl),\quad a_2^* \approx -0.1829\,F/(kl),\quad a_3^* \approx 0.0164\,F/(kl),$$
$$a_1^{**} \approx 0.5793\,F/(kl). \tag{d}$$

The deflection is shown in Figure (b). At the midpoint, $\bar{v}(l/2) \approx 1.894\,F/(kl)$. This is more accurate than the result by the conventional Ritz method in Example D.17 employing nine basis functions.

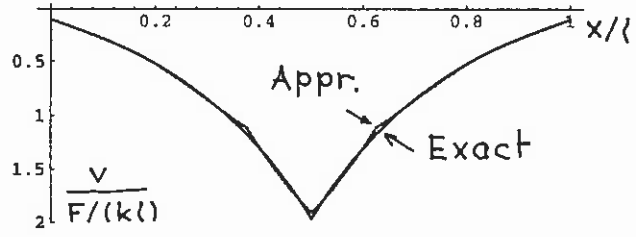


Figure (b)

D.3.7 Finite difference method

We assume in the following some knowledge from the reader of the finite difference method. The most common way to apply the *finite difference method* (differenssimenetelmä) is to simulate problems expressed in differential equation forms. However, it is possible to use the method also in connection with functionals. It is then sometimes called the *variational finite difference method* (variaatiodifferenssimenetelmä). The finite difference method resembles the finite element method in the respect that the discrete variables are again approximations to the unknown function values at certain points, called usually *gridpoints* (hilapiste). However, contrary to the finite element method, no unique representation between the gridpoints is assumed. The derivatives appearing in functionals are replaced by pointwise difference quotients and integrations by numerical quadratures. According to the literature, Courant and Hilbert (1953), Euler applied the finite difference method already in 1744 to the study of functionals. Here we do not emphasize the use of the variational finite difference method and give just one example.

Example D.25. The stretched string on an elastic foundation is considered once more. The grid is uniform and consists of five grid points (Figure (a)).

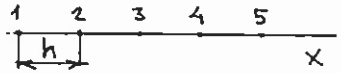


Figure (a)

The potential energy functional is

$$V(v) = \int_0^l [\frac{1}{2}S(v')^2 + \frac{1}{2}kv^2]\,dx - Fv(\frac{l}{2}) \tag{a}$$

or as S is constant and k is assumed to be constant,

$$V(v) = \frac{1}{2}S\int_0^l (v')^2\,dx + \frac{1}{2}k\int_0^l v^2\,dx - Fv(\frac{l}{2}). \tag{b}$$

This is approximated by

$$\begin{aligned}\tilde{V}(\{a\}) = {} & \frac{1}{2}S[(\frac{v_2 - v_1}{h})^2 \cdot h + (\frac{v_3 - v_2}{h})^2 \cdot h + \\ & + (\frac{v_4 - v_3}{h})^2 \cdot h + (\frac{v_5 - v_4}{h})^2 \cdot h] + \\ & + \frac{1}{2}k[(\frac{v_1 + v_2}{2})^2 \cdot h + (\frac{v_2 + v_3}{2})^2 \cdot h + \\ & + (\frac{v_3 + v_4}{2})^2 \cdot h + (\frac{v_4 + v_5}{2})^2 \cdot h] - Fv_3. \end{aligned} \tag{c}$$

For v', a conventional difference quotient expression between two gridpoint values is used and this is assumed to be valid for the whole interval. Similarly, v is approximated as the mean value between two gridpoint values and again assumed to be valid for the whole interval. Obviously, these selections are not the only possibilities to obtain a reasonable approximation.

We form $\partial\tilde{V}/\partial v_2 = 0$ as an example for a typical system equation. We have

$$\frac{\partial \tilde{V}}{\partial v_2} = \frac{1}{2} S[2\frac{v_2 - v_1}{h}\frac{1}{h}h - 2\frac{v_3 - v_2}{h}\frac{1}{h}h] +$$
$$+\frac{1}{2}k[2\frac{v_1 + v_2}{2}\frac{1}{2}h + 2\frac{v_2 + v_3}{2}\frac{1}{2}h]$$
$$= \frac{S}{h}(-v_1 + 2v_2 - v_3) + \frac{kh}{4}(v_1 + 2v_2 + v_3) = 0. \qquad \text{(d)}$$

The rest of the system equations are formed similarly. There is obtained ($\sqrt{k/S}\,l = 4$)

$$\frac{kl}{16}\begin{bmatrix} -3 & 10 & -3 & 0 & 0 \\ 0 & -3 & 10 & -3 & 0 \\ 0 & 0 & -3 & 10 & -3 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{Bmatrix} = F\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}. \qquad \text{(e)}$$

The equations corresponding to v_1 and v_5 have not been formed as they are replaced by the information from the boundary conditions:

$$v_1 = \bar{v} = \frac{1}{10}\frac{F}{kl}, \qquad v_5 = \bar{v} = \frac{1}{10}\frac{F}{kl}. \qquad \text{(f)}$$

When these are taken into account in (e), we have as the final system equations

$$\frac{kl}{16}\begin{bmatrix} 10 & -3 & 0 \\ -3 & 10 & -3 \\ 0 & -3 & 10 \end{bmatrix}\begin{Bmatrix} v_2 \\ v_3 \\ v_4 \end{Bmatrix} = \frac{F}{160}\begin{Bmatrix} 3 \\ 160 \\ 3 \end{Bmatrix}. \qquad \text{(g)}$$

The solution is

$$v_2 = v_4 \approx 0.622\,F/(kl), \quad v_3 = 1.973\,F/(kl). \qquad \text{(h)}$$

The exact value at the midpoint is $v(l/2) = 1.955\,F/(kl)$. Thus, contrary to the earlier results, the corresponding approximate value v_3 here overestimates the deflection.

Let us consider the typical discrete equation (d) somewhat more. After division by h and with a change of sign it reads

$$\frac{S}{h^2}(v_1 - 2v_2 + v_3) - \frac{k}{4}(v_1 + 2v_2 + v_3) = 0. \qquad \text{(i)}$$

This clearly a finite difference approximation to the governing differential equation

$$Sv'' - kv = 0 \qquad \text{(j)}$$

at gridpoint 2.

D.4 REFERENCES

Bazaraa M. S., H. D. Sherali and C. M. Shetty, (1993): *Nonlinear Programming, Theory and Algorithms*, Wiley, New York.

Courant, R. (1923): Über ein konvergenzerzeugendes Prinzip in der Variationsrechnung, *Nachrichten von der Könichlichen Gesellschaft der Wissenschaften zu Göttingen*, Mathematisch-physikalische Klasse aus dem Jahre 1922, pp. 144-150.

Courant, R. (1943): Variational methods for the solution of problems of equilibrium and vibrations, Bull. Amer. Math. Soc. 49, pp. 2165-2186.

Courant, R. and D. Hilbert, (1953): *Methods of Mathematical Physics*, Interscience Publishers, New York.

Crandall, S. (1956): *Enginering Analysis, A Survey of Numerical Procedures*, McGraw-Hill, New York.

Finlayson, B. A. and L. E. Scriven, (1967): "On the search for variational principles," *J. Heat Mass Transfer*, Vol. 10, pp. 799-821.

Lanczos, C. (1974): *The Variational Principles of Mechanics*, University of Toronto Press, Toronto.

Langhaar, H. L. (1962): *Energy Methods in Applied Mechanics*, Wiley, New York.

Salonen, E.-M. and J. Freund, (1994): Good weighting functions for diffusion-convection-reaction problems, *Seventh Nordic Seminar on Computational Mechanics*, The Norwegian Institute of Technology, Ed. K. Bell, pp. 1-4.

Salonen, E.-M. (1995): "*Mekaniikan matematiikkaa*", Helsinki University of Technology, Computational Dynamics, Report 42.

Mote, C. D. (1971): "Global-local finite element," *Int. J. num. Meth. Engng.* Vol.22, pp. 565-574.

Ritz, W. (1909): "Über eine neue Methode zur Lösung gewisser Variations-Probleme der mathematischen Physik," *J. Reine Angew. Math.*, Vol. 135, pp. 1-61.

Zienkiewicz, O. C. and R.L. Taylor (1989): *The Finite Element Method*, 4th ed. Vol. 1, McGraw-Hill, London.

LIST OF SYMBOLS

Some of the notations used in the text are collected here.

Sets

$\cup$	union (yhdiste)
$\cap$	intersection (leikkaus)
$\varnothing$	empty set
$]a,b[$	open interval
$[a,b]$	closed interval

Matrices

$[\,]$	square or rectangular matrix
$\{\,\}$	column matrix called column vector (pystyvektori)
$[\,]^{\mathrm{T}}$	matrix transpose
$[\,]^{-1}$	matrix inverse
$[\,]^{-\mathrm{T}}$	$([\,]^{-1})^{\mathrm{T}} = ([\,]^{\mathrm{T}})^{-1}$
$\{a\}$	column vector of undetermined parameter
$\{b\}$	right-hand side column vector of given quantities
$[D]$	stress-strain matrix
$[E]$	equilibrium operator matrix
$[K]$	stiffness matrix
$[J]$	inertia matrix
$[M]$	mass matrix
$[N]$	shape function matrix
$\{R\}$	equation residual column vector
$[S]$	strain-displacement operator matrix
$\{w\}$	weighting function column vector
$[\alpha]$	weight factor matrix
$[\tau]$	sensitizing parameter matrix

Latin symbols

a,b,c	Cartesian coordinates in Lagrangian description
$\boldsymbol{a}$	acceleration
A	plane surface, cross section area
c	coefficient of viscous damping
C	Young's modulus
$\boldsymbol{e}$	unit vector
E	Young's modulus, internal energy
$\boldsymbol{E}$	Green strain tensor
$\boldsymbol{f}$	internal force
$\boldsymbol{F}$	force

g	acceleration of gravity
G	shear modulus
$\boldsymbol{G}_i$	basis vector in large displacement theory
$\boldsymbol{G}$	Green's deformation tensor
h	scale factor, element length
$\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$	Cartesian unit basis vectors with coordinates a,b,c or x,y,z
$I_{xx},\ldots$	moment of inertia
$I_{xy},\ldots$	product of inertia
$\boldsymbol{I},\boldsymbol{J},\boldsymbol{K}$	Cartesian unit basis vectors with coordinates X,Y,Z
J	Jacobian determinant
k	spring constant, foundation modulus, thermal conductivity
K	kinetic energy
l	length
L	Lagrangian function, linear operator,
$\boldsymbol{L}$	moment of momentum
m	mass, mass per unit length
$\boldsymbol{M}$	moment
$\boldsymbol{n}$	outward unit normal vector
N	normal force, shape function
$\boldsymbol{p}$	momentum
P	power
q	generalized coordinate, loading per unit length
Q	generalized force, shearing force
r	moment per unit length
$\boldsymbol{r}$	position vector
R	residual
$\boldsymbol{R}$	resultant force
s	curve length, bar length
S	curved surface, internal scalar force, string tension
$\boldsymbol{S}$	Kirchhoff stress tensor
t	time, thickness
$\boldsymbol{t}$	stress vector (traction)
T	kinetic energy, temperature
$\boldsymbol{T}$	pseudo-stress vector (pseudo-traction)
u,v,w	Cartesian displacement components
$\boldsymbol{u}$	displacement
$\boldsymbol{v}$	velocity
V	volume, potential energy
w	weighting function
$\boldsymbol{w}$	vector weighting function
W	work
x,y,z	Cartesian coordinates
X,Y,Z	Cartesian coordinates, inertial, local auxiliary

Greek symbols

α	penalty parameter
$\boldsymbol{\alpha}$	angular acceleration
α, β, γ	curvilinear coordinates, coefficients
δ	variation symbol,
δ_{ij}	Kronecker delta
ε_{ijk}	permutation symbol
ε	small parameter in variational calculus
$\boldsymbol{\varepsilon}$	small strain tensor
ρ	mass density
θ	polar angle, cross-section rotation
Π	functional
ϕ, θ, ψ	Euler angles
ϕ	typical unknown function
φ	trial function
γ	shearing strain, constant of gravitation
λ	Lagrange multiplier, Lame parameter
ν	Poisson's ratio
τ	shear stress, tuning parameter
$\boldsymbol{\sigma}$	Cauchy stress tensor
Ω	general domain symbol (alue)
$\overline{\Omega}$	closure of Ω = domain and its boundary
Γ	general boundary of Ω (alueen reuna)
$\boldsymbol{\omega}$	angular velocity
ξ, η, ζ	auxiliary coordinates with Euler angles, natural coordinates

Superscripts

$()^{A}$	apparent
$()^{c}$	conservative
$()^{e}$	quantity connected with eth element
$()^{I}$	inertia
$()^{k}$	constitutive
$()^{n}$	nonconservative
$()^{r}$	constraint
$\dot{()}$	time derivative
$\dot{()}$	material time derivative
$\bar{()}$	approximation, finite dimensional, skew-symmetric
$\bar{()}$	given quantity, connected with center of mass
$\hat{()}$	amplitude, dimensionless quantity

${}^{0}()$	reference state, initial state

Subscripts

$()_{D}$	Dirichlet
$()_{ext}$	external
$()_{int}$	internal
$()_{n}$	outward normal
$()_{N}$	Neumann
$()_{Q}$	connected with heat transfer
$()_{r}$	relative
$()_{s}$	sensitized
$()_{t}$	given traction
$()_{u}$	given displacement

Miscellaneous

bt	terms arising from boundary
EI	bending stiffness
GA	shearing stiffness
$\delta' W$	virtual work
$:=$	substitution symbol

Mat-5.160 Variational principles of mechanics, exercise 1

1. Write down the Taylor series of function $f = f(x,y)$ at $x = y = 0$. Use notations $x \to x_1$, $y \to x_2$, $f_{,i} = \partial f / \partial x_i$ and the summation convention. First, write the series with respect to x only and after that with respect to both variables x and y.

2. Let A_i be a vector, δ_{ij} the Kronecker delta and ε_{ijk} the permutation symbol. Show that equations

   (a) $\delta_{ij}\delta_{ij} = 3$, (b) $\varepsilon_{ijk}\varepsilon_{jki} = 6$, (c) $\varepsilon_{ijk}A_jA_k = 0$

   hold when $i,j,k \in \{1...3\}$. To verify equation (b) you need the relationship $\varepsilon_{ijk}\varepsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{ks}\delta_{jt}$. Also, verify equations (b) and (c) by direct calculation with the *Mathematica* program (you need to define the permutation symbol ε_{ijk} as a list of three indices, say).

3. Prove the following formulas

   (a) $\mathrm{div}(\phi\mathbf{v}) = \phi\,\mathrm{div}(\mathbf{v}) + \mathbf{v}\cdot\mathrm{grad}(\phi)$,

   (b) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$,

   (c) $\mathbf{u}\cdot(\nabla\mathbf{u}) = \nabla(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times(\nabla\times\mathbf{u})$.

   Hint: You may need the relationship $\varepsilon_{ijk}\varepsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{ks}\delta_{jt}$.

4. Use *Mathematica* program to solve the following system of linear equations: $x+3y+z=2$, $-x-2y+z=5$ and $3x+7y+z=-3$. Apply the *Mathematica* function (a) **LinearSolve**, (b) **Solve** and (c) **FindMinimum**.

5. What is the detailed expression for $\partial f_P / \partial\{x\}$, if

$$f_P(\{x\}) = \underset{1\times n}{\frac{1}{2}\{x\}^T} \underset{n\times n}{[A]} \underset{n\times 1}{\{x\}} + \underset{1\times n}{\{x\}^T} \underset{n\times 1}{\{a\}} + c + \underset{m\times n}{\frac{1}{2}(}\underset{m\times n}{[G]}\underset{n\times 1}{\{x\}} + \underset{m\times 1}{\{d\}})^T \underset{m\times m}{[\alpha]} (\underset{m\times n}{[G]}\underset{n\times 1}{\{x\}} + \underset{m\times 1}{\{d\}}),$$

   where $[A]$ and $[\alpha]$ are symmetric and where $[A]$, $[\alpha]$, c, $[G]$ and $\{d\}$ are independent of $\{x\}$?

Problem 1.1

Write down the Taylor series of function $f = f(x,y)$ at $x = y = 0$. Use notations $x \to x_1$, $y \to x_2$, $f_{,i} = \partial f / \partial x_i$ and the summation convention. First, write the series with respect to x only and after that with respect to both variables x and y.

Solution

The well-known formula gives when written using the index notation

$$f = f(0,x_2) + f(0,x_2)_{,1}\frac{x_1}{1!} + f(0,x_2)_{,11}\frac{x_1x_1}{2!}\ldots$$

or with the summation convention and $i,j,k\ldots \in \{1\}$

$$f = f(0,x_2) + f(0,x_2)_{,i}\frac{x_i}{i!} + f(0,x_2)_{,ij}\frac{x_ix_j}{(i+j)!}\ldots$$

The Taylor series with respect to both variables can be produced simply by changing the index set (try it) $i,j,k\ldots \in \{1,2\}$. Now, it is quite obvious how to proceed in the cases of three or more variables.

Problem 1.2

Let A_i be a vector, δ_{ij} the Kronecker delta and ε_{ijk} the permutation symbol. Show that equations

(a) $\delta_{ij}\delta_{ij} = 3$, (b) $\varepsilon_{ijk}\varepsilon_{jki} = 6$, (c) $\varepsilon_{ijk}A_jA_k = 0$

hold when $i,j,k \in \{1...3\}$. To verify equation (b) you need the relationship $\varepsilon_{ijk}\varepsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{ks}\delta_{jt}$. Also, verify equations (b) and (c) by direct

calculation with the *Mathematica* program (you need to define the permutation symbol ε_{ijk} as a list of three indices, say).

Solution

(a) The brute force approach works in this case:

$$\delta_{ij}\delta_{ij} = \sum_{i,j\in\{1..3\}} \delta_{ij}\delta_{ij} = \delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13} +$$

$$+\delta_{21}\delta_{21} + \delta_{22}\delta_{22} + \delta_{23}\delta_{23} + \delta_{31}\delta_{31} + \delta_{32}\delta_{32} + \delta_{33}\delta_{33} = 3.$$

(b) Since the summation convention is to be applied with respect to all the indices, formula $\varepsilon_{ijk}\varepsilon_{irs} = (\delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr})$ applies:

$$\varepsilon_{ijk}\varepsilon_{jki} = -\varepsilon_{ijk}\varepsilon_{ikj} = -(\delta_{jk}\delta_{kj} - \delta_{jj}\delta_{kk}) = -(\delta_{jj} - \delta_{jj}\delta_{kk}) = -(3-9) = 6.$$

(c) A trick based on symmetry:

$$\varepsilon_{ijk}A_jA_k = \frac{1}{2}\varepsilon_{ijk}A_jA_k + \frac{1}{2}\varepsilon_{ijk}A_jA_k = \frac{1}{2}\varepsilon_{ijk}A_jA_k - \frac{1}{2}\varepsilon_{ikj}A_jA_k =$$

$$= \frac{1}{2}\varepsilon_{ijk}A_jA_k - \frac{1}{2}\varepsilon_{ikj}A_kA_j = \frac{1}{2}\varepsilon_{ijk}A_jA_k - \frac{1}{2}\varepsilon_{ijk}A_jA_k = 0.$$

The *Mathematica* solution follows.

```
In :=
(* Let us define first the permutation symbol as a table with three indices. After that
equations (b) and (c) can be verified by direct summing. *)

eps = Table[0, {i,1,3},{j,1,3},{k,1,3}];
eps[[1,2,3]] = eps[[2,3,1]] = eps[[3,1,2]] = 1;
eps[[2,1,3]] = eps[[3,2,1]] = eps[[1,3,2]] = -1;

Sum[ eps[[i,j,k]]*eps[[j,k,i]],{i,1,3},{j,1,3},{k,1,3}]

i = 1;
Simplify[Sum[eps[[i,j,k]]*A[j]*A[k],{j,1,3},{k,1,3}]]
i = 2;
```



```
Simplify[Sum[eps[[i,j,k]]*A[j]*A[k],{j,1,3},{k,1,3}]]
i = 3;
Simplify[Sum[eps[[i,j,k]]*A[j]*A[k],{j,1,3},{k,1,3}]]


Out :=
6
0
0
0
```

Problem 1.3

Prove the following formulas

(a) $\mathrm{div}(\phi\mathbf{v}) = \phi\,\mathrm{div}(\mathbf{v}) + \mathbf{v}\cdot\mathrm{grad}(\phi)$,

(b) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$,

(d) $\mathbf{u}\cdot(\nabla\mathbf{u}) = \nabla(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times(\nabla\times\mathbf{u})$.

Hint: You may need the relationship $\varepsilon_{ijk}\varepsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{ks}\delta_{jt}$.

Solution

(a) $\mathrm{div}(\phi\mathbf{v}) = \frac{\partial}{\partial x_i}(\phi v_i) = \phi\frac{\partial}{\partial x_i}v_i + v_i\frac{\partial}{\partial x_i}\phi = \phi\,\mathrm{div}(\mathbf{v}) + \mathbf{v}\cdot\mathrm{grad}(\phi)$

(b) Let us consider a typical component:

$$[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]_i = \varepsilon_{ijk}a_j(\mathbf{b}\times\mathbf{c})_k = \varepsilon_{ijk}a_j\varepsilon_{kmn}b_mc_n =$$

$$= -\varepsilon_{kji}\varepsilon_{kmn}a_jb_mc_n = -(\delta_{jm}\delta_{in} - \delta_{jn}\delta_{im})a_jb_mc_n =$$

$$= -(a_jb_jc_i - a_jb_ic_j) = (\mathbf{a}\cdot\mathbf{c})[\mathbf{b}]_i - (\mathbf{a}\cdot\mathbf{b})[\mathbf{c}]_i.$$

(c) Let us start from $\mathbf{u}\times(\nabla\times\mathbf{u})$ and consider the typical component:

$$[\mathbf{u}\times(\nabla\times\mathbf{u})]_i=\varepsilon_{ijk}u_j(\nabla\times\mathbf{u})_k=\varepsilon_{ijk}u_j(\varepsilon_{kmn}\frac{\partial u_n}{\partial x_m})=(\varepsilon_{ijk}\varepsilon_{kmn})u_j\frac{\partial u_n}{\partial x_m}=$$

$$=-(\delta_{jm}\delta_{in}-\delta_{jn}\delta_{im})u_j\frac{\partial u_n}{\partial x_m}=-\delta_{jm}\delta_{in}u_j\frac{\partial u_n}{\partial x_m}+\delta_{jn}\delta_{im}u_j\frac{\partial u_n}{\partial x_m}=$$

$$=-u_j\frac{\partial u_i}{\partial x_j}+u_j\frac{\partial u_j}{\partial x_i}=-u_j\frac{\partial u_i}{\partial x_j}+\frac{1}{2}\frac{\partial}{\partial x_i}(u_ju_j)=-(\mathbf{u}\cdot\nabla u_i)+\frac{1}{2}\frac{\partial}{\partial x_i}(\mathbf{u}\cdot\mathbf{u}).$$

Problem 1.4

Use *Mathematica* program to solve the following system of linear equations: $x+3y+z=2$, $-x-2y+z=5$ and $3x+7y+z=-3$. Apply the *Mathematica* function (a) **LinearSolve,** (b) `Solve` and (c) **FindMinimum.**

Solution

(a)

```
In :=
matrix = {{1,3,1},{-1,-2,1},{3,7,1}};
vector = {2,5,-3};
LinearSolve[matrix,vector] // N

Out :=
{-6.5, 2., 2.5}
```

(b)

```
In :=
eqs := {1*x+3*y+1*z == 2,-1*x-2*y+1*z == 5,
     3*x+7*y+1*z == -3};
vars :={x,y,z};
Solve[eqs , vars] //N
```



```
Out :=
{{x -> -6.5, y -> 2., z -> 2.5}}
```

(c)

```
In :=
fun := (1*x+3*y+1*z - 2)^2+(-1*x-2*y+1*z - 5)^2+
       (3*x+7*y+1*z + 3)^2;

FindMinimum[fun, {x,0},{y,0},{z,0}] //N

Out :=
{1.75285 10^-18, {x -> -6.5, y -> 2., z -> 2.5}}
```

Problem 1.5

What is the detailed expression for $\partial f_P / \partial\{x\}$, if

$$f_P(\{x\})=\frac{1}{2}\underset{1\times n}{\{x\}^T}\underset{n\times n}{[A]}\underset{n\times 1}{\{x\}}+\underset{1\times n}{\{x\}^T}\underset{n\times 1}{\{a\}}+c+\frac{1}{2}\underset{m\times n}{([G]\{x\}+\{d\})^T}\underset{m\times m}{[\alpha]}\underset{m\times n}{([G]\{x\}+\{d\})}\underset{m\times 1}{}$$

where $[A]$ and $[\alpha]$ are symmetric and where $[A]$, $[\alpha]$, c, $[G]$ and $\{d\}$ are independent of $\{x\}$?

Solution

Let us use the result $\partial\{x\}^T / \partial\{x\}=\mathrm{I}$, which follows directly from the definition of the derivative with respect to a vector (if the index notation is applied, the result can be written as $\partial x_i / \partial x_j=\delta_{ij}$), and consider each term separately. The first term takes the form

$$\frac{1}{2}\frac{\partial}{\partial\{x\}}(\{x\}^{\mathrm{T}}[A]\{x\}) = \frac{1}{2}\frac{\partial}{\partial\{x\}}(\{x\}^{\mathrm{T}}[A]\{x\}_c) + \frac{1}{2}\frac{\partial}{\partial\{x\}}(\{x\}_c^{\mathrm{T}}[A]\{x\}) =$$

$$= \frac{1}{2}\frac{\partial}{\partial\{x\}}(\{x\}^{\mathrm{T}}[A]\{x\}_c) + \frac{1}{2}\frac{\partial}{\partial\{x\}}(\{x\}^{\mathrm{T}}[A]^{\mathrm{T}}\{x\}_c) =$$

$$= \frac{1}{2}[A]\{x\} + \frac{1}{2}[A]^{\mathrm{T}}\{x\} = [A]\{x\},$$

the second term reduces into

$$\frac{\partial}{\partial\{x\}}\{x\}^{\mathrm{T}}\{a\} = \{a\}$$

and the third gives a zero contribution. The fourth term can be manipulated as follows

$$\frac{\partial}{\partial\{x\}}\frac{1}{2}([G]\{x\} + \{d\})^{\mathrm{T}}[\alpha]([G]\{x\} + \{d\}) =$$

$$= \frac{1}{2}([G]^{\mathrm{T}}[\alpha]([G]\{x\} + \{d\}) + \frac{1}{2}([G]^{\mathrm{T}}[\alpha]^{\mathrm{T}}([G]\{x\} + \{d\}) =$$

$$= [G]^{\mathrm{T}}\frac{1}{2}([\alpha] + [\alpha]^{\mathrm{T}})([G]\{x\} + \{d\}) = [G]^{\mathrm{T}}[\alpha]([G]\{x\} + \{d\}).$$

Altogether the result is $\partial f_{\mathrm{P}} / \partial\{x\} = [A]\{x\} + [G]^{\mathrm{T}}[\alpha]([G]\{x\} + \{d\})$.

Mat-5.160 Variational principles of mechanics, exercise 2

1. Derive formula

$$\nabla\cdot\mathbf{f}=\frac{1}{h_\alpha h_\beta}[\frac{\partial}{\partial\alpha}(h_\beta f_\alpha)+\frac{\partial}{\partial\beta}(h_\alpha f_\beta)]$$

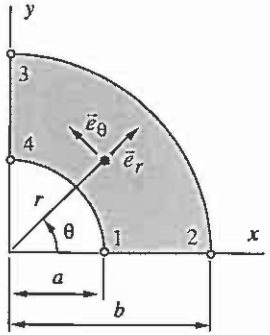
for the expression of divergence in a rectangular curvilinear coordinate system.

2. Show that the functions $f=r, g=\theta$ satisfy the integration by parts formulas

$$\int_{r,\theta} f\frac{\partial g}{\partial r}\mathrm{d}r\mathrm{d}\theta=-\int_{r,\theta}\frac{\partial f}{\partial r}g\mathrm{d}r\mathrm{d}\theta+\int_s fg\frac{n_r}{r}\mathrm{d}s\,,$$

$$\int_{r,\theta} f\frac{\partial g}{\partial\theta}\mathrm{d}r\mathrm{d}\theta=-\int_{r,\theta}\frac{\partial f}{\partial\theta}g\mathrm{d}r\mathrm{d}\theta+\int_s fgn_\theta\mathrm{d}s$$

for polar coordinates in the domain of the figure.



3. Has the function $f(x,y)=-x^2y^2+x^2+y^2$ a minimum value 0 at $x=0,\ y=0$?

4. Find the greatest and the least values of z on the ellipse formed by the intersection of the plane $x+y+z=1$ and the ellipsoid $16x^2+4y^2+z^2=16$. Then, apply the *Mathematica* function **Solve** to find the values numerically.

5. Repeat the second part of problem 4 (i.e.apply *Mathematica*) with the penalty method.

Problem 2.1

Derive formula

$$\nabla\cdot\mathbf{f}=\frac{1}{h_\alpha h_\beta}[\frac{\partial}{\partial\alpha}(h_\beta f_\alpha)+\frac{\partial}{\partial\beta}(h_\alpha f_\beta)]$$

for the expression of divergence in a rectangular curvilinear coordinate system.

Solution

Let us apply the basic formulas of Appendix C for a rectangular curvilinear coordinate system (summation convention is not employed).

$$\frac{\partial}{\partial X}=\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\,,\qquad \frac{\partial}{\partial Y}=\frac{1}{h_\beta}\frac{\partial}{\partial\beta}\,,\qquad \frac{\partial\mathbf{e}_\alpha}{\partial\alpha}=-\frac{1}{h_\beta}\frac{\partial h_\alpha}{\partial\beta}\mathbf{e}_\beta\,,$$

$$\frac{\partial\mathbf{e}_\alpha}{\partial\beta}=\frac{1}{h_\alpha}\frac{\partial h_\beta}{\partial\alpha}\mathbf{e}_\beta\,,\qquad \frac{\partial\mathbf{e}_\beta}{\partial\alpha}=\frac{1}{h_\beta}\frac{\partial h_\alpha}{\partial\beta}\mathbf{e}_\alpha\,,\qquad \frac{\partial\mathbf{e}_\beta}{\partial\beta}=-\frac{1}{h_\alpha}\frac{\partial h_\beta}{\partial\alpha}\mathbf{e}_\alpha.$$

The definition gives

$$\nabla\cdot\mathbf{f}=(\frac{\partial\mathbf{f}}{\partial X}\cdot\mathbf{i}+\frac{\partial\mathbf{f}}{\partial Y}\cdot\mathbf{j})=$$

$$=\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(f_\alpha\mathbf{e}_\alpha+f_\beta\mathbf{e}_\beta)\cdot\mathbf{e}_\alpha+\frac{1}{h_\beta}\frac{\partial}{\partial\beta}(f_\alpha\mathbf{e}_\alpha+f_\beta\mathbf{e}_\beta)\cdot\mathbf{e}_\beta=$$

$$=\frac{1}{h_\alpha}(\frac{\partial f_\alpha}{\partial\alpha}+0+0+f_\beta\frac{1}{h_\beta}\frac{\partial h_\alpha}{\partial\beta})+\frac{1}{h_\beta}(0+f_\alpha\frac{1}{h_\alpha}\frac{\partial h_\beta}{\partial\alpha}+\frac{\partial f_\beta}{\partial\beta}+0)=$$

$$=\frac{1}{h_\alpha}\frac{\partial f_\alpha}{\partial\alpha}+f_\beta\frac{1}{h_\beta h_\alpha}\frac{\partial h_\alpha}{\partial\beta}+f_\alpha\frac{1}{h_\alpha h_\beta}\frac{\partial h_\beta}{\partial\alpha}+\frac{1}{h_\beta}\frac{\partial f_\beta}{\partial\beta}=$$

$$=\frac{1}{h_\alpha h_\beta}[\frac{\partial}{\partial\alpha}(f_\alpha h_\beta)+\frac{\partial}{\partial\beta}(f_\beta h_\alpha)].$$

The second row is a direct translation of the definition of the Cartesian system on the first row to a curvilinear system. On the third row we have used the given formulas.

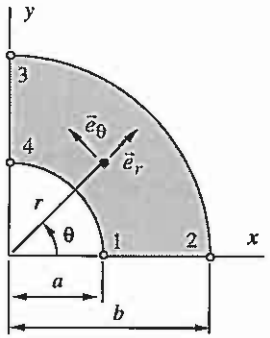
Problem 2.2

Show that the functions $f = r, g = \theta$ satisfy the integration by parts formulas

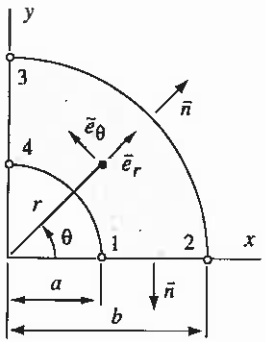
$$\int_{r,\theta} f\frac{\partial g}{\partial r}\mathrm{d}r\mathrm{d}\theta = -\int_{r,\theta} \frac{\partial f}{\partial r} g\mathrm{d}r\mathrm{d}\theta + \int_s fg\frac{n_r}{r}\mathrm{d}s\,,$$

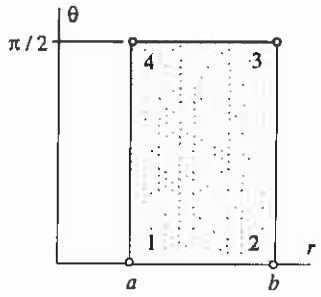
$$\int_{r,\theta} f\frac{\partial g}{\partial \theta}\mathrm{d}r\mathrm{d}\theta = -\int_{r,\theta} \frac{\partial f}{\partial \theta} g\mathrm{d}r\mathrm{d}\theta + \int_s fgn_\theta \mathrm{d}s$$

for polar coordinates in the domain of the figure.



Solution





The figure shows the domain in the r,θ -system where the formulas are to be applied. The unit outward normal to the boundary (appearing in the formulas) has the components:

side $1-2$: $n_r = 0,\ n_\theta = -1$,

side $2-3$: $n_r = 1,\ n_\theta = 0$,

side $3-4$: $n_r = 0,\ n_\theta = 1$,

side $4-1$: $n_r = -1,\ n_\theta = 0$,

The first equation gives

$$\int_{r,\theta} r\cdot 0\mathrm{d}r\mathrm{d}\theta = -\int_{r,\theta} \theta\mathrm{d}r\mathrm{d}\theta + \int_s n_r\theta\mathrm{d}s =$$

$$= -\int_0^{\pi/2}\int_a^b \theta\mathrm{d}r\mathrm{d}\theta + \int_2^3 \theta b\mathrm{d}\theta - \int_1^4 \theta a\mathrm{d}\theta =$$

$$= -(b-a)\int_0^{\pi/2} \theta\mathrm{d}\theta + \int_0^{\pi/2} \theta b\mathrm{d}\theta - \int_0^{\pi/2} \theta a\mathrm{d}\theta = 0.$$

The second equation can be verified in the way similar to that used above:

$$\int_{r,\theta} r\mathrm{d}r\mathrm{d}\theta = -\int_{r,\theta} 0\cdot\theta\mathrm{d}r\mathrm{d}\theta + \int_s r\theta n_\theta \mathrm{d}s =$$

$$= -\int_1^2 r\theta\mathrm{d}r + \int_4^3 r\theta\mathrm{d}r = \int_{r,\theta} r\mathrm{d}r\mathrm{d}\theta.$$

Problem 2.3

Has the function $f(x,y) = -x^2y^2 + x^2 + y^2$ a minimum value 0 at $x = 0,\ y = 0$?

Solution

The function has a local minimum at a point inside the domain if the partial derivatives vanish and if the Hessian is positive definite at that point. Let us first verify that the given point is stationary

$$\frac{\partial}{\partial x} f(x,y) = -2xy^2 + 2x = 0,$$

$$\frac{\partial}{\partial y} f(x,y) = -2x^2 y + 2y = 0.$$

A solution is clearly $x = 0, y = 0$. Then, let us verify the second condition

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x} f(x,y) = -2y^2 + 2, \qquad \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y) = -4xy$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) = -4xy, \qquad \frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x,y) = -2x^2 + 2$$

At $x = 0, y = 0$ the Hessian

$$[H] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

is clearly positive definite. Conclusion: the function takes a minimum at the origin of the coordinate system.

Problem 2.4

Find the greatest and the least values of z on the ellipse formed by the intersection of the plane $x + y + z = 1$ and the ellipsoid $16x^2 + 4y^2 + z^2 = 16$. Then, apply the *Mathematica* function `Solve` to find the values numerically.

Solution

The problem is to minimize $f = z$ under the constraints $g_1 = x + y + z - 1 = 0$ and $g_2 = 16x^2 + 4y^2 + z^2 - 16 = 0$. Alternatively one may seek the extremum value for

$$f_L = f + \lambda_i g_i = z + \lambda_1 (x + y + z - 1) + \lambda_2 (16x^2 + 4y^2 + z^2 - 16) .$$

At the point where the function takes the extremum value, the partial derivatives with respect to the variables satisfy

$$\frac{\partial}{\partial x} f_L = \lambda_1 + 32\lambda_2 x = 0, \qquad \frac{\partial}{\partial y} f_L = \lambda_1 + 8\lambda_2 y = 0,$$

$$\frac{\partial}{\partial z} f_L = 1 + \lambda_1 + 2\lambda_2 z = 0, \qquad \frac{\partial}{\partial \lambda_1} f_L = x + y + z - 1 = 0,$$

$$\frac{\partial}{\partial \lambda_2} f_L = 16x^2 + 4y^2 + z^2 - 16 = 0.$$

In general, a non-linear equation system (even only quadratic in the variables) may be very difficult to solve. In this case one may use, however, the fact that the two first equations are consistent only if $y = 4x$. Then the second last equation gives $x = (1 - z)/5$. Finally, the using of the relationships to eliminate x, y from the last equation results into

$$21z^2 - 32z - 64 = 0$$

giving $z = 8/3$ and $z = -8/7$ as the maximum and minimum values.

The mathematica solution follows:

In :=

(* Solve finds the solution to a given system of equations *)

```
equations := {L1+32*L2*x == 0, L1+8*L2*y == 0,
1+L1+2*L2*z == 0,x+y+z-1 == 0,16*x^2+4*y^2+z^2-16 == 0};
variables :={L1,L2,x,y,z};
Solve[equations,variables] //N
```

Out :=

```
{{L1 -> -0.857143, L2 -> 0.0625, x -> 0.428571, y -> 1.71429, z -> -1.14286}, {L1 -> -
0.666667, L2 -> -0.0625, x -> -0.333333, y -> -1.33333, z -> 2.66667}}
```

In:=

```
(* A more convenient way is to define a function  for finding the extremum point *)
GRAD[exp_,var_]:= Map[D[exp,#]&,Flatten[var]];
function := z + L1*(x+y+z-1)+L2*(16*x^2+4*y^2+z^2-16);
variables :={L1,L2,x,y,z};
equations = GRAD[function,variables];
Solve[equations == 0,variables] //N
```

Out:=

```
{{L1 -> -0.857143, L2 -> 0.0625, x -> 0.428571, y -> 1.71429, z -> -1.14286}, {L1 -> -
0.666667, L2 -> -0.0625, x -> -0.333333, y -> -1.33333, z -> 2.66667}}
```

Problem 2.5

Repeat the second part of problem 4 (i.e. apply *Mathematica*) with the penalty method.

Solution

In :=

```
(* some function definitions are needed here *)

GRAD[fun_,var_]:= Map[D[fun,#]&,var];
```



```
RULE[a_,b_] := Table[a[[i]] -> b[[i]],{i,1,Length[a]}];

EXTREMIZE[fun_,var_,var0_]:= Module[{FUN, HESS},
      FUN = GRAD[fun,var]; HESS = GRAD[FUN,var];
      FixedPoint[(# - LinearSolve[(HESS /. RULE[var,#]),
      (FUN  /. RULE[var,#])])&,
      var0, SameTest -> (Max[Abs[#1-#2]] < 10^-5&)]];

(* then the definitions related to problem 2.5 *)

alfa = beta = 100;
fun = z + alfa*(x+y+z-1)^2 + beta*(16*x^2+4*y^2+z^2-16)^2;
var = {x,y,z};
var0 = {1.,1.,-1.};

EXTREMIZE[fun,var,var0]
```

Out :=

```
{0.428563, 1.71424, -1.14323}
```

Mat-5.160 Variational principles of mechanics, exercise 3

1. Write a *Mathematica* program for the Lagrange multiplier/penalty method algorithm. Then, use the program to solve problem 4 of exercise 2.

2. Derive the Euler-Lagrange equation corresponding to functional

$$\Pi(\phi) = \int_a^b [p(x)(\phi')^2 + q(x)\phi^2 + 2r(x)\phi]\mathrm{d}x ,$$

where p, q and r given continuous functions (a) by using formula $\frac{\partial f}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial f}{\partial \phi'}) = 0$, (b) by performing the variation directly on the functional.

3. Derive the Euler-Lagrange equation and the natural boundary condition for the functional

$$V(u) = \int_0^l [\frac{AE}{2}(u')^2 - uq]\mathrm{d}x - Pu(l)$$

with the essential boundary condition $u(0) = 0$. The quantities AE and q are independent of u but they may depend on x and P is constant.

4. Deflection v of a simply supported elastic beam can be obtained from the stationarity of the functional

$$V(v) = \int_0^l [\frac{EI}{2}(v'')^2 - vq]\mathrm{d}x$$

in the set of smooth functions satisfying the essential boundary conditions $v(0) = v(l) = 0$. Knowing that, derive the corresponding boundary value problem. Assume that EI and q are given functions of x.

5. Use *Mathematica* program to find an approximate solution to the deflection problem above. Substitute the polynomial approximation

$$v = \sum_{i\in\{1..n\}} a_i (x/l)^i (x/l - 1),$$

where the a_i:s are free parameters and minimize the function (after substitution the left hand side is an ordinary function of the a_i:s) with respect to them. Consider the cases $q = x/l \cdot \sin(x/l)$, $EI = 1$, $l = 1$ and $n \in \{2..4\}$.

Problem 3.1

Write a *Mathematica* program for the Lagrange multiplier/penalty method algorithm. Then, use the program to solve problem 4 of exercise 2.

Solution

See the examples section of ALP.

Problem 3.2

Derive the Euler-Lagrange equation corresponding to functional

$$\Pi(\phi) = \int_a^b [p(x)(\phi')^2 + q(x)\phi^2 + 2r(x)\phi]\mathrm{d}x ,$$

where p, q and r given continuous functions

(a) using formula $\frac{\partial f}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial f}{\partial \phi'}) = 0$

(b) by performing the variation directly on the functional.

Solution

(a) The expression for the integrand is $f = p(x)(\phi')^2 + q(x)\phi^2 + 2r(x)\phi$. Thus

$$\frac{\partial f}{\partial \phi} = 2q(x)\phi + 2r(x), \quad \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial f}{\partial \phi'}) = \frac{\mathrm{d}}{\mathrm{d}x}(2p(x)\phi') = (2p(x)\phi')'$$

giving as the Euler-Lagrange equation $(2p(x)\phi')' + 2q(x)\phi + 2r(x) = 0$.

(b) The first variation of the functional is

$$\delta\Pi = \int_a^b [2p(x)\phi'\delta\phi' + 2q(x)\phi\delta\phi + 2r(x)\delta\phi]\mathrm{d}x =$$

$$= \int_a^b [-(2p(x)\phi')' + 2q(x)\phi + 2r(x)]\delta\phi\mathrm{d}x + \sum_{x\in\{a,b\}}(2p(x)\phi' n_x)\delta\phi = 0.$$

Let us first consider the set of functions whose members vanish at the boundaries. Then the last term is zero and the standard argument gives the Euler-Lagrange equation

$$(2p(x)\phi')' + 2q(x)\phi + 2r(x) = 0 \quad x \in]a,b[\ .$$

Knowing that, one may omit the first integral and consider the set of functions satisfying the field equation only. Then, the natural boundary condition is seen to be

$$2p(x)\phi' n_x = 0 \quad x \in \{a,b\} \ .$$

Problem 3.3

Derive the Euler-Lagrange equation and the natural boundary condition for the functional

$$V(u) = \int_0^l [\frac{AE}{2}(u')^2 - uq]\mathrm{d}x - Pu(l)$$

with the essential boundary condition $u(0) = 0$. The quantities AE and q are independent of u but they may depend on x and P is a constant.

Solution

The first variation of the functional gives

$$\delta V = \int_0^l [EAu'\delta u' + q\delta u]\mathrm{d}x - P\delta u =$$

$$= \int_a^b [-(AEu')' + q]\delta u\mathrm{d}x + [(EAu' n_x - P)\delta u]_{x=l} = 0$$

which should be satisfied for any selection of the variation δu restricted by $\delta u(0) = 0$. Let us first consider the set of functions whose members vanish at $x = l$ also. Then the last term is zero and the Euler-Lagrange equation is seen to be

$$-(AEu')' + q = 0 \quad x \in]0,l[\ .$$

Knowing that, one may omit the first integral and consider the set of functions satisfying the field equation. This gives the natural boundary condition

$$EAu' n_x - P = 0 \quad x = l \ .$$

Problem 3.4

Deflection v of an elastic beam can be obtained from the stationarity of the functional

$$V(v) = \int_0^l [\frac{EI}{2}(v'')^2 - vq]\mathrm{d}x$$

in the set of smooth functions satisfying the essential boundary conditions $v(0) = v(l) = 0$. Knowing that, derive the corresponding boundary value problem. Assume that EI and q are given functions of x.

Solution

The first variation of the functional gives

$$\delta V = \int_0^l [EIv''\delta v'' + q\delta v]dx =$$

$$= \int_0^l [-(EIv'')'\delta v' + q\delta v]dx + \sum_{x\in\{0,l\}} [EIv''n_x\delta v'] =$$

$$= \int_0^l [(EIv'')'' + q]\delta v dx + \sum_{x\in\{0,l\}} [EIv''n_x\delta v'] = 0$$

since $\delta v(0) = \delta v(l) = 0$. The equation should be satisfied for any selection of the variation δv restricted by $\delta v(0) = \delta v(l) = 0$. Let us first consider the set of functions with the property $\delta v'(0) = \delta v'(l) = 0$. Then the last term is zero and the the Euler-Lagrange equation is clearly

$$(EIv'')'' + q = 0 \quad x \in]0,l[\ .$$

After that one may consider the set of functions satisfying the field equation and discard the first integral. The remaining equation implies the natural boundary condition

$$EIv''n_x = 0 \quad x \in \{0,l\} \ .$$

Problem 3.5

Use *Mathematica* program to find an approximate solution to the deflection problem above. Substitute the polynomial approximation

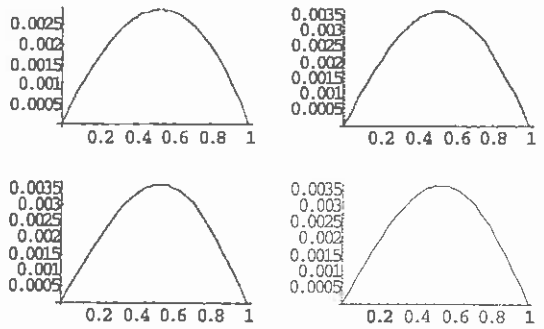
$$v = \sum_{i\in\{1..n\}} a_i (x/l)^i (x/l - 1),$$

where the a_i:s are free parameters and minimize the function (after substitution the left hand side is an ordinary function of the a_i:s) with respect to them. Consider the cases $q = x/l \cdot \sin(x/l)$, $EI = 1$, $l = 1$ and $n \in \{2..4\}$.

Solution

```
Plo = {};
EI  = 1;
l   = 1;
f   = x/l*Sin[x/l];
w[k_]:= (1-x/l)*(x/l)^k;
Do[
    Y = Sum[w[k]*a[k],{k,1,n}];
    V  = Integrate[EI/2*D[y,{x,2}]^2-y*f,{x,0,1}];
    B  = Solve[Table[D[V,a[k]] == 0,{k,1,n}], Table[a[k],{k,1,n}]];
    z[n] = N[(y /. B)[[1]]];
    Plo  = Append[Plo,Plot[z[n],{x,0,1}, DisplayFunction -> Identity]],
{n,2,5}];

Show[GraphicsArray[Partition[Plo,2]], DisplayFunction -> $DisplayFunction];
```



Mat-5.160 Variational principles of mechanics, exercise 4

1. Derive the weak formulation for the diffusion-convection-reaction equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}(k\frac{\mathrm{d}T}{\mathrm{d}x})+u\frac{\mathrm{d}T}{\mathrm{d}x}+cT-s=0 \quad x\in]0,L[,$$

where the diffusion coefficient k, velocity u and the sink factor c are independent of the unknown function T and the boundary condition is $T-\overline{T}=0$ $x\in\{0,L\}$ ($\overline{T}$ is the given value on the boundary).

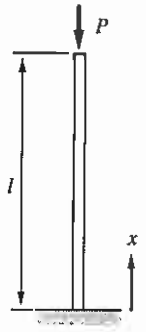
2. Let us consider the displacement field u of an elastic body Ω in the case where the displacement is known on the boundary Γ_u, $\overline{\Gamma}_u\cup\overline{\Gamma}_t=\Gamma$, $\Gamma_u\cap\Gamma_t=\varnothing$. The potential energy functional is

$$V(\mathrm{u})=\frac{1}{2}\int_\Omega C_{ijkl}u_{i,j}u_{k,l}\mathrm{d}\Omega,$$

in which the quantities C_{ijkl} are independent of u. Write down the boundary value problem whose solution makes the functional stationary.

3. Write down the functional for the problem of finding the domain $\Omega\subset\mathbb{R}^2$ with minimum boundary length when the area is given. Also, derive the corresponding boundary value problem and verify that a circular boundary curve is the solution.

4. The potential energy for an axially inextensible column subjected to a compressing force P is



$$V(v)=\int_0^l[\frac{EI}{2}(v'')^2]\mathrm{d}x-P\int_0^l[\frac{1}{2}(v')^2\mathrm{d}x.$$

Find the minimum value for P causing the column to buckle.

5. What is the underlying boundary value problem of the weak formulation: find $\phi-\overline{\phi}\in V$, such that

$$\int_\Omega(D_{\alpha\beta}\frac{\partial w}{\partial x_\alpha}\frac{\partial\phi}{\partial x_\beta}-wf)\mathrm{d}\Omega+\int_{\Gamma_N}wh\mathrm{d}\Gamma=0 \quad \forall w\in V,$$

where $V=\{v: v\in C^0(\Omega), v|\Gamma_D=0\}$ and $\Gamma=\overline{\Gamma}_D\cup\overline{\Gamma}_N$, $\Gamma_D\cap\Gamma_N=\varnothing$?

Problem 4.1

Derive the weak formulation for the diffusion-convection-reaction equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}(k\frac{\mathrm{d}T}{\mathrm{d}x})+u\frac{\mathrm{d}T}{\mathrm{d}x}+cT-s=0 \quad x\in]0,L[,$$

where the diffusion coefficient k, velocity u and the sink factor c are independent of the unknown function T and the boundary condition is $T-\overline{T}=0$ $x\in\{0,L\}$ ($\overline{T}$ is the given value on the boundary).

Solution

The starting point is the field equation multiplied by the weighting function. Integration over the solution domain gives

$$\int_0^L w[-\frac{\mathrm{d}}{\mathrm{d}}(k\frac{\mathrm{d}T}{\mathrm{d}x})+u\frac{\mathrm{d}T}{\mathrm{d}x}+cT-s]\mathrm{d}x=0.$$

or when integration by parts (assuming a smooth enough integrand) is performed

$$\int_0^L[\frac{\mathrm{d}w}{\mathrm{d}x}k\frac{\mathrm{d}T}{\mathrm{d}x}+wu\frac{\mathrm{d}T}{\mathrm{d}x}wcT-ws]\mathrm{d}x-(wk\frac{\mathrm{d}T}{\mathrm{d}x})_{x=L}+(wk\frac{\mathrm{d}T}{\mathrm{d}x})_{x=0}=0. \qquad (1)$$

When the boundary condition is of the Dirichlet type, one usually sets $w=0$ at the corresponding part of the boundary. Here $T-\overline{T}=0$ $x\in\{0,L\}$ which means the selection $w(0)=w(L)=0$ makes the boundary terms to disappear. Thus the weak form of the problem reads: find $T(x)$ satisfying $T(0)=\overline{T}_0$ and $T(L)=\overline{T}_L$, such that

$$\int_0^L[\frac{\mathrm{d}w}{\mathrm{d}x}k\frac{\mathrm{d}T}{\mathrm{d}x}+wu\frac{\mathrm{d}T}{\mathrm{d}x}wcT-ws]\mathrm{d}x=0$$

for any $w(x)$ satisfying $w(0)=0$ and $w(L)=0$.

Problem 4.2

Let us consider the displacement field $\mathbf{u}$ of an elastic body Ω in the case where the displacement is known on the boundary Γ_u, $\overline{\Gamma}_u \cup \overline{\Gamma}_t = \Gamma$, $\Gamma_u \cap \Gamma_t = \emptyset$. The potential energy functional is

$$V(\mathbf{u}) = \frac{1}{2}\int_\Omega C_{ijkl} u_{i,j} u_{k,l} \mathrm{d}\Omega ,$$

where the quantities C_{ijkl} are independent of $\mathbf{u}$. Write down the boundary value problem whose solution makes the functional stationary.

Solution

The functional takes a stationary value when the first variation vanishes i.e.

$$\delta V(\mathbf{u}) = \frac{1}{2}\int_\Omega C_{ijkl}(\delta u_{i,j} u_{k,l} + u_{i,j}\delta u_{k,l})\mathrm{d}\Omega =$$

$$= \frac{1}{2}\int_\Omega -C_{ijkl}(\delta u_i u_{k,lj} + u_{i,jl}\delta u_k)\mathrm{d}\Omega + \frac{1}{2}\int_{\Gamma_t} C_{ijkl}(\delta u_i n_j u_{k,l} + u_{i,j} n_l \delta u_k)\mathrm{d}\Omega =$$

$$= \frac{1}{2}\int_\Omega -(C_{ijkl}u_{k,lj} + C_{kjil}u_{k,jl})\delta u_i \mathrm{d}\Omega + \frac{1}{2}\int_{\Gamma_t} (C_{ijkl} n_j u_{k,l} + C_{kjil} u_{k,j} n_l)\delta u_i \mathrm{d}\Omega = 0$$

The original domain of the boundary integral $\Gamma = \overline{\Gamma}_u \cup \overline{\Gamma}_t$ has been changed to Γ_t since $\delta u_i|\Gamma_u = 0$. Let us consider first variations that vanish everywhere on the boundary. The necessary condition for the expression to vanish is then

$$C_{ijkl}u_{k,lj} + C_{kjil}u_{k,jl} = 0 \quad \mathbf{x} \in \Omega.$$

Using the standard argument the natural boundary is found to be

$$C_{ijkl} n_j u_{k,l} + C_{kjil} u_{k,j} n_l = 0 \quad \mathbf{x} \in \Gamma_t .$$

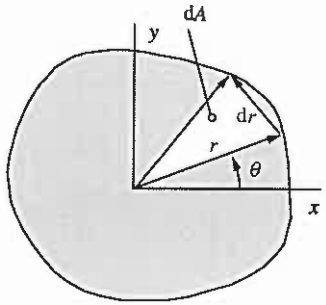
Finally, to complete the system of equations, the restriction

$$u_i - \overline{u}_i = 0 \quad \mathbf{x} \in \Gamma_n$$

where $\overline{u}_i$ is known, has to be included.

Problem 4.3

Write down the functional for the problem of finding the domain $\Omega \subset \mathbb{R}^2$ with minimum boundary length when the area is given. Also, derive the corresponding boundary value problem and verify that a circular boundary curve is the solution.



Solution

The notations to be used are shown in the figure. In this case it is convenient to use the polar coordinate system. Then

$$\bar{r} = r\cos\theta \bar{i} + r\sin\theta \bar{j} ,$$

$$d\bar{r} = (dr\cos\theta - r\sin\theta\, d\theta)\bar{i} + (dr\sin\theta + r\cos\theta\, d\theta)\bar{j} ,$$

$$\mathrm{d}A = \frac{1}{2}|\bar{r} \times \mathrm{d}\bar{r}| = \frac{1}{2}r^2 \mathrm{d}\theta ,$$

$$\mathrm{d}S = \sqrt{\mathrm{d}\bar{r} \cdot \mathrm{d}\bar{r}} = \sqrt{r'^2 + r^2}\,\mathrm{d}\theta .$$

The independent variable is chosen to be θ. Let us minimize the length of the boundary curve under the constraint that the area of the domain inside the curve is constant A, say. The corresponding functional is

$$\Pi(r,\lambda)=\int_0^{2\pi}\sqrt{r'^2+r^2}\,d\theta+\lambda[\int_0^{2\pi}\frac{1}{2}r^2d\theta-A].$$

At the stationary point the first variation vanishes

$$\delta\Pi(r,\lambda)=\int_0^{2\pi}[\frac{r'\delta r'+r\delta r}{\sqrt{r'^2+r^2}}+\lambda r\delta r]d\theta+\delta\lambda[\int_0^{2\pi}\frac{1}{2}r^2d\theta-A]=$$

$$=\int_0^{2\pi}[-\frac{d}{dr}(\frac{r'}{\sqrt{r'^2+r^2}})+\frac{r}{\sqrt{r'^2+r^2}}+\lambda r]\delta r d\theta+$$

$$+\sum_{\theta\in\{0,2\pi\}}(\frac{r'n_\theta\delta r}{\sqrt{r'^2+r^2}})+\delta\lambda[\int_0^{2\pi}\frac{1}{2}r^2d\theta-A]=0$$

where the boundary terms vanishes if one assumes that $r\in C^2$, i.e. that the boundary curve is smooth. The system of equations (together with the smoothness assumptions) is

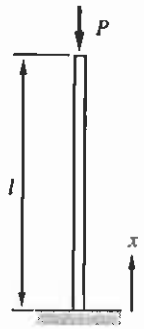
$$-\frac{d}{dr}(\frac{r'}{\sqrt{r'^2+r^2}})+\frac{r}{\sqrt{r'^2+r^2}}+\lambda r=0\quad\theta\in]0,2\pi[,$$

$$r'(0)=r'(2\pi),\quad r(0)=r(2\pi),$$

$$\int_0^{2\pi}\frac{1}{2}r^2d\theta-A=0.$$

The easiest way to proceed from that point on is simply to use a guess $r=R=\text{constant}$ giving after some simple manipulations $A=\pi R^2$ and $\lambda=-1/R$. With these selections all the equations are satisfied which means that the solution is found.

Problem 4.4



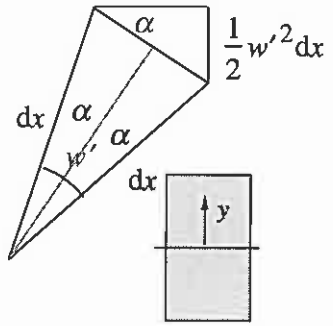
The potential energy for an axially inextensible column subjected to a compressing force P is

$$V(v)=\int_0^l[\frac{EI}{2}(v'')^2]dx-P\int_0^l[\frac{1}{2}(v')^2dx.$$

Find the minimum value for P causing the column to buckle.

Solution

Before going into the actual subject a few words about the physical background. The potential energy of the column consists of the internal and the external parts V^{int} and V^{ext}, respectively. The external part i.e. the potential energy of the external force P follows from the vertical displacement of the upper end when the displacements of the column elements (see the figure) are summed giving



$$V^{\text{ext}}(v)=-P\int_0^l[\frac{1}{2}(v')^2dx$$

The internal part having to do with the stress field, follows from the basic formula

$$V^{\text{int}}(v)=\int_V\frac{1}{2}\varepsilon_{ij}\sigma_{ij}dV$$

when the usual kinematical and stress-strain relationships $\sigma_{xx}=E\varepsilon_{xx}$, $\varepsilon_{xx}=du/dx$, $u=-ydv/dx$ are substituted and integration over the cross section is performed (see the figure) .

Now, let us minimize the functional in the function set whose members satisfy the essential conditions $v(0)=v'(0)=0$.

$$\delta V = \int_0^l [EIv''\delta v'' - Pv'\delta v']\mathrm{d}x =$$

$$= \int_0^l [-(EIv'')'\delta v' + Pv''\delta v)\mathrm{d}x + [EIv''\delta v' n_x - Pv' n_x \delta v]_{x=l} =$$

$$= \int_0^l [(EIv'')'' + Pv'']\delta v \mathrm{d}x + [-(EIv'')'\delta v n_x + EIv''\delta v' n_x - Pv' n_x \delta v]_{x=l} = 0.$$

The boundary terms at $x=0$ vanish since $\delta v(0) = \delta v'(0) = 0$. The Euler-Lagrange equations with the natural and the essential conditions are then

$$(EIv'')'' + Pv'' = 0 \qquad x \in]0, l[\,,$$

$$(EIv'')' + Pv' = 0 \qquad x = l\,,$$

$$EIv'' = 0 \qquad x = l\,,$$

$$v = v' = 0 \qquad x = 0.$$

The solution to the problem is of the form $v = a + bx + c\sin\alpha x + d\cos\alpha x$ where $\alpha = \sqrt{P/EI}$. When substituted to the boundary condition, the solution gives

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & -\alpha^2\sin\alpha l & -\alpha^2\cos\alpha l \\ 0 & \alpha^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

For a non-trivial solution the determinant of the matrix must vanish. Thus $\cos\alpha l = 0$ giving $\alpha l = n\pi/2$ and $P = EI/l^2 \cdot (\pi/2)^2$.

Problem 4.5

What is the underlying boundary value problem of the weak formulation: find $\phi - \overline{\phi} \in V$, such that

$$\int_\Omega (D_{\alpha\beta}\frac{\partial w}{\partial x_\alpha}\frac{\partial \phi}{\partial x_\beta} - wf)\mathrm{d}\Omega + \int_{\Gamma_N} wh\mathrm{d}\Gamma = 0 \quad \forall w \in V,$$

where $V = \{v : v \in C^0(\Omega), v|\Gamma_D = 0\}$ and $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \varnothing$?

Solution

Assuming that the solution to the problem is smooth, integration by parts gives first

$$\int_\Omega (-\frac{\partial}{\partial x_\alpha}(D_{\alpha\beta}\frac{\partial \phi}{\partial x_\beta}) - wf)\mathrm{d}\Omega + \int_{\Gamma_N \cup \Gamma_D} wn_\alpha D_{\alpha\beta}\frac{\partial \phi}{\partial x_\beta}\mathrm{d}\Gamma + \int_{\Gamma_N} wh\mathrm{d}\Gamma = 0,$$

or, since $w|\Gamma_D = 0$,

$$\int_\Omega (-\frac{\partial}{\partial x_\alpha}(D_{\alpha\beta}\frac{\partial \phi}{\partial x_\beta}) - wf)\mathrm{d}\Omega + \int_{\Gamma_N} w(n_\alpha D_{\alpha\beta}\frac{\partial \phi}{\partial x_\beta} + h)\mathrm{d}\Gamma = 0\,.$$

Thus the boundary value problem consists of equations

$$-\frac{\partial}{\partial x_\alpha}(D_{\alpha\beta}\frac{\partial \phi}{\partial x_\beta}) - f = 0 \qquad \text{in } \Omega,$$

$$n_\alpha D_{\alpha\beta}\frac{\partial \phi}{\partial x_\beta} + h = 0 \qquad \text{on } \Gamma_N,$$

$$\phi - \overline{\phi} = 0 \qquad \text{on } \Gamma_D.$$

Mat-5.160 Variational principles of mechanics, exercise 5
(to be returned before March 12)

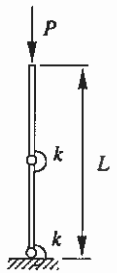
1. Find the ratio of radius R to height H that will minimize the total surface area of a cylinder of fixed volume. Use the Lagrange multiplier method.

2. The length S of a curve between points (x_1, y_1) and (x_2, y_2) is given by

   $$S = \int_{(x_1,y_1)}^{(x_2,y_2)} \mathrm{d}s,$$

   where $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. Show that the minimizer of S is a straight line between the end points.

3. What is the minimum value of the compressing force P causing the system of the figure to buckle? The bars are rigid and weightless, and the springs linear.



4. Use the Ritz's method with

   $$v = \sum_{i\in\{2\ldots n\}} a_i x^i$$

   to find an approximation to the buckling force of problem 4.4. Consider cases $n \in \{2..5\}$.

5. Find an approximation to function $f(x) = 5x^2 \sin(\pi x)$ $x \in [0,1]$ by using the weak formulation

   $$\int_0^1 w(\phi - f)\mathrm{d}x = 0 \quad \forall w \in V,$$

   where $V = \{v : v = \sum_{i\in\{0\ldots 4\}} a_i x^i\}$ (polynomials up to and including degree 4).

Problem 5.1
Find the ratio of radius R to height H that will minimize the total surface area of a cylinder of fixed volume. Use the Lagrange multiplier method.

Solution
The total surface area and the volume of the cylinder are given by

$$A = \pi R^2 + \pi R^2 + 2\pi RH,$$

$$V = \pi R^2 H,$$

respectively. The functional to be extremized

$$\Pi(R, H, \lambda) = 2\pi R(R + H) + \lambda(V - \pi R^2 H)$$

gives

$$\frac{\partial \Pi}{\partial R} = 2\pi(2R + H) - \lambda 2\pi RH = 0,$$

$$\frac{\partial \Pi}{\partial H} = 2\pi R - \lambda \pi R^2 = 0,$$

$$\frac{\partial \Pi}{\partial \lambda} = V - \pi R^2 H = 0.$$

By eliminating the Lagrange multiplier from the two first equations we get

$$2R + H - 2H = 0$$

or

$$R / H = 1/2.$$

Problem 5.2

The length S of a curve between points (x_1, y_1) and (x_2, y_2) is given by

$$S = \int_{(x_1,y_1)}^{(x_2,y_2)} \mathrm{d}s,$$

where $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. Show that the minimizer of S is a straight line between the end points.

Solution

Let us consider x as the independent variable. Then, the length to be extremized is

$$S(y) = \int_{x_1}^{x_2} \sqrt{1+y'}\,dx,$$

where $y(x_1) = y_1$ and $y(x_2) = y_2$. The variation of the functional should vanish i.e. (the boundary terms arising from integration by parts vanish since $\delta y = 0$ at the end points)

$$\delta S(y) = \int_{x_1}^{x_2} \frac{1}{2}\frac{\delta y'}{\sqrt{1+y'}}\mathrm{d}x = -\int_{x_1}^{x_2} \frac{1}{2}\frac{d}{dx}\left(\frac{1}{\sqrt{1+y'}}\right)\delta y\,\mathrm{d}x + \left[\frac{1}{2}\left(\frac{1}{\sqrt{1+y'}}\right)\delta y\right] = 0$$

giving

$$\frac{d}{dx}\left(\frac{1}{\sqrt{1+y'}}\right) = 0.$$

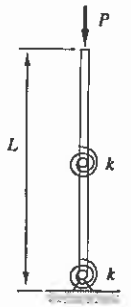
The solution to this equation is $y = ax + b$. The two constants follow from the boundary conditions. Therefore the minimizer is given by

$$y = \frac{x - x_2}{x_1 - x_2} y_1 + \frac{x - x_1}{x_2 - x_1} y_2,$$

which represents a straight line between the given points.

Problem 5.3

Determine the force P causing the system of the figure to buckle. The bars are assumed totally rigid and massless, and torsion springs linear i.e. $M = k\Delta\theta$.

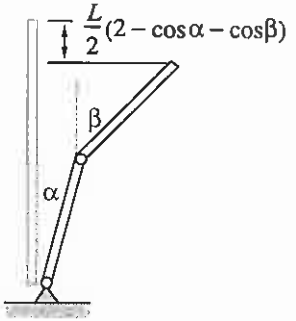


Solution

The potential energies (see the figure) of the point force and the torsion springs are given by

$$V^1 = \frac{PL}{2}(2 - \cos\alpha - \cos\beta),$$

$$V^2 = \frac{1}{2}k\alpha^2 + \frac{1}{2}k(\beta - \alpha)^2,$$



respectively. At the equilibrium the derivatives of $V = V^1 + V^2$ with respect to the generalized coordinates α and β vanish. So

$$\frac{\partial V}{\partial \alpha} = k\alpha - k(\alpha - \beta) - \frac{PL}{2}\sin\alpha = 0,$$

$$\frac{\partial V}{\partial \beta} = k(\beta - \alpha) - \frac{PL}{2}\sin\beta = 0.$$

When the problem is linearized at the obvious equilibrium position $\alpha=\beta=0$, the result is

$$\begin{bmatrix} 2k-PL/2 & -k \\ -k & k-PL/2 \end{bmatrix}\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}=0$$

having other solutions than $\alpha=\beta=0$ only if the determinant of the matrix vanishes. This gives $(2k-PL/2)(k-PL/2)-k^2=0$ or

$$(PL)^2-6kPL-4k^2=0 \quad \Rightarrow PL=(6k\pm\sqrt{36k^2-4\cdot 4k^2})/2=(3\pm\sqrt{5})k\,.$$

Conclusion: the minimum force causing the system to loose its stability is $P=(3-\sqrt{5})k/L$.

Problem 5.4
Use the Ritz's method with

$$v=\sum_{i\in\{2..n\}} a_i x^i$$

to find an approximation to the buckling force of problem 4.4. Consider cases $n\in\{2..5\}$.

Solution
Let us extremize the functional

$$V(v)=\int_0^l [\frac{EI}{2}(v'')^2]\mathrm{d}x-P\int_0^l [\frac{1}{2}(v')^2\mathrm{d}x$$

in the set of polynomials. When the approximation is substituted, the result is an ordinary equation taking a stationary value, if the partial derivatives with respect to the parameters vanish i.e.

$$\sum_{j\in\{2..n\}}(A_{ij}-\frac{Pl^2}{D}B_{ij})a_j=0 \quad \forall i\in\{2..n\},$$

where

$$A_{ij}=\int_0^l [ij(i-1)(j-1)(x/l)^{i+j-4}]\mathrm{d}x/l^4=\frac{1}{l^3}\frac{ij(i-1)(j-1)}{i+j-3},$$

$$B_{ij}=\int_0^l [ij(x/l)^{i+j-2}]\mathrm{d}x/l^2=\frac{1}{l}\frac{ij}{i+j-1},$$

and $D=EI$. The linear system of equations has a non-trivial solution, if one the eigenvalues of the matrix $\mathbf{B}^{-1}\mathbf{A}$ is Pl^2/D. The mathematica solution follows:

```
Do[
    A = Table[i*(i-1)*j*(j-1)*(1/(i+j-3)),{i,2,n},{j,2,n}];
    B = Table[i*j/(i+j-1),{i,2,n},{j,2,n}];
    Print["n = ",n,"    P/D*l^2 =",
    Min[Eigenvalues[Inverse[N[B]] . A]]],
{n,2,6}]
```

```
n = 2      P/D*l^2 =3.
n = 3      P/D*l^2 =2.48596
n = 4      P/D*l^2 =2.46774
n = 5      P/D*l^2 =2.4674
```

Problem 5.5
Find an approximation to function $f(x)=5x^2\sin(\pi x)\;\; x\in[0,1]$ by using the weak formulation

$$\int_0^1 w(\phi - f)\mathrm{d}x = 0 \quad \forall w \in V,$$

where $V = \{v : v = \sum_{i\in\{0..4\}} \; a_i x^i\}$ (polynomials up to and including degree 4).

Solution

Substituting the approximation and selecting $w = x^i$ gives

$$\int_0^1 x^i(\sum_{j\in\{0..4\}} \; a_j x^j - f)\mathrm{d}x = 0 \quad \forall i \in \{0..4\} \quad \Leftrightarrow$$

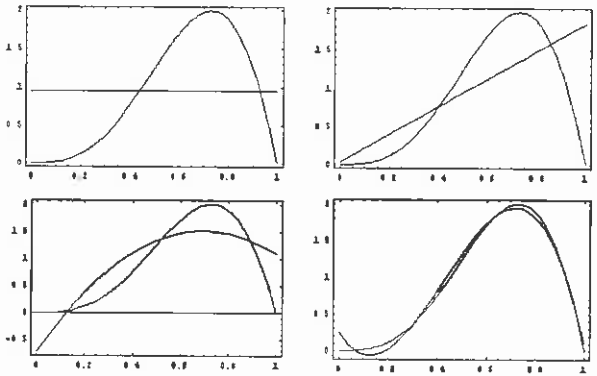
$$\sum_{j\in\{0..4\}} K_{ij} a_j - b_i = 0 \quad \forall i \in \{0..4\},$$

where

$$K_{ij} = \int_0^1 x^i x^j \mathrm{d}x = 1/(i+j+1),$$

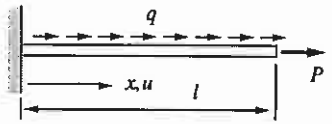
$$b_i = \int_0^1 x^i \cdot 5x^2 \sin(\pi x)\mathrm{d}x.$$

The latter integral can be calculated for example with the aid of *Mathematica* program. The linear equation system above gives then the unknown a_j:s of the approximation $\tilde{\phi} = \sum_{i\in\{0..4\}} a_i x^i$. The approximation is compared with the exact solution in the figures below when $n \in \{0..3\}$.



Mat-5.160 Variational principles of mechanics, exercise 6

1. The potential energy of an elastic bar is given by



$$V(u)=\int_0^l [\frac{EA}{2}(\frac{du}{dx})^2-uq]dx-P\cdot u(l),$$

where $l=1$, $P=1$, $q=1$, $E=1$ and $A=1$. Let us apply the Ritz's method with a piecewise continuous finite element approximation $\tilde{u}=\sum_{j\in\{1...n\}} N_j(x)u_j$. The basis functions on a regularly spaced set of points $x_j=l\cdot(j-1)/(n-1)$ $j\in\{1...n\}$ are

$$N_j(x)=\begin{cases}(x-x_{j-1})/h, & x\in\{x_{j-1},x_j\}\\(x_{j+1}-x)/h, & x\in\{x_j,x_{j+1}\}\\0, & \text{otherwise}\end{cases}.$$

Write down the typical difference equation $\partial\tilde{V}/\partial u_i=0$ and solve it analytically.

2. The potential energy of a stretched string on an elastic foundation is given by

$$V(v)=\int_0^1 [\frac{1}{2}(\frac{dv}{dx})^2+\frac{1}{2}v^2]dx,$$

where $v(0)=0$ and $v(1)=1$. Knowing that the exact solution to the problem is the minimizer of the functional, solve the problem by the finite element method. The approximation is the same as in problem 6.1. *Hint:* write down the typical difference equation $\partial\tilde{V}/\partial v_i=0$ and solve it analytically.

3. The potential energy of a certain stretched string on a non-linearly elastic foundation can be written as

$$V(v)=\int_0^1 [\frac{10}{2}(\frac{dv}{dx})^2+\frac{1}{4}v^4+\frac{1}{2}v^2]dx,$$

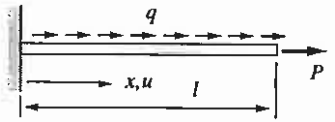
where $v(0)=v(1)=1$. If the problem is solved numerically with the approximation of problem 6.1, write down the typical difference equation $\partial\tilde{V}/\partial v_i=0$.

4. Solve problems 6.1, 6.2 and 6.3 numerically by the finite element method and approximation with $n=5$. Use the *Mathematica* function **Extremize** of the **Lmp** notebook.

5. Derive the detailed Euler-Lagrange equations and initial conditions for the functions $a_j(t)$ of Example D.13 in the case of $\varphi_j(x)=\sin j\pi x/l$ $j\in\{1,2,3\}$, $v_0(t)=v_1(t)=0$, $g(x)=x(1-x)$ and $h(x)=0$ $(l=1)$.

Problem 6.1

The potential energy of an elastic bar is given by



$$V(u)=\int_0^l [\frac{EA}{2}(\frac{du}{dx})^2-uq]dx-P\cdot u(l)$$

where $l=1$, $P=1$, $q=1$, $E=1$ and $A=1$. Let us apply the Ritz's method with a piecewise continuous finite element approximation $\tilde{u}=\sum_{j\in\{1...n\}} N_j(x)u_j$. The basis functions on a regularly spaced set of points $x_j=l\cdot(j-1)/(n-1)$ $j\in\{1...n\}$ are

$$N_j(x)=\begin{cases}(x-x_{j-1})/h, & x\in\{x_{j-1},x_j\}\\(x_{j+1}-x)/h, & x\in\{x_j,x_{j+1}\}\\0, & \text{otherwise}\end{cases}.$$

Write down the typical difference equation $\partial\tilde{V}/\partial u_i=0$ and solve it analytically.

Solution

Variation of the potential energy functional with $k=EA$ gives

$$\int_0^l (\frac{d\delta\tilde{u}}{dx}k\frac{d\tilde{u}}{dx}-\delta\tilde{u}q)dx-P\delta\tilde{u}(l)=0,$$

when also the approximation is substituted there. In this case the variation $\delta\tilde{u}$ can be taken to be the typical basis function N_i (having the support $]x_{i-1},x_{i+1}[$). Thus assuming that node i is not located at the boundary

$$\int_{x_{i-1}}^{x_{i+1}} (\frac{dN_i}{dx} k \frac{d\bar{u}}{dx} - N_i q)dx =$$

$$\int_{x_{i-1}}^{x_i} (\frac{1}{h} k \frac{u_i - u_{i-1}}{h} dx + \int_{x_i}^{x_{i+1}} [-\frac{1}{h} k \frac{u_{i+1} - u_i}{h}]dx +$$

$$-\int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{h} q dx - \int_{x_i}^{x_{i+1}} \frac{x_{i+1} - x}{h} q dx = 0 \quad .$$

If $i = n$ the corresponding equation is

$$\int_{x_{n-1}}^{x_n} (\frac{dN_n}{dx} k \frac{d\bar{u}}{dx} - N_n q)dx - P = \int_{x_{n-1}}^{x_n} (\frac{1}{h} k \frac{u_n - u_{n-1}}{h} dx +$$

$$-\int_{x_{n-1}}^{x_n} \frac{x - x_{n-1}}{h} q dx - P = 0 \quad .$$

After integrations, the equations give an ordinary second order difference problem

$$u_1 - \bar{u} = 0 ,$$

$$\frac{k}{h}(-u_{i-1} + 2u_i - u_{i+1}) - qh = 0 \quad i \in \{2 \ldots n-1\},$$

$$\frac{k}{h}(u_n - u_{n-1}) - qh/2 - P = 0,$$

assuming that the grid is uniform and the coefficients constants. The discrete problem can be treated in the same manner as a continuous problem consisting of a differential equation and boundary conditions. For a particular solution we assume that $u_i = Ai^2$, to get

$$-A(i^2 - 2i + 1) + 2Ai^2 - A(i^2 + 2i + 1) - h^2 = 0 \quad \Rightarrow \quad A = -h^2/2 .$$

For the solution to the homogeneous problem we assume that $u_i = Br^i$, to get

$$r^2 - 2r + 1 = 0 \quad \Rightarrow \quad r = 1.$$

Due to the double root, the solution to the homogeneous problem is of the form $u_i = B + Ci$ and therefore the solution reads

$$u_i = B + Ci - i^2 h^2/2 .$$

The two constants B and C can be determined with the aid of the boundary conditions:

$$u_1 = 0,$$

$$u_n - u_{n-1} - h^2/2 - h = 0$$

giving $C = h^2/2 \cdot [n^2 - (n-1)^2 + 1] + h$ and $B = h^2/2 - C$.

Problem 6.2

The potential energy of a stretched string on an elastic foundation is given by

$$V(v) = \int_0^1 [\frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}v^2]dx,$$

where $v(0) = 0$ and $v(1) = 1$. Knowing that the exact solution to the problem is the minimizer of the functional, solve the problem by the finite element method. The approximation is the same as in problem 6.1. *Hint:* write down the typical difference equation $\partial \bar{V} / \partial v_i = 0$ and solve it analytically.

Solution

Variation of the potential energy functional gives

$$\int_0^1 (\frac{d\delta\bar{v}}{dx}\frac{d\bar{v}}{dx}+\bar{v}\delta\bar{v})dx=0,$$

when the approximation is substituted there. Also now, the variation $\delta\bar{v}$ can be taken as the typical basis function N_i (having the support $]x_{i-1},x_{i+1}[$). Thus assuming that node i is not located at the boundary

$$\int_{x_{i-1}}^{x_{i+1}} (\frac{dN_i}{dx}\frac{d\bar{v}}{dx}+N_i\bar{v})dx=$$

$$=\int_{x_{i-1}}^{x_i} (\frac{1}{h}\frac{v_i-v_{i-1}}{h}+\frac{x-x_{i-1}}{h}(\frac{x_i-x}{h}v_{i-1}+\frac{x-x_{i-1}}{h}v_i)dx+$$

$$+\int_{x_i}^{x_{i+1}} [-\frac{1}{h}\frac{v_{i+1}-v_i}{h}+\frac{x_{i+1}-x}{h}(\frac{x_{i+1}-x}{h}v_i+\frac{x-x_i}{h}v_{i+1})]dx=0 \ .$$

The outcome is the difference problem

$$\frac{1}{h}(-v_{i-1}+2v_i-v_{i+1})+\frac{h}{6}(v_{i-1}+4v_i+v_{i+1})=0 \quad i\in\{2\ldots n-1\},$$

$$v_1=0\ ,\ v_n=0$$

assuming that the grid is uniform. Note: By using the shorthand notation $\alpha=(1/h-h/6)/(2/h+4h/6)$ the difference equation can also be written as $-\alpha v_{i-1}+v_i-\alpha v_{i+1}=0$. Let us assume that $v_i=Ar^i$, to get

$$-\alpha+r-\alpha r^2=0 \quad\Leftrightarrow\quad r=(1\pm\sqrt{1-4\alpha^2})/2\alpha.$$

Because there are two possible r:s the solution is of the form $v_i=A_1(r_1)^i+A_2(r_2)^i$. The two constants A_1 and A_2 follow from the boundary conditions:

$$\begin{cases} v_1\equiv A_1r_1+A_2r_2=0 \\ v_n\equiv A_1(r_1)^n+A_2(r_2)^n=1 \end{cases} \quad\Leftrightarrow\quad \begin{bmatrix} r_1 & r_2 \\ (r_1)^n & (r_2)^n \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the solution can be written as

$$v_i=[r_1^i \quad r_2^i]\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}=[r_1^i \quad r_2^i]\begin{bmatrix} r_1 & r_2 \\ r_1^n & r_2^n \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad i\in\{1\ldots n\}.$$

Problem 6.3

The potential energy of a certain stretched string on a non-linearly elastic foundation can be written as

$$V(v)=\int_0^1 [\frac{10}{2}(\frac{dv}{dx})^2+\frac{1}{4}v^4+\frac{1}{2}v^2]dx\ ,$$

where $v(0)=v(1)=1$. If the problem is solved numerically with the approximation of problem 6.1, write down the typical difference equation $\partial\bar{V}/\partial v_i=0$.

Solution

The functional contains now higher than quadratic terms, which means that the equation system obtained by the Ritz's method is non-linear in the unknown parameters. Let us take the first variation before substituting the approximation

$$\delta V(v)=\int_0^1 [10\frac{dv}{dx}\frac{d\delta v}{dx}+v^3\delta v+v\delta v]dx=0\ .$$

The next step is to replace the non-linear term by its truncated Taylor series at reference solution $\bar{v}$, i.e., $v^3\to\bar{v}^3+3\bar{v}^2\Delta v$. Similar decomposition $v=\bar{v}+\Delta v$ is applied also in the other terms. The correction Δv is chosen to satisfy the homogeneous essential boundary conditions and, at this phase, $\bar{v}$ is a smooth extension of the boundary conditions $\bar{v}=1$, say. Then

$$\int_0^1 [10\frac{d(\bar{v}+\Delta v)}{dx}\frac{d\delta\Delta v}{dx}+(\bar{v}^3+3\bar{v}^2\Delta v)\delta\Delta v+(\bar{v}+\Delta v)\delta\Delta v]dx=$$

$$=\int_0^1 [10\frac{d\Delta v}{dx}\frac{d\delta\Delta v}{dx}+3\bar{v}^2\Delta v\delta\Delta v+\Delta v\delta\Delta v]dx+$$

$$+\int_0^1 [10\frac{d\bar{v}}{dx}\frac{d\delta\Delta v}{dx}+\bar{v}^3\delta\Delta v+\bar{v}\delta\Delta v]dx=0 \quad .$$

Since $\bar{v}$ is assumed to be known (temporarily), the result is a linear system of equations (note that the variation $\delta\Delta v$ can be chosen to be the typical shape function) whose typical equation i is given by

$$\int_0^1 [10\frac{d\Delta\bar{v}}{dx}\frac{dN_i}{dx}+3\bar{v}^2\Delta\bar{v}N_i+\Delta\bar{v}N_i]dx+\int_0^1 [10\frac{d\bar{v}}{dx}\frac{dN_i}{dx}+\bar{v}^3N_i+\bar{v}N_i]dx=0 \; .$$

The non-linear problem can be solved now by starting from $\bar{v}=1$ and solving the correction $\Delta\bar{v}$. After that $\bar{v}\to\bar{v}+\Delta\bar{v}$ is chosen as the new reference solution. This is repeated until the norm of the change $\Delta\bar{v}$ falls below a given limit.

Problem 6.4

Solve problems 6.1, 6.2 and 6.3 numerically by the finite element method and approximation with $n=5$. Use the *Mathematica* function **Extremize** of the **Lmp** notebook.

Solution

See the examples section of *Mathematica* notebook **Ritz.ma**.

Problem 6.5

Derive the detailed Euler-Lagrange equations and initial conditions for the functions $a_j(t)$ of Example D.13 in the case of $\varphi_j(x)=\sin j\pi x/l \quad j\in\{1,2,3\}$, $v_0(t)=v_1(t)=0$, $g(x)=x(1-x)$ and $h(x)=0$ $(l=1)$.

Solution

Hamilton's principle for the problem of the example reads

$$\Pi(v)=\int_{t_1}^{t_2}\int_0^1 [\frac{1}{2}\rho(\frac{\partial v}{\partial t})^2-\frac{1}{2}S(\frac{\partial v}{\partial x})^2]dxdt \; .$$

In this case the approximation is of the form

$$\bar{v}(x,t)=\sin(j\pi x)\cdot a_j(t) \; ,$$

where one may use the index set $j\in\{1,3\}$, since the initial conditions are symmetric with respect to the point $x=1/2$. Substituting the expression and performing the integrations with respect to the spatial coordinate gives the following reduced principle (see the remark)

$$\Pi(\mathbf{a})=\int_{t_1}^{t_2}[\frac{1}{2}\rho\frac{1}{2}\frac{\partial a_j}{\partial t}\frac{\partial a_j}{\partial t}-\frac{1}{2}S(j\pi)^2\frac{1}{2}a_ja_j]dt \; .$$

The Euler-Lagrange equations are obtained by considering the first variation (one has to assume that the variation vanishes at the end points of the domain)

$$2\cdot\delta\Pi=\int_{t_1}^{t_2}[\rho\frac{da_j}{dt}\frac{d\delta a_j}{dt}-S(j\pi)^2a_j\delta a_j]dt=$$

$$=\int_{t_1}^{t_2}[-\rho\frac{d^2a_j}{dt^2}-S(j\pi)^2a_j]\delta a_jdt=0 \quad ,$$

which implies

$$\rho\frac{d^2a_j}{dt^2}+Sj^2\pi^2a_j=0 \quad j\in\{1,3\}.$$

The two initial conditions needed are obtained for example by projecting the original initialconditions onto the subset spanned by the basis functions. Equivalently one may minimize the least-squares functionals

$$\Pi(v)=\int_0^1 \frac{1}{2}(v-g)^2dx,$$

$$\Pi(v)=\int_0^1 \frac{1}{2}(\frac{\partial v}{\partial t}-h)^2dx,$$

giving

$$a_j(t_1)=2\int_0^1 \sin(j\pi x)gdx=2\int_0^1 \sin(j\pi x)x(1-x)dx,$$

$$\frac{d}{dt}a_j(t_1)=0.$$

Remark: $\int_0^1 \sin(i\pi x)\cdot\sin(j\pi x)dx=\delta_{ij}/2$, $\int_0^1 \cos(i\pi x)\cdot\cos(j\pi x)dx=\delta_{ij}/2$.

Mat-5.160 Variational principles of mechanics, exercise 7

1. A system consists of three particles 1,2,3 with masses $m_1 = 2\text{kg}$, $m_2 = 3\text{kg}$ and $m_3 = 4\text{kg}$ with the following position vectors, velocities and external forces acting on them:

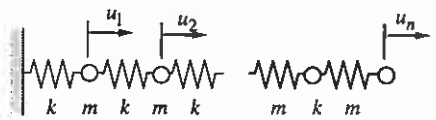
   $\mathbf{r}_1 = (2\mathbf{j}+4\mathbf{k})\text{m}, \quad \mathbf{v}_1 = (1\mathbf{i}+1\mathbf{k})\text{m/s}, \quad \mathbf{F}_1 = 2\text{kN},$

   $\mathbf{r}_2 = (1\mathbf{i}+3\mathbf{j})\text{m}, \quad \mathbf{v}_2 = (1\mathbf{j}-1\mathbf{k})\text{m/s}, \quad \mathbf{F}_2 = 1\text{jN},$

   $\mathbf{r}_3 = (1\mathbf{i}+2\mathbf{k})\text{m}, \quad \mathbf{v}_3 = (2\mathbf{i}-3\mathbf{j})\text{m/s}, \quad \mathbf{F}_3 = 4\text{jN}.$

   Determine (a) the position vector $\mathbf{r}_C$ of the mass center of the system, (b) the velocity $\mathbf{v}_C$ of the mass center, (c) the acceleration $\mathbf{a}_C$ of the mass center, (d) the rate of change of the momentum $\mathrm{d}\mathbf{p}/\mathrm{d}t$ of the system, (e) the rate of change of the moment of momentum $\mathrm{d}\mathbf{L}/\mathrm{d}t$ of the system with respect to the origin and (f) the kinetic energy K of the system.

2. Determine the displacements $u_i(t)$ of the particles of the system of the figure. The initially zero spring forces are proportional to the extension with the spring constant k, $u_i(0) = 0$ and $\dot{u}_i(0) = \varepsilon$, and the particles are identical with masses m.



3. The potential energy of a stretched string on an elastic foundation and subjected to distributed force q is

   $$V(v) = \int_0^1 [\frac{1}{2}(v')^2 + \frac{1}{2}(v)^2 - qv]\mathrm{d}x.$$

   The displacements at the end points are $v(0) = v(1) = 1$ and the distributed force is zero except in the neighborhood of the centerpoint where $q = 10$ $x \in (0.4,0.6)$. The domain is divided into 10 elements of equal size and the global-local approximation is

   $$\tilde{v} = 1 + a_0 \sin(\pi x) + a_5 N_5 + a_6 N_6 + a_7 N_7,$$

   where a_0, a_5, a_6 and a_7 are the unknowns and the N_i:s are the usual piecewise linear shape functions. Write down the linear equation system for the unknowns.

4. Solve problem 7.3 with the aid of the *Mathematica* notebook **RITZ.ma**.

5. Write down the typical difference equation corresponding to the variational finite difference method and problem 7.3 with $q = 0$. Use a regular grid with 11 points.

Problem 7.1

A system consists of three particles 1,2,3 with masses $m_1 = 2\text{kg}$, $m_2 = 3\text{kg}$ and $m_3 = 4\text{kg}$ with the following position vectors, velocities and external forces acting on them:

$\mathbf{r}_1 = (2\mathbf{j}+4\mathbf{k})\text{m}, \quad \mathbf{v}_1 = (1\mathbf{i}+1\mathbf{k})\text{m/s}, \quad \mathbf{F}_1 = 2\text{kN},$

$\mathbf{r}_2 = (1\mathbf{i}+3\mathbf{j})\text{m}, \quad \mathbf{v}_2 = (1\mathbf{j}-1\mathbf{k})\text{m/s}, \quad \mathbf{F}_2 = 1\text{jN},$

$\mathbf{r}_3 = (1\mathbf{i}+2\mathbf{k})\text{m}, \quad \mathbf{v}_3 = (2\mathbf{i}-3\mathbf{j})\text{m/s}, \quad \mathbf{F}_3 = 4\text{jN}.$

Determine (a) the position vector $\mathbf{r}_C$ of the mass center of the system, (b) the velocity $\mathbf{v}_C$ of the mass center, (c) the acceleration $\mathbf{a}_C$ of the mass center, (d) the rate of change of the momentum $\mathrm{d}\mathbf{p}/\mathrm{d}t$ of the system, (e) the rate of change of the moment of momentum $\mathrm{d}\mathbf{L}/\mathrm{d}t$ of the system with respect to the origin and (f) the kinetic energy K of the system.

Solution

(a) The mass center is defined by $\sum m_i \mathbf{r}_C = \sum m_i \mathbf{r}_i$, where the sum extends over the particles of the system. Thus

$$(2+3+4)\mathbf{r}_C = 2(2\mathbf{j}+4\mathbf{k}) + 3(1\mathbf{i}+3\mathbf{j}) + 4(1\mathbf{i}+2\mathbf{k}) = (7\mathbf{i}+13\mathbf{j}+16\mathbf{k}) \Leftrightarrow$$

$$\mathbf{r}_C = \frac{1}{9}(7\mathbf{i}+13\mathbf{j}+16\mathbf{k}) \quad [\text{m}].$$

(b) The formula for the velocity of the mass center is obtained by taking derivatives on both sides of $\sum m_i\mathbf{r}_C = \sum m_i\mathbf{r}_i$ giving $\sum m_i\mathbf{v}_C = \sum m_i\mathbf{v}_i$. Thus

$$(2+3+4)\mathbf{v}_C = 2(\mathbf{i}+\mathbf{k})+3(\mathbf{j}-\mathbf{k})+4(2\mathbf{i}-3\mathbf{j}) = 10\mathbf{i}-9\mathbf{j}-\mathbf{k} \Leftrightarrow$$

$$\mathbf{v}_C = \frac{1}{9}(10\mathbf{i}-9\mathbf{j}-\mathbf{k}) \quad [\text{ms}^{-1}].$$

(c) Since for each particle $m_i\mathbf{a}_i = \mathbf{F}_i + \mathbf{f}_i$, the formula for the velocity of the mass center is obtained by taking derivatives on both sides of $\sum m_i\mathbf{v}_C = \sum m_i\mathbf{v}_i$ giving $\sum m_i\mathbf{a}_C = \sum m_i\mathbf{a}_i = \sum \mathbf{F}_i$ (note that the internal forces add to zero). Thus

$$(2+3+4)\mathbf{a}_C = 2\mathbf{k}+\mathbf{j}+4\mathbf{j} = 5\mathbf{j}+2\mathbf{k} \quad \Leftrightarrow \quad \mathbf{a}_C = \frac{1}{9}(5\mathbf{j}+2\mathbf{k}) \quad [\text{ms}^{-2}].$$

(d) The momentum of the system is defined by $\mathbf{p} = \sum m_i\mathbf{v}_i$. By taking derivatives on both sides of $\sum m_i\mathbf{v}_C = \sum m_i\mathbf{v}_i$ we get $\dot{\mathbf{p}} = \sum m_i\dot{\mathbf{v}}_i = \sum \mathbf{F}_i$, (see also (4.3.4)). So

$$\dot{\mathbf{p}} = 2\mathbf{k}+\mathbf{j}+4\mathbf{j} = 5\mathbf{j}+2\mathbf{k} \quad [\text{N}].$$

(e) The moment of momentum of the system is defined by $\mathbf{L} = \sum m_i\mathbf{r}_i \times \mathbf{v}_i$, where $\mathbf{r}_i$ is the position vector of the particle with respect to the origin. By taking derivatives on both sides we get $\dot{\mathbf{L}} = \sum m_i\mathbf{r}_i \times \mathbf{a}_i = \sum \mathbf{r}_i \times \mathbf{F}_i$ (the moments of the internal forces add to zero, see also (4.4.4)). So in this particular case

$$\dot{\mathbf{L}} = \sum \mathbf{r}_i \times \mathbf{F}_i = (2\mathbf{j}+4\mathbf{k})\times 2\mathbf{k} + (1\mathbf{i}+3\mathbf{j})\times 1\mathbf{j} + (1\mathbf{i}+2\mathbf{k})\times 4\mathbf{j} =$$

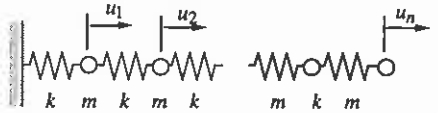
$$= 4\mathbf{i}+\mathbf{k}+4\mathbf{k}-8\mathbf{i} = (-4\mathbf{i}+5\mathbf{k}) \quad [\text{Nm}].$$

(f) The kinetic energy of the system is defined by $K = \sum m_i\mathbf{v}_i \cdot \mathbf{v}_i / 2$. So in this case

$$K = [2(1+1)+3(1+1)+4(4+9)]/2 = 31 \quad [\text{Nm}].$$

Problem 7.2

Determine the displacements $u_i(t)$ of the particles of the system of the figure. The initially zero spring forces are proportional to the lengthening with the spring constant k, $u_i(0) = 0$ and $\dot{u}_i(0) = \varepsilon$, and the particles are identical with masses m.



Solution

The kinematical constraint and free body diagrams of the typical particle and the last particle give equations

$$u_i = 0 \quad i = 0,$$

$$m\ddot{u}_i = k(u_{i+1} - 2u_i + u_{i-1}) \quad i \in \{1 \ldots n-1\},$$

$$m\ddot{u}_i = k(u_{i-1} - u_i) \quad i = n$$

or when written using the matrix notation

$$m[M]\{\ddot{u}\} + k[K]\{u\} = 0 \text{ , where}$$

$$[M]=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & \ddots & & \\ \vdots & & 1 & 0\\ 0 & & 0 & 1\end{bmatrix},\quad [K]=\begin{bmatrix}2 & -1 & & 0\\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1\\ 0 & & -1 & 1\end{bmatrix}.$$

The solution to the problem is of the form

$$\{u\}=\{a\}\sin\sqrt{[A]}t+\{b\}\cos\sqrt{[A]}t\ ,$$

where $[A]=[M]^{-1}[K]$. Since $[A]$ is symmetric, it can be written in the form $[A]=[x][\lambda][x]^{\mathrm{T}}$ where $[\lambda]$ is diagonal and $[x][x]^{\mathrm{T}}=[1]$. Here $[K]$ is positive definite and therefore the eigenvalues in $[\lambda]$ are positive and real. By taking into account the first initial condition $\{u\}=0$, the solution can be written as

$$\{u\}=[x]\sin(\sqrt{[\lambda]}t)[x]^{\mathrm{T}}\{a\}.$$

The second initial condition $\{\dot{u}\}=\varepsilon$ gives

$$\{\dot{u}\}=[x]\sqrt{[\lambda]}[x]^{\mathrm{T}}\{a\}=\{\varepsilon\}\ \Leftrightarrow\ \{a\}=[x][\lambda]^{-1/2}[x]^{\mathrm{T}}\{\varepsilon\}.$$

Then, the solution to the problem is

$$\{u\}=[x]\sin(\sqrt{[\lambda]}t)[\lambda]^{-1/2}[x]^{\mathrm{T}}\{\varepsilon\}\ .$$

Problem 7.3

The potential energy of a stretched string on an elastic foundation and subjected to distributed force q is

$$V(v)=\int_0^1\left[\frac{1}{2}(v')^2+\frac{1}{2}(v)^2-qv\right]\mathrm{d}x.$$

The displacements at the end points are $v(0)=v(1)=1$ and the distributed force is zero except in the neighborhood of the centerpoint where $q=10\ \ x\in\{0.4,0.6\}$. The domain is divided into 10 elements of equal size and the global-local approximation is

$$\bar{v}=1+a_0\sin(\pi x)+a_5N_5+a_6N_6+a_7N_7,$$

where a_0, a_5, a_6 and a_7 are the unknowns and the N_i:s are the usual piecewise linear shape functions. Write down the linear equation system for the unknowns and solve it with the aid of *Mathematica*.

Solution

Let us write the approximation in the form

$$\bar{v}=\sum_{i\in\{-1\ldots7\}}b_iN_i\ ,$$

where the non-positive indices denote the global modes ($N_{-1}=1$, $b_{-1}=1$, $N_0=\sin(\pi x)$, $b_0=a_0$) and the positive ones the usual piecewise polynomial 'local' modes. In this particular case we choose $b_i=0\ i\in\{1\ldots4,8\ldots11\}$. Note that the boundary condition is accounted for correctly by this selection. Minimization of the functional in the set spanned by the approximation gives the linear equation system

$$\int_0^1\left[N_i'\bar{v}'+N_i\,\bar{v}-N_i\,q\right]\mathrm{d}x=0\quad i\in\{0,5,6,7\}\ .$$

Since the other parameters are known, minimization should not be performed with respect to them. Let us divide the approximation $\bar{v}\to\bar{v}+\tilde{v}$ into parts of which the first consists of the known modes and the second of the unknown modes. Then the linear equation system can be written as

$$K_{ij}b_j-F_i=0\quad i\in\{0,5,6,7\}\ ,$$

where $K_{ij} = \int_0^1 [N_i' N_j' + N_i N_j]dx$, $F_i = -\int_0^1 [N_i' \bar{v}' + N_i \bar{v} - N_i q]dx$.

Problem 7.4

Solve problem 7.3 with the aid of the *Mathematica* notebook **RITZ.ma** (you can get the notebook from the homepage http://www.hut.fi/HUT/Dynamics/ of the Laboratory of Computational Dynamics) .

Solution

See the examples section of the *Mathematica* notebook **RITZ.ma**

Problem 7.5

Write down the typical difference equation corresponding to the variational finite difference method and problem 7.3 with $q = 0$. Use a regular grid with 11 points.

Solution

The starting point is the minimization principle

$$V(v) = \int_0^1 [\frac{1}{2}(v')^2 + \frac{1}{2}(v)^2]dx,$$

which is approximated by replacing the integrand by simple difference formulas and performing the integration after that. Thus assuming that the grid points are uniformly spaced with the distance h (the number of points is n)

$$\bar{V} = \sum_{j\in\{1..n\}} [\frac{1}{2}(\frac{v_{j+1} - v_j}{h})^2 h + \frac{1}{2}(\frac{v_{j+1} + v_j}{2})^2 h].$$

At the stationary point the partial derivatives with respect to the nodal values vanish

$$\frac{\partial \bar{V}}{\partial v_i} = \sum_{j\in[1..n]} [2\frac{1}{2}(\frac{v_i - v_{i-1}}{h^2})h - 2\frac{1}{2}(\frac{v_{i+1} - v_i}{h^2})h +$$

$$+ 2\frac{1}{2}(\frac{v_i + v_{i-1}}{4})h + 2\frac{1}{2}(\frac{v_{i+1} + v_i}{4})h] = 0 \quad .$$

Thus the result is the difference equation

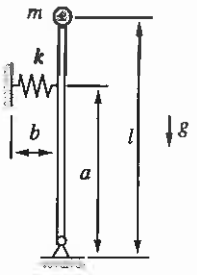
$$\frac{1}{h}(-v_{i-1} + 2v_i - v_{i+1}) + \frac{h}{4}(v_{i-1} + 2v_i + v_{i+1}) = 0.$$

Mat-5.160 Variational principles of mechanics, exercise 8

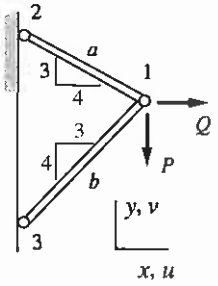
1. (a) Show that if the principle of balance of moment of momentum is valid with respect to a fixed point (origin), it is valid with respect to the center of mass of a system.
   (b) Derive formula (4.4.11) for the moment of momentum of a rigid body.

2. Derive the equations of (plane) motion of a particle in the polar coordinates r,θ using Lagrange's equations.

3. Derive the equations of motion of the system of the figure consisting of a particle, rigid bar (assumed massless) and a linear spring. Use Lagrange's equation of motion with the angle of the rotation of the bar as the generalized coordinate.



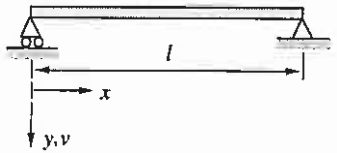
4. Derive the equations of motion for the system of the figure using Lagrange's equations. The generalized coordinates are the horizontal and vertical displacements u,v of point 1. The bars are homogeneous with the total mass m, the cross sectional area is constant A and the force displacement relationship for a bar is assumed to be $F = AE\cdot\Delta l/l$, where E is the Young's modulus, l is the original length of the bar and Δl is the elongation. The components P,Q of the force at point 1 are constants.



5. Derive the discrete equation of the motion for the vibration of a simply supported uniform Bernoulli beam with the mass per length ρ and with the bending stiffness EI using the Lagrange's equations with

   $\bar{v} = a_j(t)\cdot\sin(j\pi\cdot x/l) \quad j\in\{1\ldots n\}$.

   Consider the $a_j(t)$:s as generalized coordinates. The inertia forces associated with the rotation of beam cross sections are neglected.



Problem 8.1

(a) Show that if the principle of balance of moment of momentum is valid with respect to a fixed point (origin), it is valid with respect to the center of mass of a system.

(b) Derive formula (4.4.11) for the moment of momentum of a rigid body.

Solution

(a) Let us assume that the balance of moment of momentum is valid with respect to origin, say. Then

$\dot{\mathbf{L}} = \mathbf{M}$, where $\mathbf{L} = \sum_{i\in\{1..n\}} \mathbf{r}_i \times m_i\mathbf{v}_i$, $\mathbf{M} = \sum_{i\in\{1..n\}} \mathbf{r}_i \times (\mathbf{f}_i + \mathbf{F}_i)$.

The position vector of the typical particle can be written as $\mathbf{r}_i = \mathbf{r}_C + \mathbf{r}_{i/C}$. When substituted in the formula for the moment of momentum and in the formula for the moment

$$\mathbf{L} = \sum_{i\in\{1..n\}} (\mathbf{r}_C + \mathbf{r}_{i/C})\times m_i\mathbf{v}_i = \mathbf{r}_C \times \mathbf{P} + \sum_{i\in\{1..n\}} \mathbf{r}_{i/C}\times m_i\mathbf{v}_i ,$$

$$\mathbf{M} = \sum_{i\in\{1..n\}} (\mathbf{r}_C + \mathbf{r}_{i/C})\times(\mathbf{f}_i + \mathbf{F}_i) =$$

$$= \sum_{i\in\{1..n\}} \mathbf{r}_C \times \mathbf{F}_i + \sum_{i\in\{1..n\}} \mathbf{r}_{i/C}\times\mathbf{F}_i \quad .$$

Since $\dot{\mathbf{r}}_C \times \mathbf{P} = \dot{\mathbf{r}}_C \times m\dot{\mathbf{r}}_C = 0$, the balance of moment of momentum takes the form

$$\mathbf{r}_C \times \dot{\mathbf{P}} + \frac{d}{dt}\left(\sum_{i\in\{1..n\}} \mathbf{r}_{i/C}\times m_i\mathbf{v}_i\right) = \mathbf{r}_C \times \mathbf{F} + \sum_{i\in\{1..n\}} \mathbf{r}_{i/C}\times\mathbf{F}_i .$$

Finally, using the balance law for the momentum $\dot{\mathbf{P}} = \mathbf{F}$, one obtains

$$\frac{d}{dt}\left(\sum_{i\in\{1..n\}} \mathbf{r}_{i/C}\times m_i\mathbf{v}_i\right) = \sum_{i\in\{1..n\}} \mathbf{r}_{i/C}\times\mathbf{F}_i \quad \text{or} \quad \dot{\mathbf{L}}_C = \mathbf{M}_C .$$

(b) Equation (4.4.11) states that the moment of momentum with respect to the origin (say) can be written in terms of the moment of momentum with respect to the mass center, linear momentum and the mass center by

$$\mathbf{L}=\mathbf{r}_C\times\mathbf{p}+\mathbf{L}_C\,.$$

The definition when written in terms of $\mathbf{r}=\mathbf{r}_C+\mathbf{s}$ gives

$$\mathbf{L}=\int_V\ (\mathbf{r}_C+\mathbf{s})\times\rho\dot{\mathbf{r}}dV=\mathbf{r}_C\times\int_V\ \rho\dot{\mathbf{r}}dV+\int_V\ \mathbf{s}\times\rho\dot{\mathbf{r}}dV=\mathbf{r}_C\times\mathbf{p}+\mathbf{L}_C\,.$$

Problem 8.2

Derive the equations of (plane) motion of a particle in the polar coordinates r,θ using Lagrange's equations.

Solution

The speed squared of a particle in polar coordinates is obtained from

$$\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}=\frac{\mathrm{d}}{\mathrm{d}t}(r\cos\theta\mathbf{i}+r\sin\theta\mathbf{j})\cdot\frac{\mathrm{d}}{\mathrm{d}t}(r\cos\theta\mathbf{i}+r\sin\theta\mathbf{j})=$$

$$=[\frac{\mathrm{d}}{\mathrm{d}t}(r\cos\theta)]^2+[\frac{\mathrm{d}}{\mathrm{d}t}(r\sin\theta)]^2=\dot{r}^2+r^2\dot{\theta}^2\,.$$

The kinetic energy is then

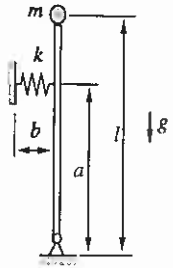
$$K=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)\,.$$

Since the generalized external forces F_θ and F_r may not be conservative, the Lagrange function is chosen to coincide with the kinetic energy i.e. $L=K$ and the equations of motion are

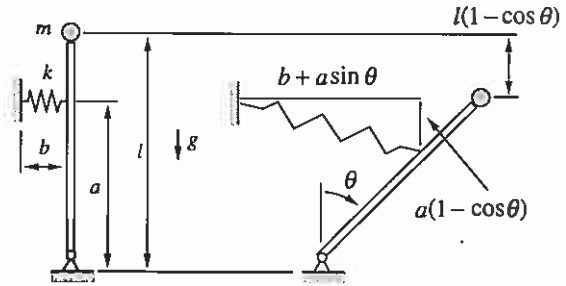
$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial K}{\partial\dot{\theta}})-\frac{\partial K}{\partial\theta}=mr^2\ddot{\theta}=F_\theta\ \text{ and }\ \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial K}{\partial\dot{r}})-\frac{\partial K}{\partial r}=m\ddot{r}-mr^2\dot{\theta}^2=F_r.$$

Problem 8.3

Derive the equations of motion of the system of the figure consisting of a particle, rigid bar (assumed massless) and a linear spring. Use Lagrange's equation of motion with the angle of the rotation of the bar as the generalized coordinate.



Solution



According to the figure the elongation of the spring is given by

$$\Delta s=\sqrt{(b+a\sin\theta)^2+(a-a\cos\theta)^2}-b\,.$$

The potential energy consists of the potential energy of the spring and the potential energy of force due to the gravity. Altogether

$$V=\frac{1}{2}k\Delta s^2-mgl(1-\cos\theta).$$

Only the point mass has to accounted for in the kinetic energy expression since the bar (and also the spring) are assumed massless:

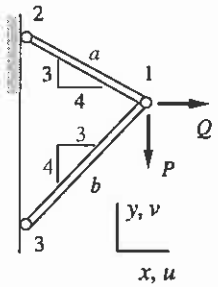
$$T=\frac{1}{2}ml^2\dot{\theta}^2.$$

The Lagrange's equation of motion $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(T-V)}{\partial\dot{\theta}}-\frac{\partial(T-V)}{\partial\theta}=0$ gives

$$ml^2\ddot{\theta}+k\Delta s\frac{\partial\Delta s}{\partial\theta}-mgl\sin\theta=0.$$

Problem 8.4

Derive the equations of motion for the system of the figure using Lagrange's equations. The generalized coordinates are the horizontal and vertical displacements u,v of point 1. The bars are homogeneous with the total mass m, the cross sectional area is constant A and the force displacement relationship for a bar is assumed to be $F=AE\cdot\Delta l/l$, where E is the Young's modulus, l is the original length of the bar and Δl is the elongation. The components P,Q of the force at point 1 are constants.

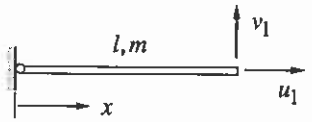


Solution

Let us first consider the horizontal bar of the figure alone. The potential and kinetic energies are (we assume that $u=x/l\cdot u_1$ and $v=x/l\cdot v_1$)

$$V=\frac{1}{2}\int_V\sigma_{ij}\varepsilon_{ij}\mathrm{d}V=\frac{1}{2}\frac{EA}{l}u_1{}^2$$

$$T=\frac{1}{2}\int_V\rho(\dot{u}^2+\dot{v}^2)\mathrm{d}V=\frac{1}{2}\frac{m}{3}(\dot{u}_1{}^2+\dot{v}_1{}^2).$$



The displacements of the bars in the direction of the axes and in the direction perpendicular to that are related to displacements by (see the original figure for the angles)

$$\begin{Bmatrix}u_1\\v_1\end{Bmatrix}_1=\frac{1}{5}\begin{bmatrix}4&3\\-3&4\end{bmatrix}\begin{Bmatrix}u\\v\end{Bmatrix}\text{ and }\begin{Bmatrix}u_1\\v_1\end{Bmatrix}_2=\frac{1}{5}\begin{bmatrix}3&-4\\-4&3\end{bmatrix}\begin{Bmatrix}u\\v\end{Bmatrix}.$$

Thus for the whole structure the potential and kinetic energies are

$$V=\frac{1}{2}\frac{EA}{a}(\frac{4}{5}u+\frac{3}{5}v)^2+\frac{1}{2}\frac{EA}{b}(\frac{3}{5}u-\frac{4}{5}v)^2-Pv-Qu$$

$$T=\frac{1}{6}\frac{am}{a+b}(\dot{u}^2+\dot{v}^2)+\frac{1}{6}\frac{bm}{a+b}(\dot{u}^2+\dot{v}^2)=\frac{1}{6}m(\dot{u}^2+\dot{v}^2)$$

or using the index notation

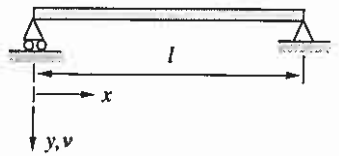
$$V=\frac{1}{2}K_{ij}u_iu_j-F_iu_i,\quad T=\frac{1}{2}M_{ij}\dot{u}_i\dot{u}_j,$$

where the definition of the stiffness and the mass matrices K_{ij} and M_{ij} can be obtained by comparing the representations without and with the index notation. The Lagrange's equation of motion gives finally (with $L=T-V$)

$$\frac{1}{2}(M_{ij}+M_{ji})\ddot{u}_j+\frac{1}{2}(K_{ij}+K_{ji})u_j-F_i=0.$$

Problem 8.5

Derive the discrete equation of the motion for the vibration of a simply supported uniform Bernoulli beam with the mass per length ρ and with the bending



stiffness EI using the Lagrange's equations with

$$\bar{v} = a_j(t) \cdot \sin(j\pi \cdot x / l) \quad k \in \{1 \ldots n\}.$$

Consider the $a_j(t)$:s as generalized coordinates. The inertia forces associated with the rotation of beam cross sections are neglected.

Solution

The kinematic assumption of the Bernoulli beam is that the cross sectional planes of the beam, originally perpendicular to the axis, remain perpendicular to the axis despite the displacement. Thus one may write

$$u(x,y) = -v'(x) \cdot y$$

if the origin of the y-axis is placed at the axis of the beam. The potential energy of the internal forces (due to the strain field) can be obtained from the general formula

$$V = \int_V \frac{1}{2} \sigma_{ij} \varepsilon_{ij} dV,$$

where now $\sigma_{xx} = E\varepsilon_{xx} = -Ev'' \cdot y$. Assuming that the Young's modulus E and the cross sectional area are constants, one obtains

$$V = \frac{EI}{2} \int_0^l (v'')^2 dx \quad \text{with } I = \int_A y^2 dA.$$

The expression for the kinetic energy is (assuming that the rotational part is negligible)

$$K = \int_V \frac{1}{2} \rho \dot{v}^2 dV = \frac{1}{2} A\rho \int_0^l \dot{v}^2 dx.$$

Then, the Lagrange's function is

$$L = K - V = \frac{1}{2} A\rho \int_0^l \dot{v}^2 dx - \frac{EI}{2} \int_0^l (v'')^2 dx,$$

which simplifies into

$$L = K - V = \frac{1}{2} A\rho \frac{l}{2} \sum_{i \in \{1..n\}} \dot{a}_i^2 - \frac{EI}{2} \sum_{i \in \{1..n\}} \left(\frac{\pi i}{l}\right)^4 a_i^2 \frac{l}{2}$$

when the series representation is substituted there and the orthogonality condition of the remark is accounted for. The Lagrange's equation

$$\frac{\partial L}{\partial a_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{a}_i} = 0$$

gives then $A\rho \ddot{a}_i + EI \left(\frac{\pi i}{l}\right)^4 a_i = 0$.

Remark. $\int_0^l \sin(i\pi \cdot x / l) \cdot \sin(j\pi \cdot x / l) dx = l \frac{1}{2} \delta_{ij}$

Mat-5.160 Variational principles of mechanics, exercise 9

1. What is the maximum mass that can be attached at the top of the rigid bar (assumed massless) supported by a frictionless hinge and a linear spring if the position of the figure is to be stable (fig.1).

2. The homogeneous rigid body of the figure consists of a cylinder and of a halfball (fig.2). Determine the value of the ratio r/h at which the upright position becomes unstable.

3. Determine the force P causing the system of figure 3 to buckle. The bars are assumed totally rigid and massless and torsion springs linear i.e. $M = k\Delta\theta$.

4. Write down the potential energy functional for the Bernoulli beam of figure 4. Also determine an approximation for the buckling force by assuming that the displacement is of the form $\bar{v} = v_1 \sin(\pi x/l)$. Compare the result with the exact solution.

5. The smallest eigenvalue of matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ is given by

$$\lambda = \min_{x \in R^3, x^T x \neq 0} \frac{x^T A x}{x^T x}.$$

(a) Use the formula above to find the smallest eigenvalue of A.
(b) Use the same idea to find an approximation for the buckling force of problem 9.4 when $\bar{v} = \sum_{i \in [1..3]} v_i \sin(i\pi x/l)$

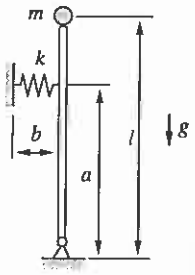


Fig.1

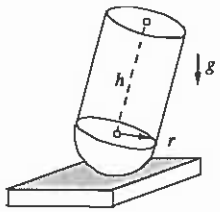


Fig.2

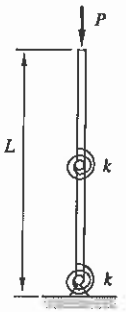


Fig.3

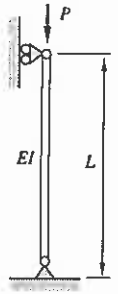
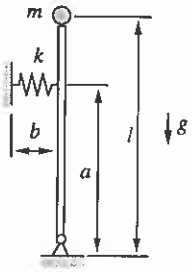


Fig.4

Problem 9.1



What is the maximum mass that can be attached at the top of the rigid bar of the figure (assumed massless) supported by a frictionless hinge and a linear spring if the position of the figure is to be stable.

Solution

The potential energy has been discussed already in connection with problem 8.3. As the equilibrium position where the buckling takes place is known to be $\theta = 0$, the setting can be simplified by replacing all the quantities involved by their Taylor series at $\theta = 0$. Then $(1 - \cos\theta \approx 1 - 1 + \sin 0 + \cos 0 \cdot \theta^2/2)$

$$V = \frac{1}{2}k(a\theta)^2 - mgl\frac{1}{2}\theta^2.$$

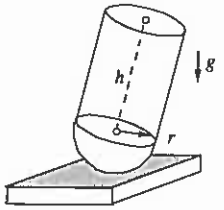
At the equilibrium, the derivative with respect to the generalized coordinate θ vanishes, i.e

$$\frac{dV}{d\theta} = (ka^2 - mgl)\theta = 0$$

giving $\theta = 0$ or $m = ka^2/gl$. The latter gives the maximum mass for the upper position to be the only solution, since if the multiplier of θ becomes zero any θ will do.

Problem 9.2

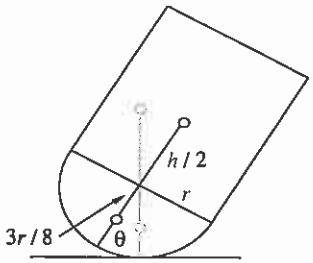
The homogeneous rigid body of the figure consists of a cylinder and of a half-ball. Determine the value of the ratio r/h at which the upright position becomes unstable.



Solution

The figure shows that the (positive upward) displacement of the mass center of the half-ball due to the change θ in the angular coordinate is

$$\Delta h = \frac{3}{8} r(1-\cos\theta)$$



(the distance of the mass center is $3r/8$ from the center of a ball). For the cylinder the displacement is

$$\Delta h = -\frac{h}{2}(1-\cos\theta).$$

Thus the potential energy of the system is given by ($c \triangleq$ cylinder, $b \triangleq$ ball)

$$V = -m^c g \frac{h}{2}(1-\cos\theta) + m^b g \frac{3}{8} r(1-\cos\theta).$$

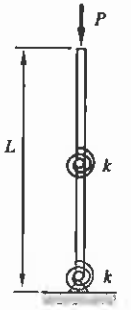
At the equilibrium, the derivative of V with respect to the generalized coordinate θ vanishes. So

$$\frac{dV}{d\theta} = -g(m^c \frac{h}{2} - m^b \frac{3}{8} r)\sin\theta = 0,$$

which gives $\sin\theta = 0$ or $m^c h/2 - 3m^b r/8 = 0$. As the masses are given by $m^c = \pi r^2 h$ and $m^b = 2\pi r^3/3$, the latter equation implies $r/h = \sqrt{2}$.

Problem 9.3

Determine the force P causing the system of the figure to buckle. The bars are assumed totally rigid and massless and torsion springs linear i.e. $M = k\Delta\theta$.

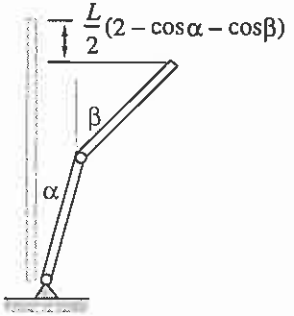


Solution

The potential energies (see the figure) of the point force and the torsion springs are given by

$$V^1 = \frac{PL}{2}(2-\cos\alpha-\cos\beta),$$

$$V^2 = \frac{1}{2}k\alpha^2 + \frac{1}{2}k(\beta-\alpha)^2,$$



respectively. At the equilibrium the derivatives of $V = V^1 + V^2$ with respect to the generalized coordinates α and β vanish. So

$$\frac{\partial V}{\partial \alpha} = k\alpha - k(\alpha-\beta) - \frac{PL}{2}\sin\alpha = 0,$$

$$\frac{\partial V}{\partial \beta} = k(\beta-\alpha) - \frac{PL}{2}\sin\beta = 0.$$

When the problem is linearized at the obvious equilibrium position $\alpha = \beta = 0$, the result is

$$\begin{bmatrix} 2k - PL/2 & -k \\ -k & k - PL/2 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = 0$$

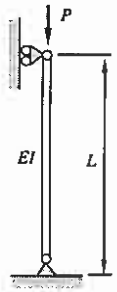
having other solutions than $\alpha=\beta=0$ only if the determinant of the matrix vanishes. So we get $(2k-PL/2)(k-PL/2)-k^2=0$ giving

$$(PL)^2-6kPL-8k^2=0 \quad \text{or} \quad PL=(6k\pm\sqrt{36k^2-4\cdot 8k^2})/2=3k\pm k.$$

Conclusion: the minimum force causing the system to loose its stability is $P=2k/L$.

Problem 9.4

Write down the potential energy functional for the Bernoulli beam of the figure. Also determine an approximation for the buckling force by assuming that the displacement is of the form $\bar{v}=v_1\sin(\pi x/l)$. Compare the result with the exact solution.



Solution

The potential energy of the beam has been discussed in connection with problem 4.4:

$$V(v)=\int_0^l \frac{EI}{2}(v'')^2\mathrm{d}x-P\int_0^l \frac{1}{2}(v')^2\mathrm{d}x.$$

In this case the function set is restricted by the essential boundary conditions $v(0)=v(l)=0$. At the equilibrium the first variation of V vanishes

$$\delta V(v)=\int_0^l EIv''\delta v''\mathrm{d}x-P\int_0^l v'\delta v'\mathrm{d}x=0.$$

Let us choose $\delta v=v$ and solve the force P to get

$$P=\int_0^l EI(v'')^2\mathrm{d}x\Big/\int_0^l (v')^2\mathrm{d}x.$$

The result is the so called Rayleigh quotient giving a good estimate for the eigenvalue P if one is able to make a reasonable guess for the corresponding eigenfunction. Here $\bar{v}=v_1\sin(\pi x/l)$ and the force is then

$$P=\frac{\int_0^l EI(\pi/l)^4\sin^2(\pi x/l)\mathrm{d}x}{\int_0^l(\pi/l)^2\cos^2(\pi x/l)\mathrm{d}x}=EI(\pi/l)^2\ ,$$

which happens to be exact (the guess for the corresponding eigenfunction is very good). The exact solution can be obtained by considering the underlying boundary value problem

$$EIv^{(4)}+Pv^{(2)}=0 \quad x\in]0,l[\ ,$$

$$v(0)=v(l)=0,$$

$$v^{(2)}(0)=v^{(2)}(l)=0.$$

The solution to the field equation is of the form

$$v=a+bx+c\sin(\sqrt{P/EI}\,x)+d\cos(\sqrt{P/EI}\,x)$$

and the boundary conditions give with $\alpha=l\sqrt{P/EI}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & l & \sin\alpha & \cos\alpha \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -\alpha^2\sin\alpha & -\alpha^2\cos\alpha \end{bmatrix}\begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}=0\ .$$

For a non-trivial solution the determinant must vanish. The final result is the same condition for the point force as given by the Rayleigh quotient.

Problem 9.5

The smallest eigenvalue of matrix $A=\begin{bmatrix}2&1&0\\1&4&1\\0&1&2\end{bmatrix}$ is given by

$$\lambda=\min_{\mathbf{x}\in\mathbf{R}^3,\mathbf{x}^T\mathbf{x}\neq 0}\frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}.$$

(a) Use the formula above to find the smallest eigenvalue of A.

(b) Use the same idea to find an approximation for the buckling force of problem 9.4 when $\bar{v}=\sum_{i\in\{1..3\}} v_i \sin(i\pi x/l)$

Solution

Mathematica program finds good use in connection with this problem.

(a) Let us write $x=\{a,b,c\}^{\mathrm{T}}$. Then

$$\lambda=\min_{\mathbf{x}\in\mathbf{R}^3,\mathbf{x}^T\mathbf{x}\neq 0}\frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}=\min_{\mathbf{x}\in\mathbf{R}^3,\mathbf{x}^T\mathbf{x}\neq 0}\frac{2a^2+4b^2+2c^2+2ab+2bc}{a^2+b^2+c^2}.$$

The mathematica solution to this problem follows:

```
In=
Min[Eigenvalues[{{2,1,0},{1,4,1},{0,1,2}}] //N]
FindMinimum[(2*a^2+4*b^2+2*c^2+2*a*b+2*b*c)/
(a^2+b^2+c^2), {a,1},{b,0},{c,0}]

Out=
1.26795
{1.26795, {a -> 1.01165, b -> -0.740581, c -> 1.01165}}
```

(b) The starting point is the Rayleigh quotient taking the form

$$P=\frac{\int_0^l EI(v'')^2\,dx}{\int_0^l (v')^2\,dx}=EI(\pi/l)^2\frac{\sum_{i\in\{1..3\}} v_i^2 i^4}{\sum_{i\in\{1..3\}} v_i^2 i^2},$$

when the approximation $\bar{v}=\sum_{i\in\{1..3\}} v_i \sin(i\pi x/l)$ is substituted there. The mathematica solution for the minimum is given by

```
In=
v = {a,b,c};
fun=Sum[v[[i]]^2*i^4,{i,1,3}]/Sum[v[[i]]^2*i^2,{i,1,3}] ;
FindMinimum[fun,{a,1},{b,0},{c,0}][[1]]

Out=
1.
```

Mat-5.160 Variational principles of mechanics, exercise 10

1. The virtual work of a bar of a truss is $\delta' W_{ij}^{\text{int}} = -N \cdot \delta s$. Another expression is

$$\delta' W_{ij}^{\text{int}} = -\int_{0_V} S\delta E \cdot \mathrm{d}^0 V,$$

where the integration is over the volume ${}^0V$ of the bar in the reference state. Assume that S and δE are constants and show that the two representations are equal.

2. Determine the vertical displacement v_1 of the elastic truss with the large deformation theory directly using (6.4.1). $a = 1\text{m}$, ${}^0A = 0.01\text{m}^2$, $C = 100\text{Nm}^{-2}$ and $F = 0.5\text{N}$ (fig.1).

3. Determine the displacement u_1, v_1 of the elastic truss with the large deformation theory directly using (6.4.1). The values of the parameters are the same as in the previous problem (fig. 2).

4. A cylinder is rolling freely on a horizontal plane. Determine the equation of (plane) motion of the cylinder using Lagrange's equation. The radius of the cylinder is R and the mass is m.

5. The particles of the figure are joined together by a massless inextensible string. Assuming that friction is negligible, determine the value of angle θ at which the system is in equilibrium (Fig. 3).

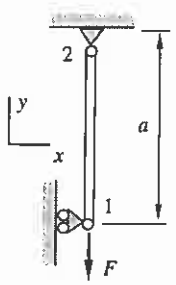


Fig. 1

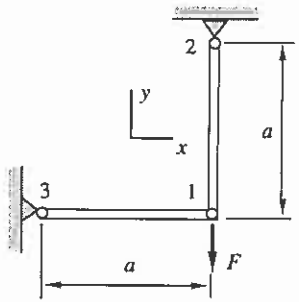


Fig. 2

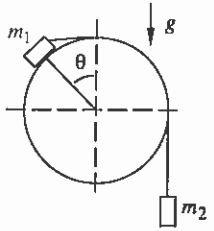


Fig. 3

Problem 10.1

The virtual work of a bar of a truss is $\delta' W_{ij}^{\text{int}} = -N \cdot \delta s$. Another expression is

$$\delta' W_{ij}^{\text{int}} = -\int_{0_V} S\delta E \cdot \mathrm{d}^0 V,$$

where the integration is over the volume ${}^0V$ of the bar in the reference state. Assume that S and δE are constants and show that the two representations are equal.

Solution

In the virtual work expression $\delta' W_{ij}^{\text{int}} = -N \cdot \delta s$ the quantities N and δs denote the force of the bar and virtual change of the length δs. The strain and the Kirchhoff stress are defined by

$$E = \frac{1}{2}\frac{s^2 - {}_0s^2}{{}_0s^2}, \quad N = \frac{s}{{}_0s} S\, {}^0A.$$

Thus

$$\delta E = \frac{s}{{}_0s^2}\delta s \quad \text{and} \quad S = \frac{{}_0s}{{}^0A\, s} N$$

giving

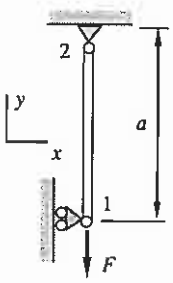
$$S\,\delta E = \frac{s}{{}_0s^2}\delta s \frac{{}_0s}{{}^0A\, s} N = \frac{1}{{}_0s\, {}^0A} N\delta s$$

and finally

$$\delta' W_{ij}^{\text{int}} = -\int_{0_V} S\delta E \cdot \mathrm{d}^0 V = -\frac{1}{{}^0_s A} N\delta s \int_{0_V} \mathrm{d}^0 V = -N\delta s\,.$$

Problem 10.2

Determine the vertical displacement v_1 of the elastic truss with the large deformation theory directly using (6.4.1). $a = 1\text{m}$, ${}^0A = 0.01\text{m}^2$, $C = 100\text{Nm}^{-2}$ and $F = 0.5\text{N}$.



Solution

The starting point is the virtual work expression $-\delta' W_{\text{int}} - \delta' W_{\text{ext}} = 0$ where in this case (v_1 is directed upwards)

$$-\delta' W_{\text{ext}} = F\delta v_1\,,$$

$$-\delta' W_{\text{int}} = \frac{C\,{}^0A}{{}^0_s}\frac{1}{2}\cdot\frac{2\,{}^0s_y\,v_1 + v_1^2}{{}^0_s{}^2}({}^0s_y + v_1)\delta v_1\,.$$

Since the virtual work equation should be satisfied with any δv_1, we get

$$F + \frac{C\,{}^0A}{{}^0_s}\frac{1}{2}\cdot\frac{2\,{}^0s_y\,v_1 + v_1^2}{{}^0_s{}^2}({}^0s_y + v_1) = 0.$$

After substituting the numerical values for the problem parameters

$$\frac{1}{2} + \frac{1}{2}\cdot(2\,v_1 + v_1^2)(1 + v_1) = 0$$

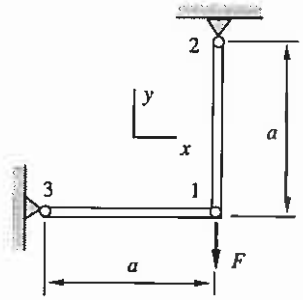
the solution to which can be obtained with the aid of *Mathematica* $\Rightarrow$

$$v_1 = -0.3247179572447461.$$

Note that the solution to the linearized 'small displacement version' is $v_1 = -0.5$.

Problem 10.3

Determine the displacement u_1, v_1 of the elastic truss with the large deformation theory directly using (6.4.1). The values of the parameters are the same as in the previous problem.



Solution

The starting point is the virtual work expression $-\delta' W_{\text{int}} - \delta' W_{\text{ext}} = 0$ where in this case

$$-\delta' W_{\text{ext}} = F\delta v_1\,.$$

$$-\delta' W_{\text{int}} = \frac{C\,{}^0A}{{}^0_s}\frac{1}{2}\cdot\frac{u_1^2 + 2\,{}^0s_y\,v_1 + v_1^2}{{}^0_s{}^2}[u_1\delta u_1 + ({}^0s_y + v_1)\delta v_1] +$$

$$+\frac{C\,{}^0A}{{}^0_s}\frac{1}{2}\cdot\frac{v_1^2 + 2\,{}^0s_x\,u_1 + u_1^2}{{}^0_s{}^2}[v_1\delta v_1 + ({}^0s_x + u_1)\delta u_1]\,.$$

Since the virtual work equation should be satisfied with any δv_1 and δu_1, we get

$$F + \frac{C\,{}^0A}{{}^0_s}\frac{1}{2}\cdot\frac{u_1^2 + 2\,{}^0s_x\,v_1 + v_1^2}{{}^0_s{}^2}({}^0s_y + v_1) + \frac{C\,{}^0A}{{}^0_s}\frac{1}{2}\cdot\frac{v_1^2 + 2\,{}^0s_x\,u_1 + u_1^2}{{}^0_s{}^2}v_1 = 0,$$

$$\frac{C\,{}^0A}{{}^0_s}\frac{1}{2}\cdot\frac{u_1^2 + 2\,{}^0s_y\,v_1 + v_1^2}{{}^0_s{}^2}u_1 + \frac{C\,{}^0A}{{}^0_s}\frac{1}{2}\cdot\frac{v_1^2 + 2\,{}^0s_x\,u_1 + u_1^2}{{}^0_s{}^2}({}^0s_x + u_1) = 0.$$

After substituting the numerical values for the problem parameters

$$\frac{1}{2}+\frac{1}{2}\cdot(u_1^2+2v_1+v_1^2)(1+v_1)+\frac{1}{2}\cdot(v_1^2+2u_1+u_1^2)v_1=0,$$

$$\frac{1}{2}\cdot(u_1^2+2v_1+v_1^2)u_1+\frac{1}{2}\cdot(v_1^2+2u_1+u_1^2)(1+u_1)=0$$

the solution to which can be obtained with the aid of *Mathematica* $\Rightarrow$

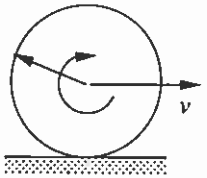
$v_1=-0.324532,\ \ u_1=-0.0383314.$

Problem 10.4

A cylinder is rolling freely on a horizontal plane. Determine the equation of (plane) motion of the cylinder using Lagrange's equation. The radius of the cylinder is R and the mass is m.

Solution

From kinematics we know that the velocity of the contact point is zero. Thus by using the notations of the figure $v-\omega R=0$. The kinetic energy of the cylinder is (for a cylinder $I=mR^2/2$)



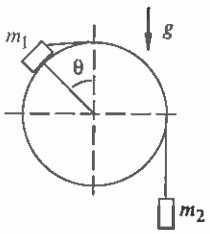
$$K=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2=\frac{1}{2}mv^2+\frac{1}{2}\frac{1}{2}mR^2(v/R)^2=\frac{3}{4}mv^2.$$

Lagranges equation of the motion gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial v}\right)=0\ \Rightarrow\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2}mv\right)=0\ \Rightarrow\ v=\text{constant}.$$

Problem 10.5

The particles of the figure are joined together by a massless inextensible string. Assuming that friction is negligible, determine the value of angle θ at which the system is in equilibrium.



Solution

The potential energy of the system is

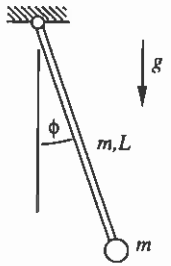
$$V=-m_1gR(1-\cos\theta)+m_2gR\theta.$$

The system is in equilibrium when $\mathrm{d}V/\mathrm{d}\theta=0$ giving

$$-m_1gR\sin\theta+m_2gR=0\ \ \Rightarrow\ \ \theta=\arcsin(m_2/m_1)\ \ \theta\in]0,\pi/2[.$$

Mat-5.160 Variational principles of mechanics, exercise 11

(to be returned before April 22)



1. The pendulum of the figure consist of a particle (mass m) and a homogeneous bar (mass m, length L). Write out the Lagrangian equation of motion of the system by using the angle of the bar ϕ as the generalized coordinate. What is the period of small oscillations ?

2. The usual way to solve non-stationary problems makes use of finite differences with respect to the time coordinate, i.e, time- derivatives are replaced by backward-, central- or forward differences.
   (a) What is the expression of the amplification coefficient T of the difference equation $\phi^{n+1} = T\phi^n + a$ ($n \in \{0,\ldots,N\}$ denotes a discrete time instant $t^n = n\Delta t$ and a is a constant), when the initial value problem

$$m\frac{d\phi}{dt} + k\phi = 0 \quad t \in]0,T], \quad \phi = \phi^0 \quad t = 0$$

   is solved by using approximations $d\phi/dt \approx (\phi^{n+1} - \phi^n)/\Delta t$ and $\phi \approx [\theta\phi^{n+1} + (1-\theta)\phi^n]$.
   (b) If the discrete solution is bounded no matter $k, m > 0$ and Δt, the numerical method can be considered absolutely stable. Under what conditions on θ is the method absolutely stable ?

3. Let us consider the implicit Euler method in connection with the linear small displacement version of problem 11.1. The first thing is to cast the problem into the form

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = 0,$$

   where $q_1 = \phi$ and $q_2 = \dot{\phi}$. Is the method thus obtained absolutely stable ? In the implicit Euler method, the approximations for the derivatives are $\dot{q}_i \approx (q_i^{n+1} - q_i^n)/\Delta t$ and $q_i \approx q_i^{n+1}$.

4. ADAMS employs an approximation to the derivative which takes the form

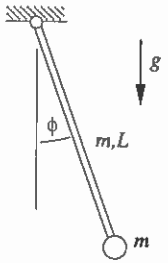
$$y^{n+1} = \sum_{i=1}^{k} \alpha^i y^{n-i+1} - \Delta t \beta_0 \dot{y}^{n+1}$$

   where y^i:s are the values of y at discrete time instants t^i and k is the order of the approximation. Derive the values for coefficients a^i and β_0 in cases $k = 1$ and $k = 2$. Assume that step size Δt is constant and use equation $\dot{y}(t^{n+1}) = \dot{p}(t^{n+1})$ ($p(t)$ is the interpolant to data (t^i, y^i) $i \in \{n-k+1 \ldots n+1\}$).

5. Write out the equation system governing the motion of the pendulum of problem 11.1 by using the ADAMS notation explained in Section 15.8.

Problem 11.1

The pendulum of the figure consist of a particle (mass m) and a homogeneous bar (mass m, length L). Write out the Lagrangian equation of motion of the system by using the angle of the bar ϕ as the generalized coordinate. What is the period of small oscillations ?



Solution

The kinetic energy of the bar can be written as

$$K = \frac{1}{2}m(L\dot{\phi})^2 + \frac{1}{2}m(r\dot{\phi})^2 = \frac{1}{2}m(L^2 + r^2)\dot{\phi}^2,$$

where $r = \sqrt{I/m}$ and $I = mL^2/3$ is the moment of inertia of the bar with respect to the end point. The expression of the potential energy is given by

$$V = L(1-\cos\phi)mg + \frac{1}{2}L(1-\cos\phi)mg = \frac{3}{2}L(1-\cos\phi)mg.$$

The partial derivatives of the Lagrange equation of motion

$$\frac{d}{dt}\left(\frac{\partial(K-V)}{\partial\dot{\phi}}\right)-\frac{\partial(K-V)}{\partial\phi}=0$$

are now

$$\frac{\partial(K-V)}{\partial\dot{\phi}}=m(L^2+r^2)\dot{\phi} \text{ and } \frac{\partial(K-V)}{\partial\phi}=-\frac{3}{2}mgL\sin\phi.$$

Therefore the equation of motion is

$$m(L^2+r^2)\ddot{\phi}+\frac{3}{2}mgL\sin\phi=0.$$

When the angle is small i.e. $\sin\phi\approx\phi$, we get

$$m(L^2+r^2)\ddot{\phi}+\frac{3}{2}mgL\phi=0$$

whose solutions are $\sin\omega t$ and $\cos\omega t$ with

$$\omega=\sqrt{\frac{3}{2}\frac{gL}{r^2+L^2}}.$$

Thus the period of small oscillation is given by $T=2\pi/\omega$.

Problem 11.2

The usual way to solve non-stationary problems makes use of finite differences with respect to the time coordinate, i.e, time- derivatives are replaced by backward-, central- or forward differences.

(a) What is the expression of the amplification coefficient T of the difference equation $\phi^{n+1}=T\phi^n+a$ ($n\in\{0,\ldots,N\}$ denotes a discrete time instant $t^n=n\Delta t$ and a is a constant), when the initial value problem

$$m\frac{d\phi}{dt}+k\phi=0 \quad t\in]0,T], \quad \phi=\phi^0 \quad t=0$$

is solved by using approximations $d\phi/dt\approx(\phi^{n+1}-\phi^n)/\Delta t$ and $\phi\approx[\theta\phi^{n+1}+(1-\theta)\phi^n]$.

(b) If the discrete solution is bounded no matter $k,m>0$ and Δt, the numerical method can be considered absolutely stable. Under what conditions on θ is the method absolutely stable ?

Solution

(a) When the difference approximations

$$\frac{d\phi}{dt}\approx\frac{\phi^{n+1}-\phi^n}{\Delta t} \text{ and } \phi\approx\theta\phi^{n+1}+(1-\theta)\phi^n$$

are substituted, the ordinary differential equation takes the form

$$m\frac{\phi^{n+1}-\phi^n}{\Delta t}+k[\theta\phi^{n+1}+(1-\theta)\phi^n]=0$$

or after some arrangements

$$\phi^{n+1}=\frac{1-(1-\theta)\hat{k}}{1+\theta\hat{k}}\phi^n=T\phi^n$$

where $\hat{k}=k\Delta t/m$ and T is the amplification coefficient whose expression is evident from the equation.

(b)The solution to the difference equation (use the equation recursively n-times)

$$\phi^n=T^n\phi^0$$

is bounded only if $-1 \le T \le 1$ as otherwise the left hand side grows with exponential rate in n. The restriction $T \le 1$ is always satisfied as

$$\frac{1-(1-\theta)\hat{k}}{1+\theta\hat{k}} \le 1 \;\Leftrightarrow\; 1-(1-\theta)\hat{k} \le 1+\theta\hat{k} \;\Leftrightarrow\; 0 \le 1.$$

The remaining condition $-1 \le T$ yields

$$-1 \le \frac{1-(1-\theta)\hat{k}}{1+\theta\hat{k}} \;\Leftrightarrow\; -1-\theta\hat{k} \le 1-(1-\theta)\hat{k} \;\Leftrightarrow\; -2 \le \hat{k}(2\theta-1).$$

The condition is satisfied if $2\theta - 1 \ge 0$, i.e $\theta \ge 1/2$, as then the right hand side is positive no matter the (positive) value of $\hat{k}$. The methods corresponding to selections $\theta = 1/2$ and $\theta = 1$ are called Crank-Nicholson and implicit Euler, respectively.

Problem 11.3

Let us consider the implicit Euler method in connection with the linear small displacement version of problem 11.1. The first thing is to cast the problem into the form

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = 0,$$

where $q_1 = \phi$ and $q_2 = \dot{\phi}$. Is the method thus obtained absolutely stable ? In the implicit Euler method, the approximations for the derivatives are $\dot{q}_i \approx (q_i^{n+1} - q_i^n)/\Delta t$ and $q_i \approx q_i^{n+1}$.

Solution

By introducing notations $q_2 = \dot{q}_1$, $q_1 = \phi$ and $\omega^2 = 2/3 \cdot gL/(r^2 + L^2)$, the linearized differential equation can be written as

$$\dot{q}_2 + \omega^2 q_1 = 0,$$

$$\dot{q}_1 - q_2 = 0$$

or

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} 0 & \omega^2 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = 0.$$

When the approximations $\dot{q}_i = (q_i^n - q_i^{n-1})/\Delta t$, $q_i = q_i^n$ corresponding to the implicit Euler method are substituted there, the result is

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^n = \begin{bmatrix} 1 & \Delta t\omega^2 \\ -\Delta t & 0 \end{bmatrix}^{-1} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^{n-1} = \mathbf{T} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^{n-1}.$$

For unconditional absolute stability, the magnitudes of the eigenvalues of the amplification matrix $\mathbf{T}$ should not exceed one. As the amplification matrix is given by

$$\mathbf{T} = \begin{bmatrix} 1 & \Delta t\omega^2 \\ -\Delta t & 0 \end{bmatrix}^{-1} = \frac{1}{1+\Delta t^2\omega^2} \begin{bmatrix} 1 & -\Delta t\omega^2 \\ \Delta t & 0 \end{bmatrix},$$

the characteristic polynomial

$$\mathrm{Det} \begin{bmatrix} 1-(1+\Delta t^2\omega^2)\lambda & -\Delta t\omega^2 \\ \Delta t & -(1+\Delta t^2\omega^2)\lambda \end{bmatrix} = 0$$

gives $\lambda = 1/(\Delta t\omega \pm i)$. Since the magnitudes of both eigenvalues are smaller than one no matter $\omega \ge 0$ and $\Delta t > 0$, the method is unconditionally absolutely stable.

Problem 11.4

ADAMS employs an approximation to the derivative, which takes the form

$$y^{n+1} = \sum_{i=1}^{k} \alpha^i y^{n-i+1} - \Delta t \beta_0 \dot{y}^{n+1}$$

where y^i:s are the values of y at discrete time instants t^i and k is the order of the approximation. Derive the values for coefficients a^i and β_0 in cases $k=1$ and $k=2$. Assume that step size Δt is constant and use equation $\dot{y}(t^{n+1}) = \dot{p}(t^{n+1})$ ($p(t)$ is the interpolant to data (t^i, y^i) $i \in \{n-k+1 \ldots n+1\}$).

Solution

When $k=1$ the difference equation

$$y^{n+1} = \alpha^1 y^n - \Delta t \beta_0 \dot{y}^{n+1}$$

should hold for polynomials of degree zero and one. The selections $y^n = 1$ and $y^n = n\Delta t$ give

$$1 = \alpha^1 1,$$

$$(n+1)\Delta t = \alpha^1 n\Delta t - \Delta t \beta_0 1,$$

respectively. Therefore the values of the coefficients are $\alpha^1 = 1$ and $\beta_0 = -1$. When $k=2$ the difference equation

$$y^{n+1} = \alpha^1 y^n + \alpha^2 y^{n-1} - \Delta t \beta_0 \dot{y}^{n+1}$$

should hold for polynomials of degree zero, one and two. The selections $y^n = 1$, $y^n = n\Delta t$ and $y^n = n^2 \Delta t^2$ give

$$1 = \alpha^1 1 + \alpha^2 1,$$

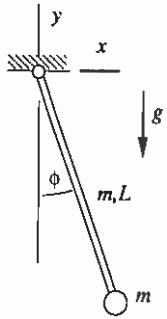
$$(n+1)\Delta t = \alpha^1 n\Delta t + \alpha^2 (n-1)\Delta t - \Delta t \beta_0 1,$$

$$(n+1)^2 \Delta t^2 = \alpha^1 n^2 \Delta t^2 + \alpha^2 (n-1)^2 \Delta t^2 - \Delta t \beta_0 2(n+1)\Delta t,$$

respectively. Therefore the values of the coefficients are $\alpha^1 = 4/3$, $\alpha^2 = -1/3$ and $\beta_0 = -2/3$ (calculated with *Mathematica*).

Problem 11.5

Write out the equation system governing the motion of the pendulum of problem 11.1 by using the ADAMS notation explained in Section 15.8.



Solution

Let us consider the bar and the particle as separate rigid bodies with generalized coordinates x_1, y_1 (particle) x_2, y_2, ϕ (bar). The kinetic energy of the system is given by

$$K = \frac{1}{2} m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} I \dot{\phi}^2$$

where $I = mL^2/12$ is the moment of inertia with respect to

the centerpoint of the bar. The potential energy of the external forces is

$V = mgy_1 + mgy_2$.

The constraints of the motion are

$\varphi_1 = y_1 - L(1-\cos\phi) = 0,$

$\varphi_2 = x_1 - L\sin\phi = 0,$

$\varphi_3 = y_2 - L/2 \cdot (1-\cos\phi) = 0,$

$\varphi_4 = x_2 - L/2 \cdot \sin\phi = 0.$

With this data, the Lagrange equations of motion

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = \sum \lambda_j \frac{\partial \varphi_j}{\partial q_i}$$

give for selections $q_1 = x_1$, $q_2 = y_1$, $q_3 = x_2$, $q_4 = y_2$ and $q_5 = \phi$

$m\ddot{x}_1 = \lambda_2,$

$m\ddot{y}_1 - mg = \lambda_1,$

$m\ddot{x}_2 = \lambda_3,$

$m\ddot{y}_2 - mg = \lambda_4,$

$I\ddot{\phi} = -\lambda_1 L\sin\phi - \lambda_2 L\cos\phi - \lambda_3 L/2 \cdot \sin\phi - \lambda_4 L/2\cos\phi,$

respectively. The equation system of the ADAMS form consists of the constraints

$q_2 - L(1-\cos q_5) = 0,$

$q_1 - L\sin q_5 = 0,$

$q_4 - L/2 \cdot (1-\cos q_5) = 0,$

$q_3 - L/2 \cdot \sin q_5 = 0$

the equations of motion

$m\dot{u}_1 - \lambda_2 = 0,$

$m\dot{u}_2 - mg - \lambda_1 = 0,$

$m\dot{u}_3 - \lambda_3 = 0,$

$m\dot{u}_4 - mg - \lambda_4 = 0,$

$I\dot{u}_5 + \lambda_1 L\sin q_5 + \lambda_2 L\cos q_5 + \lambda_3 L/2 \cdot \sin q_5 + \lambda_4 L/2\cos q_5 = 0,$

and kinematic relationships

$u_1 - \dot{q}_1 = 0,$

$u_2 - \dot{q}_2 = 0,$

$u_3 - \dot{q}_3 = 0,$

$u_4 - \dot{q}_4 = 0,$

$u_5 - \dot{q}_5 = 0.$

These are subjected to initial conditions depending on the setting.

Mat-5.160 Variational principles of mechanics, exercise 12

1. Let us consider the truss of problem 10.2 (Fig. 1) using the small displacement theory of Section 6.5. Write down the element contributions for the two- and one-noded elements of the problem. Assemble the system matrices and solve the vertical displacement.

2. Repeat problem 1 in connection with Fig. 2 (for the parameters see problem 10.2).

3. Determine the vertical displacement v_1 of the elastic truss of Fig. 1 with the large deformation theory and the incremental approach of Section 6.4 (you may need *Mathematica*). $a = 1\text{m}$, ${}^0A = 0.01\text{m}^2$, $C = 100\text{Nm}^{-2}$ and $F = 0.5\text{N}$.

4. Determine the displacement u_1, v_1 of the elastic truss of Fig. 2 with the large deformation theory and the incremental approach of Section 6.4. The values of the parameters are the same as in the previous problem.

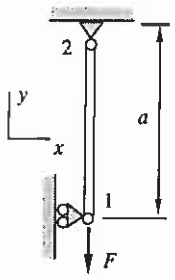


Fig. 1

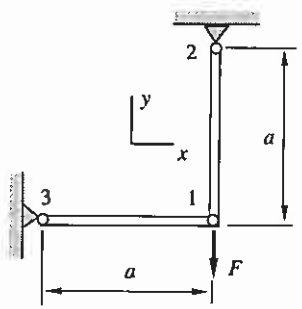


Fig. 2

Problem 12.1

Let us consider the truss of problem 10.2 (Figure 1) using the small displacement theory of Section 6.5. Write down the element contributions for the two- and one-noded elements of the problem. Assemble the system matrices and solve the vertical displacement

Solution

The formulas of the small displacement theory element contributions for two noded and one-noded elements are

$$[K] = \frac{C\,{}^0A}{s}\frac{1}{s^2}\begin{bmatrix} s_x^2 & s_x s_y & -s_x^2 & -s_x s_y \\ s_x s_y & s_y^2 & -s_x s_y & -s_y^2 \\ -s_x^2 & -s_x s_y & s_x^2 & s_x s_y \\ -s_x s_y & -s_y^2 & s_x s_y & s_y^2 \end{bmatrix}, \quad [b] = \begin{bmatrix} F_x \\ F_y \end{bmatrix}.$$

In this case the number of two noded elements is one and the number of one noded element is also one, since the upper node 2 is fixed. The problem parameters are $C = 100\,\text{Nm}^{-2}$, ${}^0A = 0.01\,\text{m}^2$, $F = 0.5\,\text{N}$, $a = 1\,\text{m}$ and the 'length' parameters of the matrix above are $s = 1\,\text{m}$, $s_x = 0\,\text{m}$, $s_y = 1\,\text{m}$. So (we apply the [m,N] system)

$$[K]^1 = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right], \quad [b]^2 = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix},$$

where the numbering denotes elements. Only one degree of freedom (the vertical displacement of node 1) is active. The connectivity tables for the element contributions are then

$[L]^1 = [0 \quad 1 \mid 0 \quad 0]$, $[L]^2 = [0 \quad 1]$.

This information is enough for building of the system equations $[K]\{q\} = \{b\}$ of one unknown. The first element gives a contribution to the system matrix

$$[0 \quad 1 \mid 0 \quad 0]$$

$$[K]^1 = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \begin{bmatrix} 0 \\ 1 \\ \hline 0 \\ 0 \end{bmatrix} \Rightarrow [K] = [1]$$

and the second element gives a contribution to the system vector

$$[b]^2 = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \{b\} = \{-0.5\}.$$

The final equation is then

$$[1]\{v_1\} = (-0.5) \Leftrightarrow v_1 = -0.5\,[\mathrm{m}].$$

Problem 12.2

Repeat problem 1 in connection with Figure 2 (for the problem parameters see also problem 10.2).

Solution

The formulas of the small displacement theory element contributions for two-noded and one-noded elements are

$$[K] = \frac{C\,{}^0A}{s}\frac{1}{s^2}\begin{bmatrix} s_x^2 & s_x s_y & -s_x^2 & -s_x s_y \\ s_x s_y & s_y^2 & -s_x s_y & -s_y^2 \\ -s_x^2 & -s_x s_y & s_x^2 & s_x s_y \\ -s_x s_y & -s_y^2 & s_x s_y & s_y^2 \end{bmatrix}, \quad [b] = \begin{bmatrix} F_x \\ F_y \end{bmatrix}.$$

In this case the number of two noded elements is two and the number of one noded element is one, since nodes 2 and 3 are fixed. The problem parameters are $C = 100\ \mathrm{Nm^{-2}}$, ${}^0A = 0.01\ \mathrm{m^2}$, $F = 0.5\ \mathrm{N}$, $a = 1\ \mathrm{m}$.

Element 1

The geometric parameters for the horizontal bar 'element 1' are $s = 1\ \mathrm{m}$, $s_x = 1\ \mathrm{m}$, $s_y = 0\ \mathrm{m}$, so

$$[K]^1 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad [L]^1 = [0 \quad 0 \mid 1 \quad 2].$$

The connectivity table can be obtained by numbering the active degrees of freedoms in the order of the global node numbering, giving the fixed ones the value zero and then picking up the numbers in the order of the local numbering of the nodes.

Element 2

The geometric parameters for the horizontal bar 'element 1' are $s = 1\ \mathrm{m}$, $s_x = 1\ \mathrm{m}$, $s_y = 0\ \mathrm{m}$. So

$$[K]^2 = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right], \quad [L]^2 = [1 \quad 2 \mid 0 \quad 0].$$

where numbering denotes elements.

Element 3

Only the vertical component of the external force is non-zero. So

$$[b]^3 = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}, \quad [L]^3 = [1 \quad 2].$$

Assembly

The element information is enough for building of the system equations $[K]\{q\} = \{b\}$ of two unknowns. The first element gives a contribution to the system matrix

$$[0 \quad 0 \mid 1 \quad 2]$$

$$[K]^1 = \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} 0 \\ \underline{0} \\ 1 \\ 2 \end{bmatrix} \Rightarrow [K] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Also the second element gives a contribution to the system matrix

$$[1 \quad 2 \mid 0 \quad 0]$$

$$[K]^2 = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \begin{bmatrix} 1 \\ \underline{2} \\ 0 \\ 0 \end{bmatrix} \Rightarrow [K] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right].$$

The third element gives a contribution to the system vector

$$[b]^3 = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \{b\} = \begin{Bmatrix} 0 \\ -0.5 \end{Bmatrix}.$$

The final system of equations is then

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.5 \end{Bmatrix} \Leftrightarrow \quad u_1 = 0\,[\mathrm{m}],\ v_1 = -0.5\,[\mathrm{m}].$$

Problem 12.3

Determine the vertical displacement v_1 of the elastic truss of Figure 1 with the large deformation theory and the incremental approach of Section 6.4 (you may need *Mathematica*). $a = 1\mathrm{m}$, ${}^0A = 0.01\mathrm{m}^2$, $C = 100\mathrm{Nm}^{-2}$ and $F = 0.5\mathrm{N}$.

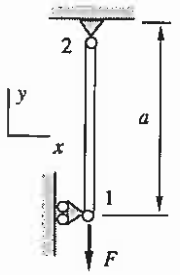


Figure 1

Solution

See the examples section of the *Mathematica* notebook **INCREMENTAL.**

Problem 12.4

Determine the displacement u_1, v_1 of the elastic truss of Figure 2 with the large deformation theory and the incremental approach of Section 6.4. The values of the parameters are the same as in the previous problem.

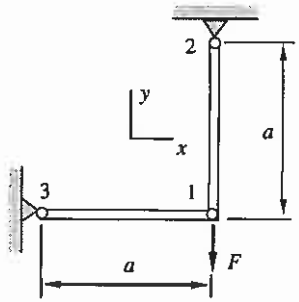


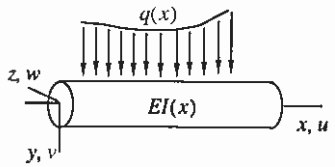
Figure 2

Solution

See the examples section of the *Mathematica* notebook **INCREMENTAL.**

Mat-5.160 Variational principles of mechanics, exercise 13

1. Derive the Euler-Lagrange equations and the natural boundary conditions for an elastic Bernoulli beam by using the principle of stationary potential energy. The kinematical assumptions are $u=-yv'(x)$, $v=v(x)$ and $w=0$ (figure). The stress-strain relations are $\sigma_x=E\varepsilon_x$ and $\tau_{xy}=G\gamma_{xy}$.



2. Repeat problem 13.1 with the Timoshenko beam kinematical assumptions $u=-y\theta(x)$, $v=v(x)$ and $w=0$.

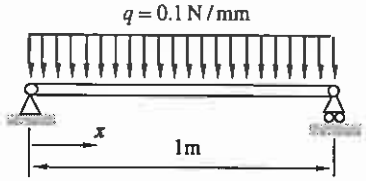
3. Let us consider the beam model of problem 13.2. In a finite element method the two unknown functions $v(x)$ and $\gamma(x)$ (here $\theta(x)=v'(x)-\gamma(x)$) are approximated in a typical element by

$$\tilde{v}(x)=\sum_{i\in\{1,2\}}\{N_i^e, N_i'^e\}\begin{Bmatrix} v_i^e \\ \theta_i^e+\bar{\gamma}\end{Bmatrix},\quad \tilde{\gamma}=\bar{\gamma}=\text{constant},$$

where the shape functions (Hermite polynomials) are defined $N_1^e=1-3\xi^2+2\xi^3$, $N_2^e=3\xi^2-2\xi^3$, $N_1'^e=h(\xi-2\xi^2+\xi^3)$ and $N_2'^e=h(-\xi^2+\xi^3)$ ($\xi=x/h$, the origin is located at node 1 and h is the element size). Derive the contribution of the typical element and eliminate (this is called static condensation) the internal parameter $\bar{\gamma}$ to get a more convenient starting point for the assembly phase. Assume that the cross-section of the beam is a rectangle with (a constant) area A and that the distributed external force is also constant q.

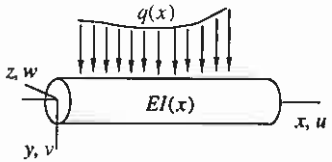
4. Repeat problem 13.3 by using piecewise linear approximations to $v(x)$ and $\theta(x)$. Use reduced integration (midpoint rule) and modification $GA\to GA\alpha$ with $\alpha=12EI/(12EI+GAh^2)$ in the shear energy term.

5. Solve the problem of the figure using the element contributions of problem 13.3. $A=100\text{mm}^2$, $E=10\cdot10^3\text{N/mm}^2$, $G=E/3$, $I=10^4\text{mm}^4$. Use two elements of equal size.



Problem 13.1

Derive the Euler-Lagrange equations and the natural boundary conditions for an elastic Bernoulli beam by using the principle of stationary potential energy. The kinematical assumptions are $u=-yv'(x)$, $v=v(x)$ and $w=0$ (figure). The stress-strain relations are $\sigma_x=E\varepsilon_x$ and $\tau_{xy}=G\gamma_{xy}$.



Solution

The potential energy can be written in the form

$$V=\int_V \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\mathrm{d}V-\int_V qv\mathrm{d}V.$$

The displacement field $u=-yv'(x)$, $v=v(x)$ and $w=0$ gives the strain and the stress components

$$\varepsilon_x=-yv'',\ \gamma_{xy}=-w'+w'=0$$

and the stress-strain relation is $\sigma_x=E\varepsilon_x$. Substituting the expressions in the potential energy expression gives

$$V(v)=\int_V \frac{1}{2}Ey^2(v'')^2\mathrm{d}V-\int_L qv\mathrm{d}x=\int_L [\frac{1}{2}D(v'')^2-qw]\mathrm{d}x,$$

where $D=\int_A Ey^2\mathrm{d}A$. The first variation gives

$$\delta V = \int_L (Dv''\delta v'' - q\delta v)\mathrm{d}x =$$

$$= \int_L [(Dv'')'' - q]\delta v\mathrm{d}x + \sum [n_x Dv''\delta v' - \sum n_x Dv'''\delta v = 0.$$

Thus the field equation and the natural boundary conditions are

$$(Dv'')'' - q = 0 \quad x \in L,$$

$$Dv'' = 0 \quad x \in \partial L,$$

$$Dv''' = 0 \quad x \in \partial L.$$

Problem 13.2

Repeat problem 13.1 with the Timoshenko beam kinematical assumptions $u = -y\theta(x)$, $v = v(x)$ and $w = 0$.

Solution

The potential energy is also now

$$V = \int_V \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\mathrm{d}V - \int_L qv\mathrm{d}x.$$

The displacement field $u = -y\theta(x)$, $v = v(x)$ and $w = 0$ gives the strain components

$$\varepsilon_x = -y\theta', \ \gamma_{xy} = -\theta + v'.$$

The stress-strain relations are $\sigma_x = E\varepsilon_x$ and $\tau_{xy} = G\gamma_{xy}$. In this case the potential energy functional simplifies into

$$V(v,\theta) = \int_V [\frac{1}{2}Ey^2(\theta')^2 + \frac{1}{2}G(v' - \theta)^2]\mathrm{d}V - \int_L qv\mathrm{d}x =$$

$$= \int_L [\frac{1}{2}D(\theta')^2 + \frac{1}{2}GA(v' - \theta)^2 - qv]\mathrm{d}x,$$

where $D = \int_A Ey^2\mathrm{d}A$. The first variation gives

$$\delta V(v,\theta) = \int_L [D\theta'\delta\theta' + GA(v' - \theta)(\delta v' - \delta\theta) - q\delta v]\mathrm{d}x =$$

$$= \int_L [-(D\theta')' - GA(v' - \theta)]\delta\theta\mathrm{d}x + \sum n_x D\theta'\delta\theta +$$

$$+ \int_L [-\{GA(v' - \theta)\}' - q]\delta v\mathrm{d}x + \sum n_x GA(v' - \theta)\delta v = 0.$$

Thus the field equation and the natural boundary conditions are

$$-(D\theta')' - GA(v' - \theta) = 0 \quad x \in L,$$

$$\{GA(v' - \theta)\}' + q = 0 \quad x \in L,$$

$$D\theta' = 0 \quad x \in \partial L,$$

$$GA(v' - \theta) = 0 \quad x \in \partial L.$$

Problem 13.3

Let us consider the beam model of problem 13.2. In the finite element method the two unknown functions $w(x)$ and $\gamma(x)$ (here $\theta(x) = v'(x) - \gamma(x)$) are approximated in a typical element by

$$\tilde{v}(x) = \sum_{i\in\{1,2\}} \{N_i^e, N_i'^e\} \begin{Bmatrix} v_i^e \\ \theta_i^e + \bar{\gamma} \end{Bmatrix}, \quad \tilde{\gamma} = \bar{\gamma} = \text{constant},$$

where the shape functions (Hermite polynomials) are defined $N_1^e = 1 - 3\xi^2 + 2\xi^3$, $N_2^e = 3\xi^2 - 2\xi^3$, $N_1'^e = h(\xi - 2\xi^2 + \xi^3)$ and $N_2'^e = h(-\xi^2 + \xi^3)$ ($\xi = x/h$, the origin is located at node 1 and h is the element size). Derive the contribution of the typical element and eliminate (this is called static condensation) the internal parameter $\bar{\gamma}$ to get a more convenient starting point for the assembly phase. Assume that the cross-section of the beam is a rectangle with (a constant) area A and that the distributed external force is also constant q.

Solution

The potential energy of problem 13.2 is

$$V(v,\theta)=\int_L [\frac{1}{2}D(v''-\gamma')^2+\frac{1}{2}GA(\gamma)^2-qv]\mathrm{d}x\,,$$

where we have applied $\theta(x)=v'(x)-\gamma(x)$ to eliminate θ. The next step is to substitute the approximations. Since only an element contribution is wanted, the domain is also chosen to be $]0,h[$. With the summation convention the result is

$$V=\int_0^h [\frac{1}{2}D([N_i^e,N_i'^e]''\begin{Bmatrix} v_i^e \\ \theta_i^e+\bar{\gamma}\end{Bmatrix})^2+\frac{1}{2}GA(\bar{\gamma})^2-q[N_i^e,N_i'^e]\begin{Bmatrix} v_i^e \\ \theta_i^e+\bar{\gamma}\end{Bmatrix}]\mathrm{d}x\,,$$

which can also be represented as

$$V=\int_0^h \mathbf{a}^{\mathrm{T}}\cdot[\frac{1}{2}D\mathbf{T}^{\mathrm{T}}\mathbf{N}''\mathbf{N}''^{\mathrm{T}}\mathbf{T}+\frac{1}{2}GA\mathbf{S}]\mathrm{d}x\cdot\mathbf{a}-\mathbf{a}^{\mathrm{T}}\cdot\int_0^h q\mathbf{T}^{\mathrm{T}}\mathbf{N}\mathrm{d}x\,,$$

by using the shorthand notations (things become simpler with the matrix notation)

$$\mathbf{T}\cdot\mathbf{a}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&1\\0&0&1&0&0\\0&0&0&1&1\end{bmatrix}\begin{Bmatrix}v_1^e\\ \theta_1^e\\ v_2^e\\ \theta_2^e\\ \bar{\gamma}\end{Bmatrix},\quad \mathbf{N}=\begin{Bmatrix}N_1^e\\ N_1'^e\\ N_2^e\\ N_2'^e\end{Bmatrix},\quad \mathbf{S}=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{bmatrix}$$

Minimization of the function with respect to $\mathbf{a}$ gives

$$\mathbf{R}=\{D\int_0^h (\mathbf{T}^{\mathrm{T}}\mathbf{N}''\mathbf{N}''^{\mathrm{T}}\mathbf{T})\mathrm{d}x+GA\int_0^h \mathbf{S}\mathrm{d}x\}\cdot\mathbf{a}-q\int_0^h \mathbf{T}^{\mathrm{T}}\mathbf{N}\mathrm{d}x\,,$$

for the element contribution if D, G, A and q are constants. The integrals can be evaluated for example by using *Mathematica*

$$\int_0^h \mathbf{S}\mathrm{d}x=h\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{bmatrix},$$

$$\int_0^h \mathbf{T}^{\mathrm{T}}\mathbf{N}\mathrm{d}x=\begin{Bmatrix}h/2\\ h^2/12\\ h/2\\ -h^2/12\\ 0\end{Bmatrix}$$

$$\int_0^h (\mathbf{T}^{\mathrm{T}}\mathbf{N}''\mathbf{N}''^{\mathrm{T}}\mathbf{T})\mathrm{d}x=\begin{bmatrix}12/h^3&6/h^2&-12/h^3&6/h^2&12/h^3\\6/h^2&4/h&-6/h^2&2/h&6/h\\-12/h^3&-6/h^2&12/h^3&-6/h^2&-12/h^2\\6/h^2&2/h&-6/h^2&4/h&6/h\\12/h^3&6/h&-12/h^2&6/h&12/h\end{bmatrix}$$

The condensed contributions can be obtained by noting that the last equation associated with $\bar{\gamma}$ is already in the final form and can then be used to eliminate $\bar{\gamma}$ from the remaining equations. The following result for the matrix is (calculated with *Mathematica*, $D\rightarrow d$)

```
{{(12*A*d*G)/(12*d*h + A*G*h^3), (6*A*d*G)/(12*d + A*G*h^2),
(-12*A*d*G)/(12*d*h + A*G*h^3), (6*A*d*G)/(12*d + A*G*h^2)},
{(6*A*d*G)/(12*d + A*G*h^2), (4*d)/h - (36*d^2)/(12*d*h + A*G*h^3),
(-6*A*d*G)/(12*d + A*G*h^2), (2*d)/h - (36*d^2)/(12*d*h + A*G*h^3)},
{(-12*A*d*G)/(12*d*h + A*G*h^3), (-6*A*d*G)/(12*d + A*G*h^2),
(12*A*d*G)/(12*d*h + A*G*h^3), (-6*A*d*G)/(12*d + A*G*h^2)},
{(6*A*d*G)/(12*d + A*G*h^2), (2*d)/h - (36*d^2)/(12*d*h + A*G*h^3),
(-6*A*d*G)/(12*d + A*G*h^2), (4*d)/h - (36*d^2)/(12*d*h + A*G*h^3)}} .
```

The corresponding vector is

q*{h/2, h^2/12, h/2, -h^2/12} .

Problem 13.4

Repeat problem 13.3 by using piecewise linear approximations to $v(x)$ and $\theta(x)$. Use reduced integration (midpoint rule) and modification $GA \rightarrow GA\alpha$ with $\alpha = 12EI/(12EI + GAh^2)$ in the shear energy term.

Solution

After the modifications, the potential energy of the Timoshenko beam

$$V(v,\theta) = \int_L [\frac{1}{2}EI(\theta')^2 + \frac{1}{2}GA(v' - \theta)^2 - qv]dx ,$$

written for a typical element takes the form

$$V(v,\theta) = \int_0^h [\frac{1}{2}EI(\theta')^2 - qv]dx + [\frac{1}{2}\alpha GA(v' - \theta)^2]_{x=h/2},$$

When the restrictions of the approximations to an element

$$\begin{Bmatrix} \tilde{v} \\ \tilde{\theta} \end{Bmatrix} = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 \\ 0 & N_1^e & 0 & N_2^e \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} \mathbf{a}$$

are substituted, the potential energy gives (assuming that GA, EI and q are constants)

$$V(\mathbf{a}) = \frac{1}{2}EI\mathbf{a}^{\mathrm{T}} \int_0^h \mathbf{M}'^{\mathrm{T}}\mathbf{M}'dx\ \mathbf{a} - q\mathbf{a}^{\mathrm{T}} \int_0^h \mathbf{N}^{\mathrm{T}}dx + \frac{1}{2}\alpha GA\mathbf{a}^{\mathrm{T}}[(\mathbf{N}'^{\mathrm{T}} - \mathbf{M}^{\mathrm{T}})(\mathbf{N}' - \mathbf{M})]_{x=h/2}\ \mathbf{a}.$$

and therefore

$$\frac{\partial V}{\partial \mathbf{a}} = EI \int_0^h \mathbf{M}'^{\mathrm{T}}\mathbf{M}'dx\ \mathbf{a} - q\int_0^h \mathbf{N}^{\mathrm{T}}dx + \alpha GA[(\mathbf{N}'^{\mathrm{T}} - \mathbf{M}^{\mathrm{T}})(\mathbf{N}' - \mathbf{M})]_{x=h/2}\ \mathbf{a}$$

giving

$$K^e = EI \int_0^h \mathbf{M}'^{\mathrm{T}}\mathbf{M}'dx + \alpha GA[(\mathbf{N}'^{\mathrm{T}} - \mathbf{M}^{\mathrm{T}})(\mathbf{N}' - \mathbf{M})]_{x=h/2} =$$

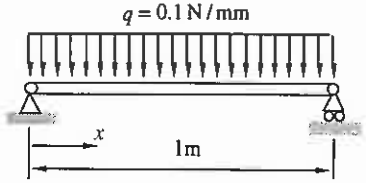
$$= \frac{EI}{h}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + GA\alpha \begin{bmatrix} 1/h & 1/2 & -1/h & 1/2 \\ 1/2 & h/4 & -1/2 & h/4 \\ -1/h & -1/2 & 1/h & -1/2 \\ 1/2 & h/4 & -1/2 & h/4 \end{bmatrix}$$

and

$$F^e = -q\int_0^h \mathbf{N}^{\mathrm{T}}dx = -\frac{qh}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix}.$$

Problem 13.5

Solve the problem of the figure using the element contributions of problem 13.3. $A = 100\text{mm}^2, E = 10 \cdot 10^3 \text{N/mm}^2$, $G = E/3$, $I = 10^4\,\text{mm}^4$. Use two elements of equal size.



Solution

After substituting the values of the problem parameters in the expressions of problem 13.3 we get the element contributions

{{9.59998617601991, 2399.996544004977, -9.59998617601991, 2399.996544004977},
{2399.996544004977, 799999.1360012442, -2399.996544004977, 399999.1360012441},
{-9.59998617601991, -2399.996544004977, 9.59998617601991, -2399.996544004977},

{2399.996544004977, 399999.1360012441, -2399.996544004977, 799999.1360012442}}

{25., 2083.333333333333, 25., -2083.333333333333}

for the matrix and the vector, respectively. The maps telling how to assemble the contributions of the two elements are

$\mathbf{L}^1 = \{0,1,2,3\}$, $\mathbf{L}^2 = \{2,3,0,4\}$.

The assembled system matrix and the vector are

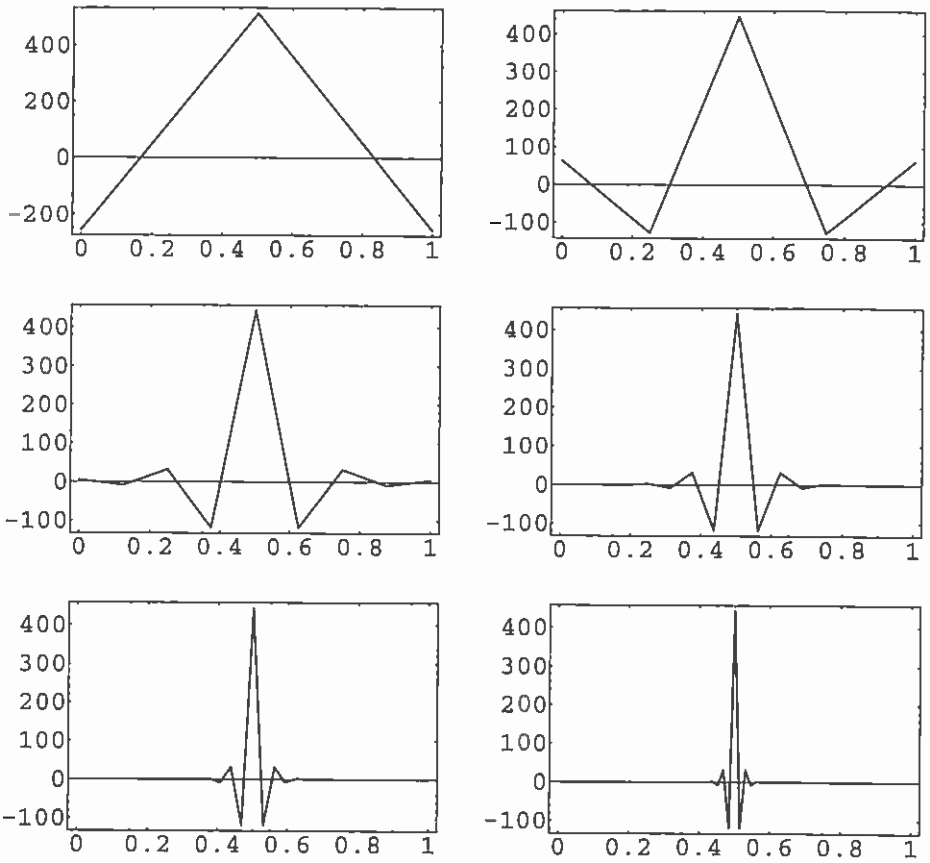
{{799999.1360012442, -2399.996544004977, 399999.1360012441, 0},
{-2399.996544004977, 19.19997235203982, 0., 2399.996544004977},
{399999.1360012441, 0., 1.599998272002488*10^6, 399999.1360012441},
{0, 2399.996544004977, 399999.1360012441, 799999.1360012442}}

{2083.333333333333, 50., 0., -2083.333333333333}

The solution to the problem is

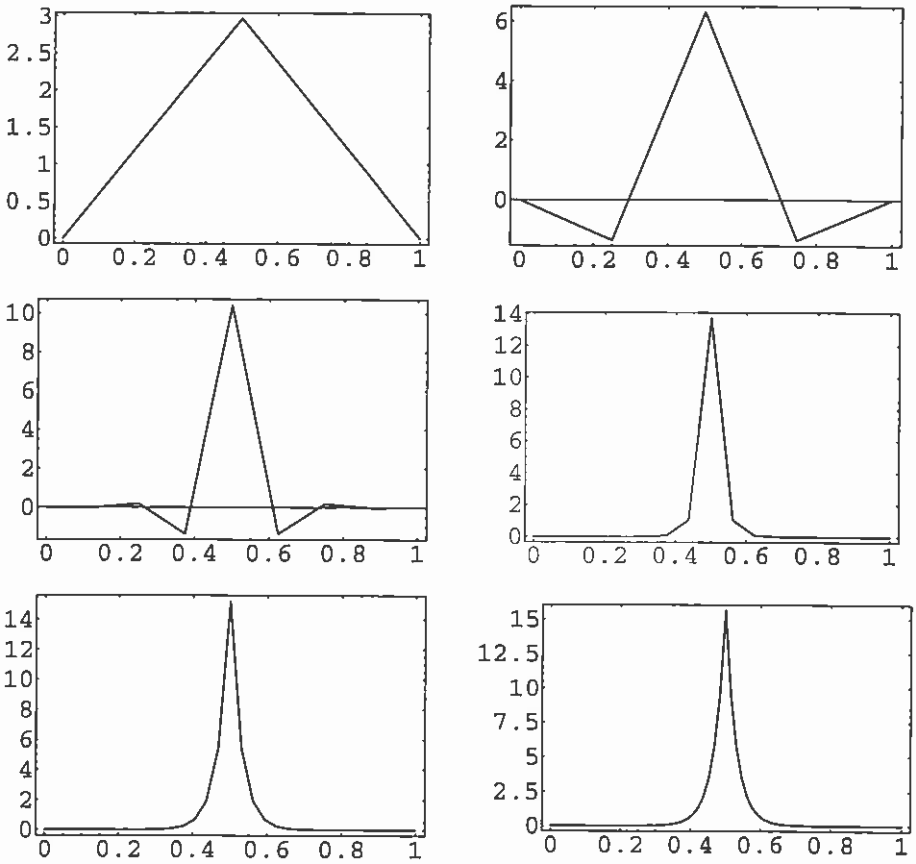
{0.04166666666666665, 13.02083708333332, 0., -0.04166666666666665}

So the deflection at the centerpoint is about 13 mm.



L_2 - projekcia $n = 2^i + 1$ $i \in \{1 \ldots 6\}$

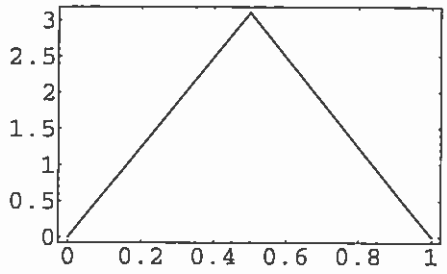
↑ solmujov 1km

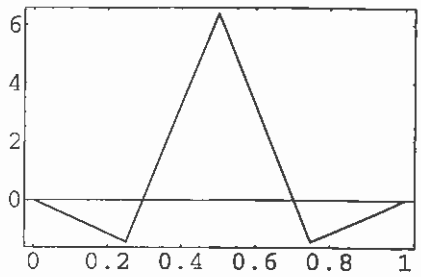


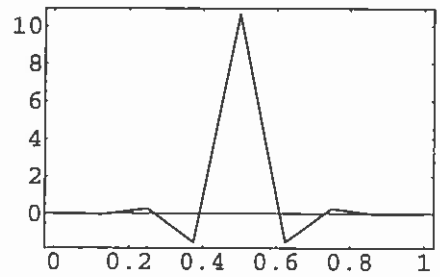
Ratkaisut: $S = 0.001$

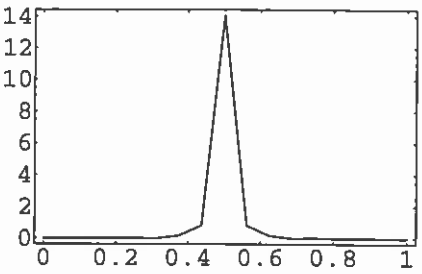
$T = 0$

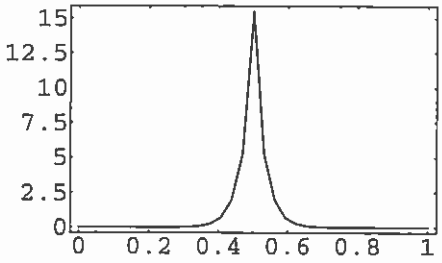
$k = F = L = 1$

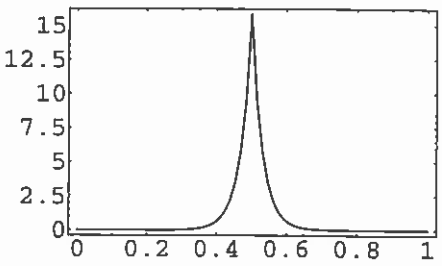








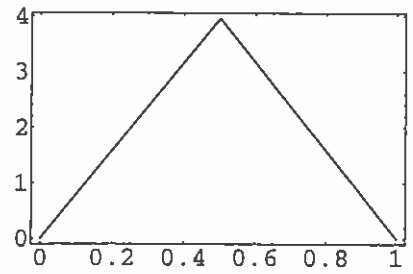


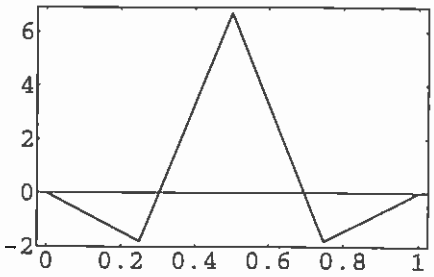


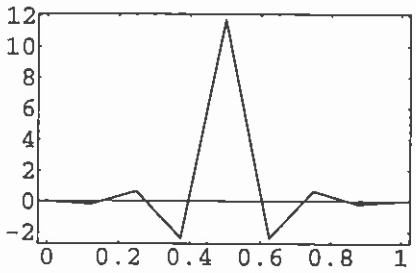
Ratkaisut: $S = 0.001$

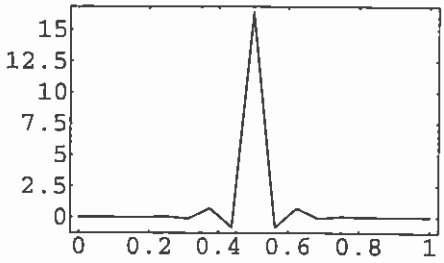
$T = T_{opt}/10$

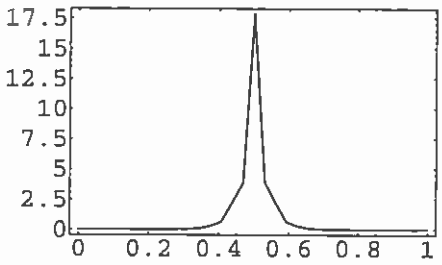
$K = F = L = 1$

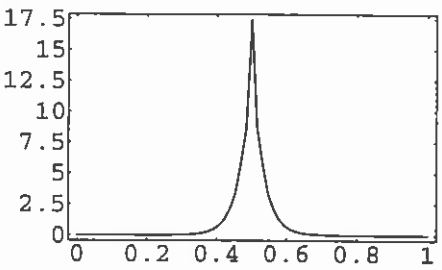








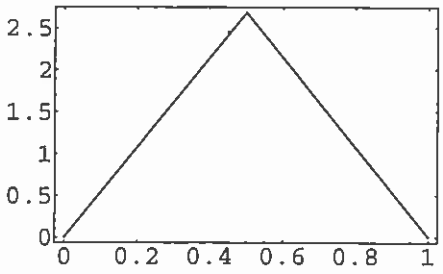


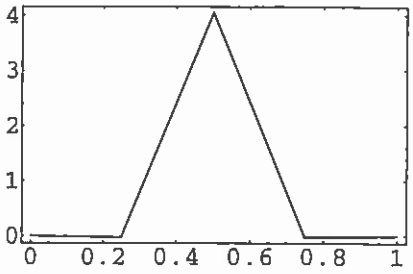


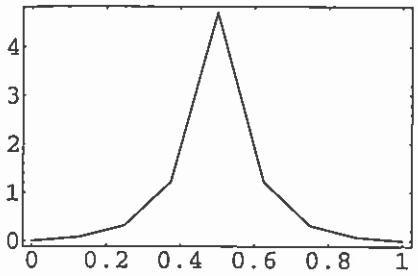
Ratkaisut : $S = 0.001$

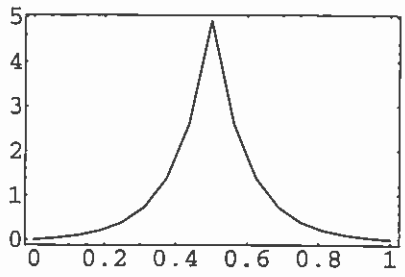
$T = T_{opt}$

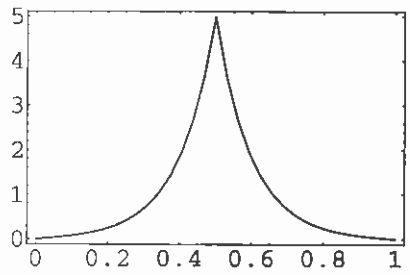
$k = F = L = 1$

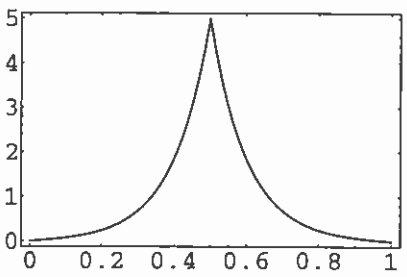










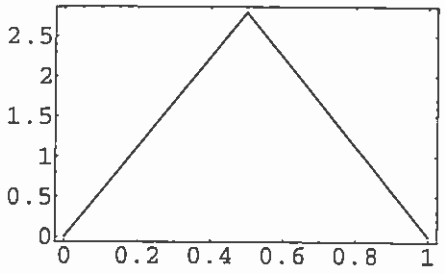


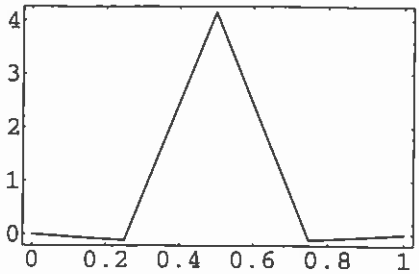
Ratkaisut:

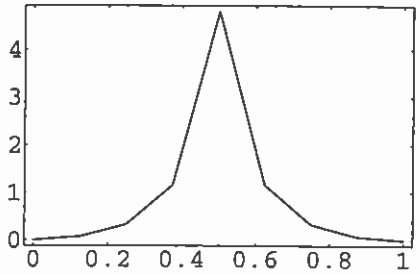
$S = 0.01$

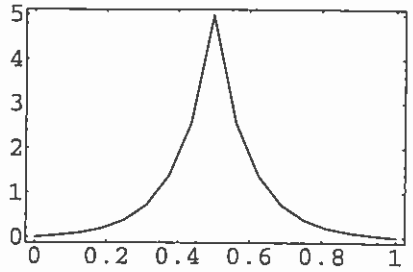
$T = 0$

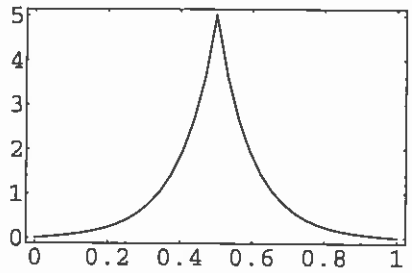
$K = F = L = 1$

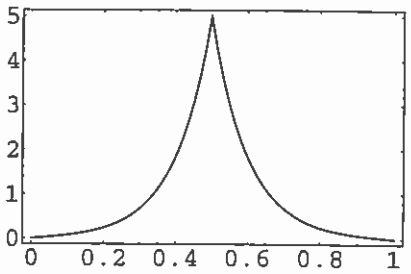










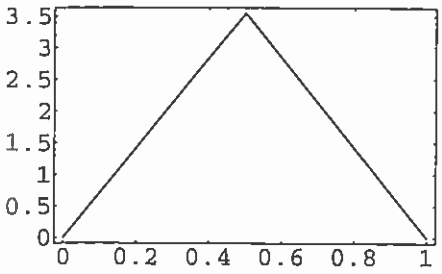


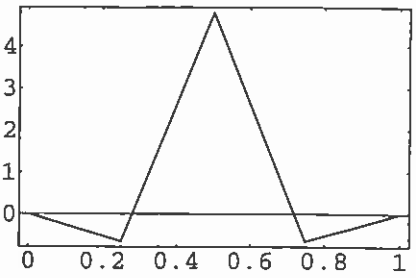
Ratkaisut :

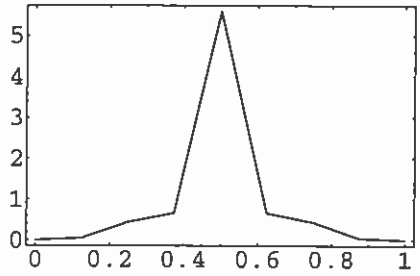
$S = 0.01$

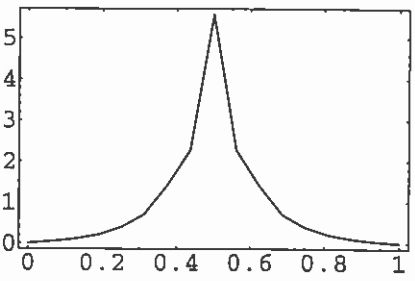
$T = T_{opt}/10$

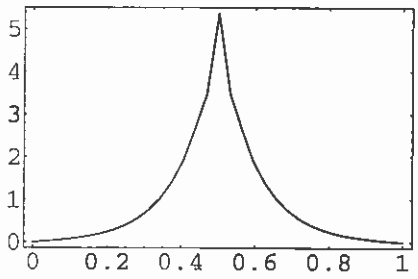
$K = F = L = 1$

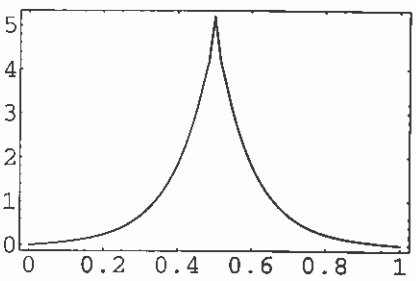












Rattkonstut: $S = 0.01$

$T = T_{opt}$

$K = F = L = 1$

Mat-5.160 Variational principles of mechanics, exercise 4

1. Derive the weak formulation for the diffusion-convection-reaction equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}(k\frac{\mathrm{d}T}{\mathrm{d}x})+u\frac{\mathrm{d}T}{\mathrm{d}x}+cT-s=0 \quad x\in]0,L[,$$

where the diffusion coefficient k, velocity u and the sink factor c are independent of the unknown function T and the boundary condition is $T-\overline{T}=0 \;\; x\in\{0,L\}$ ($\overline{T}$ is the given value on the boundary).

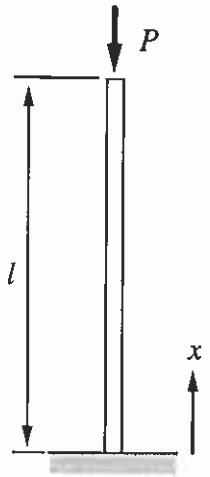
2. Let us consider the displacement field $\mathbf{u}$ of an elastic body Ω in the case where the displacement is known on the boundary Γ_u, $\overline{\Gamma}_u\cup\overline{\Gamma}_t=\Gamma$, $\Gamma_u\cap\Gamma_t=\varnothing$. The potential energy functional is

$$V(\mathbf{u})=\frac{1}{2}\int_\Omega C_{ijkl}u_{i,j}u_{k,l}\mathrm{d}\Omega\,,$$

in which the quantities C_{ijkl} are independent of $\mathbf{u}$. Write down the boundary value problem whose solution makes the functional stationary.

3. Write down the functional for the problem of finding the domain $\Omega\subset\mathbb{R}^2$ with minimum boundary length when the area is given. Also, derive the corresponding boundary value problem and verify that a circular boundary curve is the solution.

4. The potential energy for an axially inextensible column subjected to a compressing force P is



$$V(v)=\int_0^l[\frac{EI}{2}(v'')^2]\mathrm{d}x-P\int_0^l[\frac{1}{2}(v')^2\mathrm{d}x.$$

Find the minimum value for P causing the column to buckle.

5. What is the underlying boundary value problem of the weak formulation: find $\phi-\overline{\phi}\in V$, such that

$$\int_\Omega(D_{\alpha\beta}\frac{\partial w}{\partial x_\alpha}\frac{\partial\phi}{\partial x_\beta}-wf)\mathrm{d}\Omega+\int_{\Gamma_N}wh\mathrm{d}\Gamma=0 \quad \forall w\in V\,,$$

where $V=\{v: v\in C^0(\Omega), v|\Gamma_D=0\}$ and $\Gamma=\overline{\Gamma}_D\cup\overline{\Gamma}_N$, $\Gamma_D\cap\Gamma_N=\varnothing$?

TEKNILLINEN KORKEAKOULU
Mekaniikan laboratorio
http://www.hut.fi/Units/Dynamics/
Kevät 1999

Mat-5.160 MEKANIIKAN VARIAATIOPERIAATTEITA (4 ov)
39+26 (3+2) kl

Luennoitsija: Prof. Eero-Matti Salonen, huone Y334, puh. 451 3069,
e-mail eero-matti.salonen@hut.fi, vastaanotto ti 13.05 – 13.55

Luentoaika: ti 10.15 – 11.30, 12.00 – 13.00 sali N, ensimmäinen luento ti 19.1.

Kirjallisuus: Luentomoniste

Harjoitukset: Assistentti Jouni Freund, puh. 4747505,
e-mail jouni.freund@occuphealth.fi, vastaanotto huone Y334, ti 17.45 – 18.30,
puh. 451 3069.

Aika: ti 16.15 – 17.45, Y307, ensimmäinen harjoitus on ti 26.1.

Harjoitustehtävät jaetaan ennen harjoituksia luennolla ja niihin on syytä tutustua etukäteen. Pakollisessa harjoitustyössä käytetään perustana symbolisen ja numeerisen laskennan ohjelmaa (Mathematica).

Kotitehtäviä on 10 ja ne arvostellaan kukin asteikolla 0-1-2. Saatu summeerattu tulos (0-20) muunnetaan kertoimella 0.6 lopputulokseksi (0-12), joka hyvitetään tentin arvostelussa alkupisteinä, jotka lisätään itse tentin (0-30) pisteisiin.

Ilmoittautuminen: Topin kautta

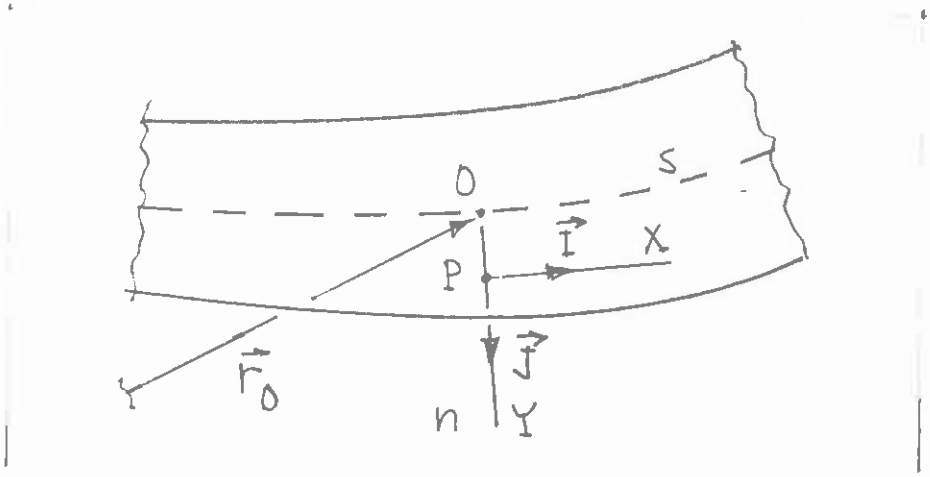
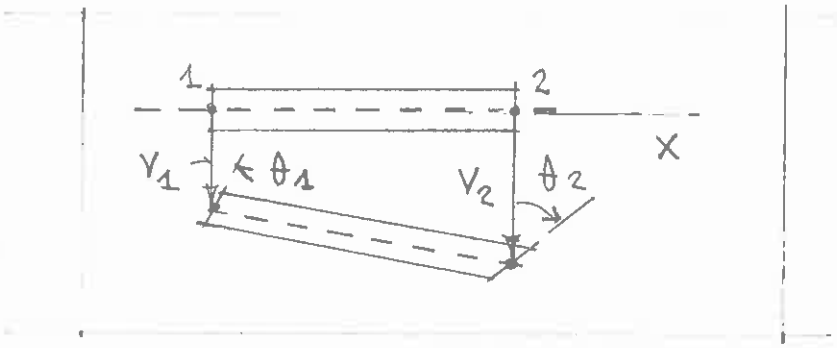
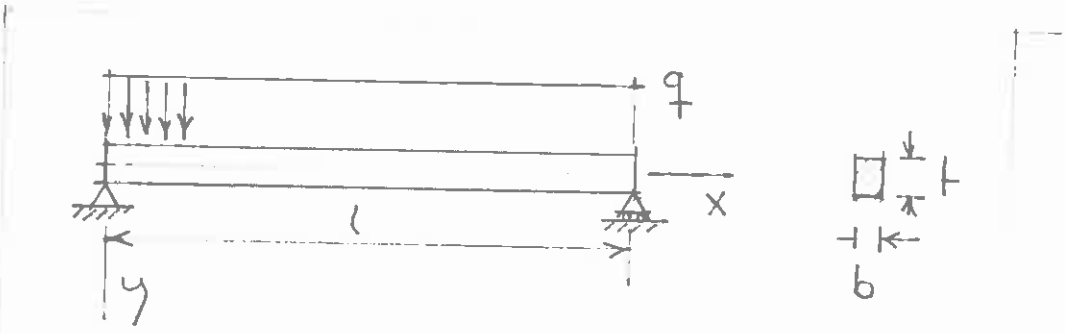


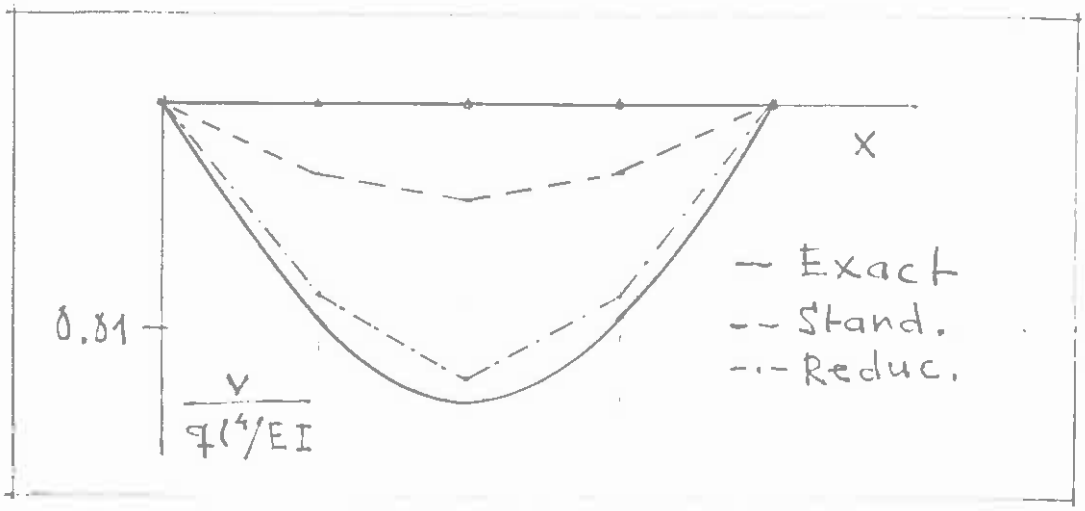
Figure 13.1



Example 13.1 , Figure (a)



Example 13,1 , Figure (b) , 85%



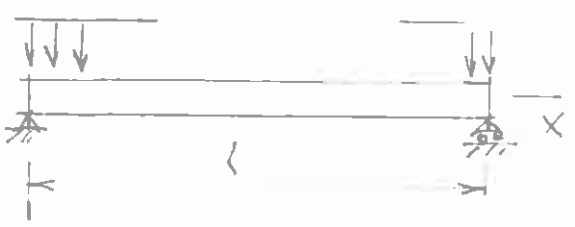
Example 13.1, Figure (c) 90%

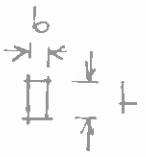
Example 14.1

$$C\begin{bmatrix} 0 & -kGA/h \\ kGA/h & -kGAN_1 \\ 0 & kGA/h \\ -kGA/h & -kGAN_2 \end{bmatrix}\begin{bmatrix} EImN_1/kGA & 0 & EImN_2/kGA & 0 \\ 0 & I_\rho N_1 & 0 & I_\rho N_2 \end{bmatrix}$$

$$= C\begin{bmatrix} 0 & -kGAI_\rho N_1/h & 0 & -kGAI_\rho N_2/h \\ EImN_1/h & -kGAI_\rho N_1N_1 & EImN_2/h & -kGAI_\rho N_1N_2 \\ 0 & kGAI_\rho N_1/h & 0 & kGAI_\rho N_2/h \\ -EImN_1/h & -kGAI_\rho N_2N_1 & -EImN_2/h & -kGAI_\rho N_2N_2 \end{bmatrix}$$

Example 13.1





$\nu = 1/3 \quad t = \frac{1}{10} l$

$$EI = E\frac{bt^3}{12} = \frac{1}{12} E b t^3 \,, \qquad kGA = \frac{5}{6}\frac{E}{2(1+\nu)} bt = \frac{5}{6}\frac{E}{2\cdot\frac{4}{3}} bt = \frac{5}{16} E b t$$

$$\hat{\varepsilon} = \frac{EI}{kGAl^2} = \frac{16\, Ebt^3}{12{\cdot}5\, Ebt\cdot l^2} = \frac{4}{15}\left(\frac{t}{l}\right)^2 = \frac{4}{1500} = \frac{1}{375}$$

Four elements, $h = l/4$

$$\hat{\varepsilon}_h = \frac{EI}{kGAh^2} = \frac{16}{375}$$

the exact solution on page 124 in Dym & Shames is

$$v = \frac{q_0}{EI}\left[\frac{x^4}{24} - \frac{l^2x^2}{16} + \frac{5l^4}{384}\right] + \frac{q_0}{2kGA}\left[\frac{l^2}{4} - x^2\right]$$

with x measured from the center of the beam. We thus put $x := x - l/2$ and obtain

$$v = \frac{q_0 l^4}{EI}\left[\frac{(x-l/2)^4}{24l^4} - \frac{(x-l/2)^2}{16l^2} + \frac{5}{384}\right] +$$

$$+\frac{q_0 l^2}{kGA}\left[\frac{1}{8} - \frac{(x-l/2)^2}{2l^2}\right],$$

We put $\xi = x/l/2$:

$$v = \frac{q_0 l^4}{EI}\left[\frac{(\xi-1)^4}{16\cdot 24} - \frac{(\xi-1)^2}{4\cdot 16} + \frac{5}{384}\right] +$$

$$+ \frac{q_0 l^2}{kGA}\left[\frac{1}{8} - \frac{(\xi-1)^2}{4\cdot 2}\right]$$

$$= \frac{q_0 l^4}{EI}\cdot\frac{1}{384}\cdot\left[5 - 6(\xi-1)^2 + (\xi-1)^4\right] +$$

$$+ \frac{q_0 l^4}{EI}\cdot\frac{EI}{kGAl^2}\cdot\frac{1}{8}\left[1-(\xi-1)^2\right]$$

$$= \frac{q_0 l^4}{EI}\cdot\left\{\frac{1}{384}\left[5 - 6(1-\xi)^2 + (1-\xi)^4\right]\right.$$

$$\left. + \frac{1}{8}\hat{\varepsilon}\left[1-(1-\xi)^2\right]\right\}$$

For the case $\hat{\varepsilon} = 1/375$, we obtain

$$v(0) = 0$$

$$v\left(\frac{1}{2}\right) = \frac{q_0 l^4}{EI}\left\{\frac{5}{384} + \frac{1}{8\cdot 375}\right\}$$

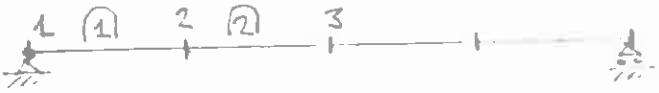
$$= \frac{q_0 l^4}{EI}\left(\frac{5}{384} + \frac{1}{3000}\right)$$

$$= \frac{q_0 l^4}{EI}\, 0{,}013'354'17$$

$$v(1) = \frac{q_0 l^4}{EI}\left\{\frac{1}{384}[5] + \frac{1}{8}\frac{1}{375}[1]\right\}$$

$$= \frac{q_0 l^4}{EI}\left\{\frac{5}{384} + \frac{1}{3000}\right\} = \frac{q_0 l^4}{EI} \cdot 0{,}013'354'17$$

Finite element solution.



Symmetry is made use of ($v_1 = 0$, $\theta_3 = 0$)

$$[K]_{6\times 6} = \begin{bmatrix} [K]_{11}\ [K]_{12} & 0 \\ [K]_{21}\ [K]_{22}\ [K]_{23} \\ 0 \quad [K]_{32}\ [K]_{33} \end{bmatrix}$$

$$\{b\}_{6\times 1} = \begin{Bmatrix} \{b\}_1 \\ \{b\}_2 \\ \{b\}_3 \end{Bmatrix}$$

$$[K]_{11} = [K]^{(1)}_{11} \qquad \{b\}_1 = \{b\}^{(1)}_1$$

$$[K]_{12} = [K]^{(1)}_{12} \qquad \{b\}_2 = \{b\}^{(1)}_2 + \{b\}^{(2)}_1$$

$$[K]_{21} = [K]^{(1)}_{21} \qquad \{b\}_3 = \{b\}^{(2)}_2$$

$$[K]_{22} = [K]^{(1)}_{22} + [K]^{(2)}_{11}$$

$$[K]_{23} = [K]^{(2)}_{12}$$

$$[K]_{32} = [K]^{(2)}_{21}$$

$$[K]_{33} = [K]^{(2)}_{22}$$

From Example 13.1 (1 red. int.

$$[K]_{11} = \frac{EI}{h}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + kGAh \begin{bmatrix} 1/h^2 & 1/2h \\ 1/2h & 1/3 \\ & (1/4) \end{bmatrix}$$

$$[K]_{12} = \frac{EI}{h}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + kGAh \begin{bmatrix} -1/h^2 & 1/2h \\ -1/2h & 1/6 \\ & (1/4) \end{bmatrix}$$

$$[K]_{21} = \frac{EI}{h}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + kGAh \begin{bmatrix} -1/h^2 & -1/2h \\ 1/2h & 1/6 \\ & (1/4) \end{bmatrix}$$

$$[K]_{22} = \frac{EI}{h}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} + kGAh \begin{bmatrix} 2/h^2 & 0 \\ 0 & 2/3 \\ & (1/2) \end{bmatrix}$$

$$[K]_{23} = \frac{EI}{h}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + kGAh \begin{bmatrix} -1/h^2 & 1/2h \\ -1/2h & 1/6 \\ & (1/4) \end{bmatrix}$$

$$[K]_{32} = \frac{EI}{h}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + kGAh \begin{bmatrix} -1/h^2 & -1/2h \\ 1/2h & 1/6 \\ & (1/4) \end{bmatrix}$$

$$[K]_{33} = \frac{EI}{h}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + kGAh \begin{bmatrix} 1/h^2 & -1/2h \\ -1/2h & 1/3 \\ & (1/4) \end{bmatrix}$$

$$\{b\}_1 = hq\begin{Bmatrix} 1/2 \\ 0 \end{Bmatrix}$$

$$\{b\}_2 = hq\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\{b\}_3 = hq\begin{Bmatrix} 1/2 \\ 0 \end{Bmatrix}$$

We denote

$$v' = \frac{v}{h} \quad 1$$

and multiply the "first" equations by h :

$$[k']_{11} = \frac{\pm\pm}{h}\left(\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + \frac{1}{\hat{\varepsilon}_h}\begin{bmatrix}1 & 1/2\\ 1/2 & \underset{(1/4)}{1/3}\end{bmatrix}\right)$$

$$[k']_{12} = \frac{\pm\pm}{h}\left(\begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix} + \frac{1}{\hat{\varepsilon}_h}\begin{bmatrix}-1 & 1/2\\ -1/2 & \underset{(1/4)}{1/6}\end{bmatrix}\right)$$

$$[k']_{21} = \frac{\pm\pm}{h}\left(\begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix} + \frac{1}{\hat{\varepsilon}_h}\begin{bmatrix}-1 & -1/2\\ 1/2 & \underset{(1/4)}{1/6}\end{bmatrix}\right)$$

$$[k']_{22} = \frac{\pm\pm}{h}\left(\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix} + \frac{1}{\hat{\varepsilon}_h}\begin{bmatrix}2 & 0\\ 0 & \underset{(1/2)}{2/3}\end{bmatrix}\right)$$

$$[k']_{23} = \frac{\pm\pm}{h}\left(\begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix} + \frac{1}{\hat{\varepsilon}_h}\begin{bmatrix}-1 & 1/2\\ -1/2 & \underset{(1/4)}{1/6}\end{bmatrix}\right)$$

$$[k']_{32} = \frac{\pm\pm}{h}\left(\begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix} + \frac{1}{\hat{\varepsilon}_h}\begin{bmatrix}-1 & -1/2\\ 1/2 & \underset{(1/4)}{1/6}\end{bmatrix}\right)$$

$$[k']_{33} = \frac{\pm\pm}{h}\left(\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + \frac{1}{\hat{\varepsilon}_h}\begin{bmatrix}1 & -1/2\\ -1/2 & \underset{(1/4)}{1/3}\end{bmatrix}\right)$$

$$\{b'\}_1 = h^2 q \mp \begin{Bmatrix}1/2\\ 0\end{Bmatrix}$$

$$\{b'\}_2 = h^2 q \mp \begin{Bmatrix}1\\ 0\end{Bmatrix}$$

$$\{b'\}_3 = h^2 q \mp \begin{Bmatrix}1/2\\ 0\end{Bmatrix}$$

In the following $\hat{\varepsilon}_h = \frac{16}{375}$

$$[k']_{11} = \frac{Et}{h} \cdot \frac{1}{96} \begin{bmatrix} 2250 & 1125 \\ 1125 & 846 \\ & (658{,}5) \end{bmatrix}$$

$$[k']_{12} = \frac{Et}{h} \cdot \frac{1}{96} \begin{bmatrix} -2250 & 1125 \\ -1125 & 279 \\ & (466{,}5) \end{bmatrix}$$

$$[k']_{21} = \frac{Et}{h} \cdot \frac{1}{96} \begin{bmatrix} -2250 & -1125 \\ 1125 & 279 \\ & (466{,}5) \end{bmatrix}$$

$$[k']_{22} = \frac{Et}{h} \cdot \frac{1}{96} \begin{bmatrix} 4500 & 0 \\ 0 & 1092 \\ & (1317) \end{bmatrix}$$

$$[k']_{23} = \frac{Et}{h} \cdot \frac{1}{96} \begin{bmatrix} -2250 & 1125 \\ -1125 & 279 \\ & (466{,}5) \end{bmatrix}$$

$$[k']_{32} = \frac{Et}{h} \cdot \frac{1}{96} \begin{bmatrix} -2250 & -1125 \\ 1125 & 279 \\ & (466{,}5) \end{bmatrix}$$

$$[k']_{33} = \frac{Et}{h} \cdot \frac{1}{96} \begin{bmatrix} 2250 & -1125 \\ -1125 & 846 \\ & (658{,}5) \end{bmatrix}$$

$$\{b'\}_1 = h^2 q \cdot \frac{1}{2} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\{b'\}_2 = h^2 q \cdot \frac{1}{2} \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$$

$$\{b'\}_3 = h^2 q \cdot \frac{1}{2} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

The system:

$$\begin{bmatrix} 2250 & 1125 & -2250 & 1125 & 0 & 0 \\ 1125 & \underset{(658,5)}{846} & -1125 & \underset{(466,5)}{279} & 0 & 0 \\ -2250 & -1125 & 4500 & 0 & -2250 & 1125 \\ 1125 & \underset{(466,5)}{279} & 0 & \underset{(1317)}{1692} & -1125 & \underset{(466,5)}{279} \\ 0 & 0 & -2250 & -1125 & 2250 & -1125 \\ 0 & 0 & 1125 & \underset{(466,5)}{279} & -1125 & \underset{(658,5)}{846} \end{bmatrix} \begin{Bmatrix} v_1' \\ \theta_1 \\ v_2' \\ \theta_2 \\ v_3' \\ \theta_3 \end{Bmatrix} = \frac{qh^3}{EI} 48 \begin{Bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$\theta_1 = \frac{160}{189} \frac{qh^3}{EI}$$

$$v_2' = \frac{18512}{23625} \frac{qh^3}{EI}, \quad v_2 = \frac{18512}{23625} \frac{qh^4}{EI} = 0,003'060'85 \frac{ql^4}{EI}$$

$$\theta_2 = \frac{16}{27} \frac{qh^3}{EI}$$

$$v_3' = \frac{8672}{7875} \frac{qh^3}{EI}, \quad v_3 = \frac{8672}{7875} \frac{qh^4}{EI} = 0,004'301'59 \frac{ql^4}{EI}$$

System with reduced integration

$$\theta_1 = 2,5 \frac{qh^3}{EI}$$

$$v_2' = 2,189 \frac{qh^3}{EI}, \quad v_2 = 2,189 \frac{qh^4}{EI} = 0,008'550'78 \frac{ql^4}{EI}$$

$$\theta_2 = 1,75 \frac{qh^3}{EI}$$

$$v_3' = 3,08533 \frac{qh^3}{EI}, \quad v_3 = 3,085'33 \frac{qh^4}{EI} = 0,012'052'07 \frac{ql^4}{EI}$$

Error

$$\frac{0,013'354 - 0,012'052'}{0,013'354} \cdot 100 = 9,7\%$$

The exact solution for θ in Kym and Shames is

$$\theta = \frac{1}{EI}\left(-\frac{q\ell^2}{8}x + \frac{q}{6}x^3\right)$$

Substitution $x : x - \ell/2$ gives

$$\theta = \frac{q\ell^3}{EI}\left(-\frac{1}{8}\frac{(x-\ell/2)}{\ell} + \frac{1}{6}\frac{(x-\ell/2)^3}{\ell^3}\right)$$

$$= \frac{q\ell^3}{EI}\left[\frac{1}{8}(1-\xi) - \frac{1}{6}\frac{(1-\xi)^3}{3}\right]$$

$$= \frac{q\ell^3}{EI}\frac{1}{48}\left[6(1-\xi) - (1-\xi)^3\right]$$

$$\theta(1) = \frac{q\ell^3}{EI}\frac{1}{48}\left[6-1\right] = \frac{5}{48}\frac{q\ell^3}{EI} = 0,104'17\,\frac{q\ell^3}{EI}$$

$$\theta(2) = \frac{q\ell^3}{EI}\frac{1}{48}\left[-6+1\right] = -\frac{5}{48}\frac{q\ell^3}{EI} =$$

One element solution

$h = \ell$

$$[K]_{4\times4} = \frac{EI}{\ell}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + kGA\ell\begin{bmatrix} 1/h^2 & 1/2h & -1/h^2 & 1/2h \\ 1/2h & 1/3 & -1/2h & 1/6 \\ -1/h^2 & -1/2h & 1/h^2 & -1/2h \\ 1/2h & 1/6 & -1/2h & 1/3 \end{bmatrix}$$

with $\hat{\varepsilon} = \frac{1}{375}$

$$[k]_{2\times 2} = \frac{EI}{l}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{\hat{\varepsilon}}\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}\right)$$

$$= \frac{EI}{l}\frac{1}{6}\left(\begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} + \begin{bmatrix} 750 & 375 \\ 375 & 750 \end{bmatrix}\right)$$

$$= \frac{EI}{l}\frac{1}{6}\begin{bmatrix} 756 & 369 \\ 369 & 756 \end{bmatrix}$$

In the standard solution loading term disappear.

In the sensitized solution

$$-\frac{12\hat{\varepsilon}}{12\hat{\varepsilon}+1} = \frac{1}{1+\frac{1}{12\hat{\varepsilon}}} = -\frac{1}{1+\frac{375}{12}} = -\frac{1}{\frac{12+375}{12}}$$

$$= \frac{12}{387}$$

$$-\frac{\hat{\varepsilon}}{12\hat{\varepsilon}+1} = -\frac{1}{12+\frac{1}{\hat{\varepsilon}}} = -\frac{1}{12+375} = -\frac{1}{387}$$

the loading term

$$\{b\}_s = -\frac{\hat{\varepsilon}}{12\hat{\varepsilon}+1} q h^2 \begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix} = -\frac{1}{387} q l^2 \begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix}$$

the system

$$\frac{EI}{\ell}\,\frac{1}{6}\cdot\frac{12^2}{387}\begin{bmatrix}756 & 369\\ 369 & 756\end{bmatrix}\begin{Bmatrix}\vartheta_1\\ \vartheta_2\end{Bmatrix}=\frac{1}{387}\,q\ell^3\begin{Bmatrix}1\\ -1\end{Bmatrix}$$

$$\begin{bmatrix}756 & 369\\ 369 & 756\end{bmatrix}\begin{Bmatrix}\vartheta_1\\ \vartheta_2\end{Bmatrix}=\frac{q\ell^3}{EI}\begin{Bmatrix}1/2\\ -1/2\end{Bmatrix}.$$

$$D = 435'375$$

$$D_1 = q\ell^3\cdot 562.5/EI$$

$$D_2 = q\ell^3\cdot(-362.5)/EI$$

$$\vartheta_1 = \frac{562.5}{435'375}\cdot\frac{q\ell^3}{EI} = 0,001'291'19\ldots\cdot\frac{q\ell^3}{EI}$$

Example 13.2

$$([E][D][B])^T[c]([E][D][B]) =$$

$$c \begin{bmatrix} 0 & -kGA/h \\ EI/h & -kGA(1-\xi) \\ 0 & kGA/h \\ -EI/h & -kGA\xi \end{bmatrix} \begin{bmatrix} 0 & kGA/h & 0 & -kGA/h \\ -kGA/h & -kGA(1-\xi) & kGA/h & -kGA\xi \end{bmatrix}$$

$$= c \begin{bmatrix} \frac{(kGA)^2}{h^2} & \frac{(kGA)^2}{h}(1-\xi) & \cdots \\ \frac{(kGA)^2}{h}(1-\xi) & \frac{EI\cdot kGA}{h^2} + (kGA)^2(1-\xi)^2 & \cdots \\ -\frac{(kGA)^2}{h^2} & -\frac{(kGA)^2}{h}(1-\xi) & \cdots \\ \frac{(kGA)^2}{h}\xi & -\frac{EI\cdot kGA}{h^2} + (kGA)^2\xi(1-\xi) & \cdots \end{bmatrix}$$

$$\begin{bmatrix} -\frac{(kGA)^2}{h^2} & \frac{(kGA)^2}{h}\xi \\ -\frac{(kGA)^2}{h}(1-\xi) & -\frac{EI\cdot kGA}{h^2} + (kGA)^2\xi(1-\xi) \\ \frac{(kGA)^2}{h^2} & -\frac{(kGA)^2}{h}\xi \\ -\frac{(kGA)}{h}\xi & \frac{EI\cdot kGA}{h^2} + (kGA)^2\xi^2 \end{bmatrix}$$

Integrating, we obtain

$$[K]_s = \int_{\Omega^e} ([E][D][B])^T[c]([E][D][B])\, d\Omega$$

$$= c(kGA)^2 \begin{bmatrix} 1/h & 1/2 & -1/h & 1/2 \\ 1/2 & \frac{EI}{kGAh} + \frac{h}{3} & -1/2 & -\frac{EI}{kGAh} + \frac{h}{6} \\ -1/h & -1/2 & 1/h & -1/2 \\ 1/2 & -\frac{EI}{kGAh} + \frac{h}{6} & -1/2 & \frac{EI}{kGAh} + \frac{h}{3} \end{bmatrix}$$

$$c(kGA)^2 = \frac{(kGA)^2}{12\frac{EI}{h^2} + kGA} = \frac{kGA}{12\frac{EI}{kGAh^2} + 1}$$

$$= \frac{kGA}{12\hat{\varepsilon}_h + 1}.$$

this means that the original stiffness matrices are multiplied instead of EI/h and $kGAh$ with

$$\frac{EI}{h} - \frac{1}{12\hat{\varepsilon}_h+1}\cdot\frac{EI}{h} = \frac{12\hat{\varepsilon}_h}{12\hat{\varepsilon}_h+1}\cdot\frac{EI}{h}$$

$$= \frac{1}{1+1/(12\hat{\varepsilon}_h)}\cdot\frac{EI}{h}$$

$$kGAh - \frac{1}{12\hat{\varepsilon}_h+1}\,kGAh = \frac{12\hat{\varepsilon}_h}{12\hat{\varepsilon}_h+1}\,kGAh$$

$$= \frac{1}{1+1/(12\hat{\varepsilon}_h)}\,kGAh$$

In the example case $\hat{\varepsilon}_h = \frac{16}{375}$

$$\frac{1}{1+\frac{1}{12\hat{\varepsilon}_h}} = \frac{1}{1+\frac{375}{12\cdot 16}} = \frac{1}{\frac{192+375}{192}}$$

$$= \frac{192}{567} \approx 0{,}338$$

the extra loading term from (11) is

$$\{b\}_s = \frac{\hat{\varepsilon}_h}{12\hat{\varepsilon}_h+1}\,qh^2\begin{Bmatrix}0\\1\\0\\-1\end{Bmatrix}$$

This produces

Here

$$\frac{\hat{\varepsilon}_h}{12\hat{\varepsilon}_h + 1} = \frac{1}{12 + \frac{1}{\hat{\varepsilon}_h}} = \frac{1}{12 + \frac{375}{16}} = \frac{1}{\frac{192+375}{16}}$$

$$= \frac{16}{567}$$

$$\{b_s'\}_1 = \frac{16}{567} q h^2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\{b_s'\}_2 = \frac{16}{567} q h^2 \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\{b_s'\}_3 = \frac{16}{567} q h^2 \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$$

We obtain

$\Phi_1 = 28$

$v_2' =$

$\Phi_2' =$

$$v_3' = \frac{168171}{141750} \frac{q h^3}{EI} \quad | \quad v_3 = \frac{168171}{141750} \frac{q h^4}{EI}$$

$$= 0{,}012'123'58 \frac{qh^4}{EI}$$

Error

$$\frac{0{,}013'354 - 0{,}012'124}{0{,}013'354} \cdot 100 = 3{,}2\,\%$$