CONCRETE STRUCTURES, ADVANCED COURSE Rak-43.3110

- Concrete-concrete composite structures and prestressed concrete structures
- Lectures, 10 times, on Mondays 14 -16 (Dos. Matti Pajari, matti.pajari@vtt.fi, p. 050-5821 736)
- Exercises 10 times on Tuesdays 12-14 (DI Pekka Häyrinen)
- Target: to understand the basics
- A collection of formulae delivered in examination, no own literature allowed
- Eurocode 2 to some extent
- See also Lin & Burns, SI version

Analysis of homogenous beam with elementary beam theory

Normal force N and bending moment M are acting at the centroidal axis, see Fig. 2.

Stress σ and strain ε are obtained from:

$$\sigma = \frac{N}{A} + \frac{M}{I} y \qquad \varepsilon = \frac{\sigma}{E} = \frac{N}{EA} + \frac{M}{EI} y$$

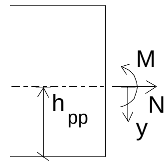


Fig. 2. Notation.

Here

A is the cross-sectional area,
I the 2nd moment of area with respect to the centroidal axis
y vertical coordinate, origin at centr. axis, positive downwards,
E elasticity modulus.

The deflection f is of the form

$$f = \frac{f(q,L,\ldots)}{EI}$$

Function f depends on loading, geometry etc.

Analysis of composite beam with elementary beam theory

Consider the cross-section shown in Fig. 3 and comprising n components.

A_k ja E_k area and elasticity modulus on part k (k = 1,…,n)
h distance to bottom fibre of section
$h_{pp,k}$ centroidal (pp) distance of part k to bottom fibre of section
y_k vertical coordinate, origin at centroid of part k , posit. downwards
I_k^x 2nd moment of area of part k around arbitrary horizontal axis x
I_k 2nd moment of area of part k around its own centroidal axis

It is well-known that:

$$A_k = \int_k dA \qquad h_{pp,k} = \frac{\int_k h dA}{A_k}$$

$$I_k^x = I_k + A_k (h_x - h_{pp,k})^2$$

(Steiner's rule)

Remember that centroidal axis and neutral axis have a different meaning

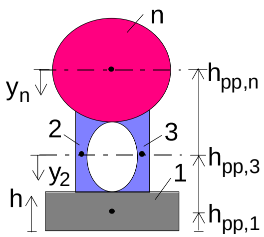


Fig. 3. Notation.

Definitions: $(EA) = \sum_k E_k A_k$ Axial stiffness of composite section

$$h_{pp} = \frac{\sum_k E_k A_k h_{pp,k}}{(EA)}$$

Distance of centroidal axis of composite section to the bottom fibre of section

I_k^{pp} 2nd moment of area of part k around centroidal axis of composite section

$$(EI) = \sum_k E_k I_k^{pp} = \sum_k \left[E_k I_k + E_k A_k (h_{pp} - h_{pp,k})^2 \right]$$

(Steiner's rule)

Bending stiffness of composite section

Axial stiffness is the cross-sectional area weighted by elasticity moduli. h_{pp} gives the position of the centroid of the weighted area.

If $E_k = E$ for all k, i.e. the whole section is homogenous,

$(EA) = E \cdot A$ and $(EI) = E \cdot I$

In other cases the symbols (EA) ja (EI) must not be interpreted as a product of E and A or I. The physical meaning of (EA), h_{pp} and (EI) becomes clear on the next few pages.

A formal example:

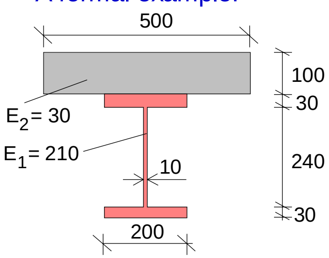


Fig. 4.

A_1 = 200x300 - 240x190 = **14400**

A_2 = 500x100 = **50000**

$h_{pp,1}$ = **150** $h_{pp,2}$ = **350**

I_1 = 200x300³/12 – 190x240³/12 = **2,3112x10⁸**

I_2 = 500x100³/12 = **4,167x10⁷**

(EA) = 210x14400 + 30x50000 = 3,024x10⁶ + 1,50x10⁶ = **4,524x10⁶**

h_{pp} = (3,024x10⁶x150 + 1,50x10⁶x350)/(4,524x10⁶) = **216,3**

$$(EI) = E_1\left[I_1 + A_1(h_{pp} - h_{pp,1})^2\right] + E_2\left[I_2 + A_2(h_{pp} - h_{pp,2})^2\right]$$

= 210x[2,311x10⁸+14400x(216,3 - 150)²] +
30x[4,167x10⁷+50000x(216,3 - 350)²]

= **8,99x10¹⁰**

In Fig. 5 normal force N and moment M act at the centroidal axis

The horizontal strain ε can be expressed using curvature κ and strain ε_0 at the centroidal axis in the form

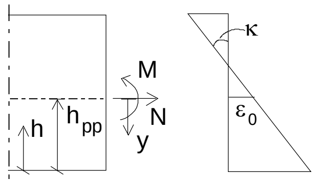


Fig. 5. Notation.

$$\varepsilon = \varepsilon_0 + \kappa y \qquad \sigma_k = E_k(\varepsilon_0 + \kappa y)$$

Then

$$\underline{N} = \sum_k \int_{A_k} E_k(\varepsilon_0 + \kappa y)dA = \varepsilon_0 \sum_k E_k \int_{A_k} dA + \cancelto{0^{*)}}{\kappa \sum_k E_k \int_{A_k} y dA} = \underline{\varepsilon_0(EA)}$$

$$\underline{M} = \sum_k \int_{A_k} E_k y(\varepsilon_0 + \kappa y)dA = \cancelto{0^{*)}}{\varepsilon_0 \sum_k E_k \int_{A_k} y dA} + \kappa \sum_k E_k \int_{A_k} y^2 dA = \underline{\kappa(EI)}$$

$$^{*)}\ \sum_k E_k \int_{A_k} y dA = \sum_k E_k \int_{A_k} (-h + h_{pp})dA = -\sum_k E_k \int_{A_k} h dA + h_{pp} \sum_k E_k \int_{A_k} dA$$

$$= -h_{pp}(EA) + h_{pp}(EA) = 0$$

$$\therefore\ \varepsilon_0 = \frac{N}{(EA)} \qquad \kappa = \frac{M}{(EI)}$$

$$\varepsilon = \frac{N}{(EA)} + \frac{M}{(EI)}y$$

Stress in material k:

$$\sigma_k = E_k \varepsilon = E_k\left[\frac{N}{(EA)} + \frac{M}{(EI)}y\right]$$

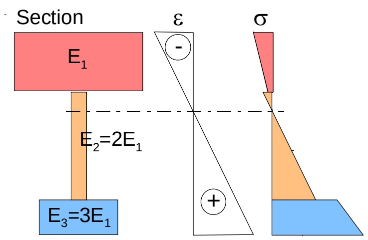


Fig. 6. Example of strain and stress distribution in composite section.

If the section is homogeneous:

$$\varepsilon = \frac{N}{EA} + \frac{M}{EI}y \qquad \sigma = E\varepsilon = \frac{N}{A} + \frac{M}{I}y$$

If N = 0, then y = 0, ε = 0 ja σ = 0 at the centroidal axis.

In this case the centroidal axis and neutral axis (σ = 0) coincide.

Transformed cross-sectional characteristics

Assume that a cross-section comprises q different E-moduli. Transformed areas $A_{k,m}$ are defined using the ratios of elasticity moduli n_k:

$$n_k = \frac{E_k}{E_0} \qquad A_{k,m} = n_k A_k$$

E_0 may be any fixed E_k, k =1,…,q *(base material)*

Definitions:

$$A_m = \frac{(EA)}{E_0} = \frac{1}{E_0}\sum_k E_k A_k = \sum_k n_k A_k = \sum_k A_{k,m}$$ Transformed area

$$I_m = \frac{(EI)}{E_0} = \sum_k n_k I_k + \sum_k n_k A_k (h_{pp} - h_{pp,k})^2$$ Transformed 2nd moment of area

It follows that $E_0 A_m = (EA)$ and $E_0 I_m = (EI)$

$$h_{pp} = \frac{\sum_k E_k A_k h_{pp,k}}{(EA)} = \frac{\sum_k E_k A_k h_{pp,k}}{E_0 A_m} = \frac{\sum_k n_k A_k h_{pp,k}}{A_m} = \frac{\sum_k A_{k,m} h_{pp,k}}{A_m}$$ Position of centroid

$\therefore$ The strain in the base material:

$$\sigma_0 = E_0\varepsilon = \frac{N}{(EA)/E_0} + \frac{M}{(EI)/E_0}y = \frac{N}{A_m} + \frac{M}{I_m}y$$

and for all materials

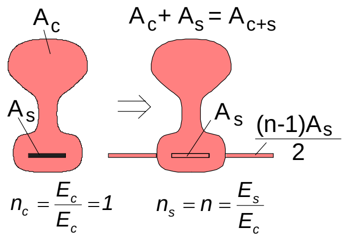
$$\sigma_k = n_k\left(\frac{N}{A_m} + \frac{M}{I_m}y\right)$$

Strains: $\varepsilon = \dfrac{\sigma_k}{E_k} = \dfrac{1}{E_k}\dfrac{E_k}{E_0}\left(\dfrac{N}{A_m} + \dfrac{M}{I_m}y\right) = \dfrac{1}{E_0}\left(\dfrac{N}{A_m} + \dfrac{M}{I_m}y\right)$

The transformed cross-sectional characteristics A_m and I_m are particularly widely used in the analysis of concrete stresses in prestressed concrete structures because, when the concrete is chosen for the base material, the calculations are similar to those used for homogenous cross-sections. The strain of the concrete and the stresses of the steel are less important in the service conditions.

Example. Reinforced concrete section . The 2nd moment of area of the rebars around their own centroidal axis is negligible. c+s refers to the zone occupied by either concrete or steel.

$$A_m = A_c + nA_s = A_c + A_s + (n-1)A_s = A_{c+s} + (n-1)A_s$$



$$h_{pp} = \frac{A_c h_{pc} + nA_s h_{ps}}{A_m} = \frac{A_c h_{pc} + A_s h_{ps} - A_s h_{ps} + nA_s h_{ps}}{A_m} = \frac{A_{c+s} h_{p,c+s} + (n-1)A_s h_{ps}}{A_m}$$

$$n_c = \frac{E_c}{E_c} = 1 \qquad n_s = n = \frac{E_s}{E_c}$$

Fig. 7. The steel section is made n_s times wider .

$$I_m = I_c + A_c(h_{pp} - h_{pc})^2 + nI_s + nA_s(h_{pp} - h_{ps})^2$$

Small

$$= I_c + A_c(h_{pp} - h_{pc})^2 + I_s + A_s(h_{pp} - h_{ps})^2 + \cancel{(n-1)I_s} + (n-1)A_s(h_{pp} - h_{ps})^2$$

$$\approx I_{c+s} + A_{c+s}(h_{pp} - h_{p,c+s})^2 + (n-1)A_s(h_{pp} - h_{ps})^2$$

N and *M* carried by different parts of composite section

A beam section is loaded by axial force N and moment M, see Fig. 8.

$$\varepsilon_0 = \frac{N}{(EA)}$$

$$\kappa = \frac{M}{(EI)}$$

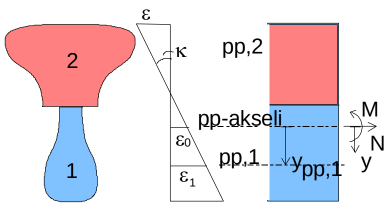


Fig. 8.

Calculate the resultants of the axial stresses of part k (k = 1,2) around their own centroidal axis using their own $(EA)_k$ and $(EI)_k$:

$$N_k = (EA)_k\varepsilon_k = (EA)_k(\varepsilon_0 + \kappa y_{pp,k})$$

$$M_k = \kappa(EI)_k$$

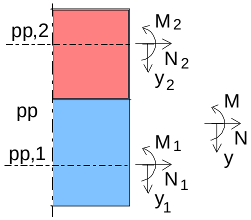


Fig. 9.

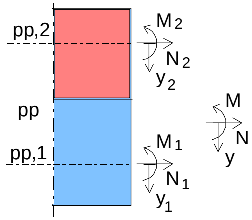
$$N = N_1 + N_2$$

$$M = M_1 + M_2 + N_1 y_{pp,1} + N_2 y_{pp,2}$$

If $N = 0$, then $N_1 = -N_2$ and

$$M = M_1 + M_2 + N_1(y_{pp,1} - y_{pp,2}) = M_1 + M_2 + N_1 e_{12}$$

Fig. 10.



The ratio

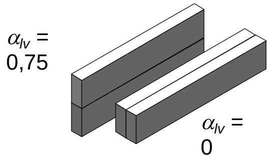
$$\alpha_{lv} = \frac{M - (M_1 + M_2)}{M} = 1 - \frac{\kappa(EI)_1 + \kappa(EI)_2}{\kappa(EI)} = 1 - \frac{(EI)_1 + (EI)_2}{(EI)}$$

is called composite stiffness coefficient. If it equals 0, there is no composite action and the bending stiffness of the section equals the sum on the stiffnesses of the parts.

Example:

- 2 identical adjacent beams, $\alpha_{lv} = 0$
- 2 identical superposed slabs,

$$\alpha_{lv} = 1 - \frac{(EI)_1 + (EI)_2}{(EI)} = 1 - \frac{h^3 + h^3}{(2h)^3} = 0{,}75$$



Kuva 11.

Calculation of deflections

Deflection f is solved from differential equation

$$f'' = -\kappa = -\frac{M}{(EI)} = -\frac{M}{E_0 I_m}$$

For a homogeneous beam

$$f'' = -\kappa = -\frac{M}{E \cdot I}$$

$\therefore$ All calculation rules developed for a homogeneous beam can be applied when the product $E \cdot I$ is replaced by (EI) or $E_0 I_m$.

E.g. Deflection due to uniformly distributed load q is $f = \frac{5}{384}\frac{qL^4}{(EI)}$

In the same way, the longitudinal deformation of a beam at the centroidal axis can be calculated using the rules developed for a homogeneous beam when the product $E \cdot A$ is replaced by axial stiffness (EA) or $E_0 A_m$.

Shear according to elementary beam theory

Consider the shear along the interface ABCD (horizontal or not) shown in Fig. 12. Indices j and k refer to the parts above and below the interface, respectively.

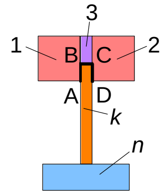


Fig. 12. Section.

The change of normal stress in distance Δx is

$$\Delta\sigma \approx \frac{d\sigma}{dx}\Delta x$$

A free body diagram for the part with length Δx below section ABCD is shown in Fig. 13. The equilibrium of horizontal forces gives for force T and shear flow $F_v = T/\Delta x$:

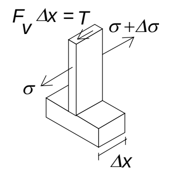


Fig. 13. Free body diagram below ABCD.

$$F_v \Delta x = T = \int_{A_{low}} \Delta\sigma dA = \sum_k \int_{A_k} \frac{d\sigma_k}{dx}\Delta x dA = \Delta x \sum_k \int_{A_k} \frac{d\sigma_k}{dx} dA$$

$$F_v = \frac{T}{\Delta x} = \sum_k \int_{A_k} \frac{d\sigma_k}{dx} dA$$

k is extended over all indices of parts below ABCD

$$\frac{d\sigma_k}{dx} = E_k\left[\frac{1}{(EA)}\frac{dN_k}{dx} + \frac{1}{(EI)}\frac{dM}{dx}y\right] = E_k\left[\frac{1}{(EA)}\frac{dN_k}{dx} + \frac{V}{(EI)}y\right]$$

When N is independent on x (this is not always the case) $\quad \frac{d\sigma_k}{dx} = E_k \frac{V}{(EI)}y$

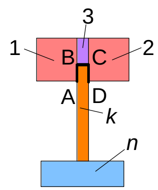


Fig. 12b.

$$F_v = \sum_k \int_{A_k} E_k \frac{V}{(EI)} y dA = \frac{V}{(EI)} \sum_k E_k \int_{A_k} y dA =$$

$$= \frac{V}{(EI)}(ES)_{low} = -\frac{V}{(EI)}(ES)_{up}$$

$$(ES)_{low} = \sum_k E_k \int_{A_k} y dA = \sum_k E_k A_k y_{pp,k}$$

$$(ES)_{up} = \sum_j E_j \int_{A_j} y dA = \sum_j E_j A_j y_{pp,j}$$

k and j are extended over all indices of parts below or above ABCD, respectively

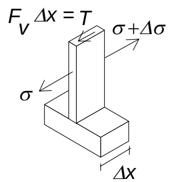


Fig. 13b.

In a horizontal interface, shear stress τ is obtained from

$$\tau = \frac{F_v}{b} = \frac{(ES)_{low}V}{(EI)b} = -\frac{(ES)_{up}V}{(EI)b}$$

Note: low $\leftrightarrow$ lower part
up $\leftrightarrow$ upper part

Part II. Prestressed structures

Prestressed structures common in Finland

- Hollow core slab
- Double tee slab (TT slab) and pitched double tee slab
- I-beam and and pitched I-beam
- Inverted T-beam
- Solid plank
- Railway sleeper
- Prestressed box girder (in bridges)
- Flat slab

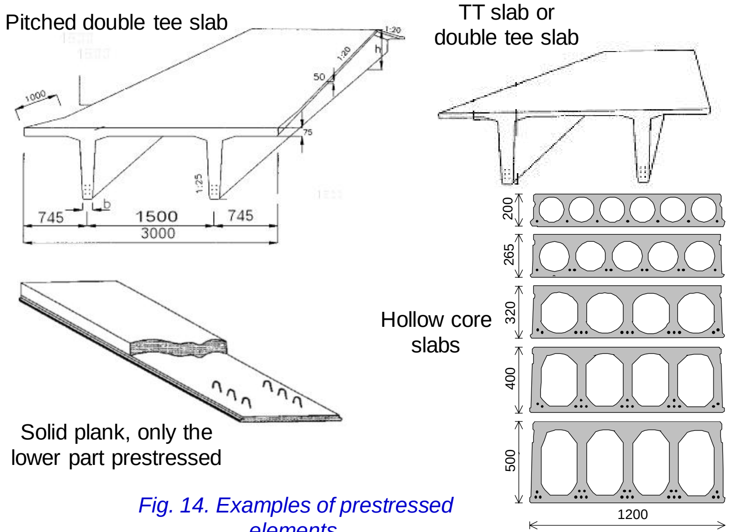


Fig. 14. Examples of prestressed elements.

Pretensioned tendons

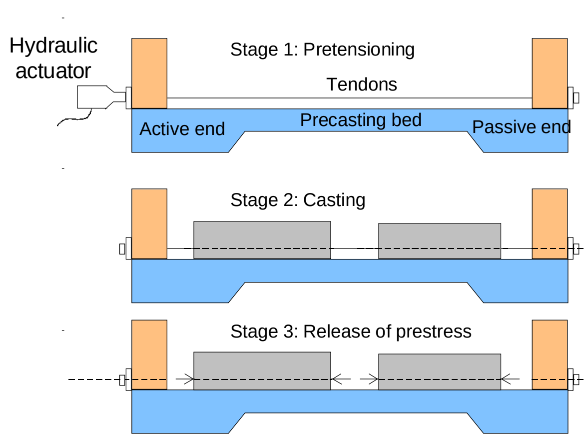


Fig. 22. According to Leskelä [BY210].

Post-tensioned tendons

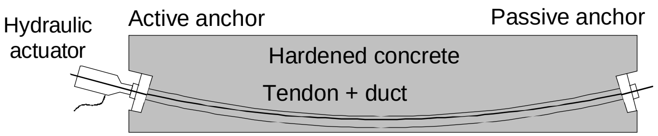


Fig. 23. According to Leskelä [BY210].

- Install the duct into the moulds with or without tendon and fix firmly
- Posttensioning after the concrete has hardened
- Place the tendon into the duct if it is not already there
- Fill the duct with grout (not for unbonded tendons)

Terminology

- Tendon: Wire, strand, bar or a bundle made of these which is tensioned simultaneously and which is treated as one component in calculations
- Pretensioning: Tensioning first, concrete is cast afterwards
- Post-tensioning : Concrete is cast first, tensioning afterwards against the hardened concrete
- Bonded tendon: There is bond between the tendon and the concrete, may be anchored tendon which is grouted or prestressed tendon without anchors
- Anchored tendon: The ends of the tendon are fixed to special anchors which transmit the tensioning force to the concrete
- Unbonded tendon: Anchored tendon which is not grouted
- Grouting: Filling the open space in a duct with pressurized, cementitious grout
- Partial prestressing: Both normal and prestressing steel are used, cracking allowed in serviceability limit state
- Active reinforcement: Prestressed steel
- Passive reinforcement: Non-prestressed steel

Why prestressing?

- To prevent cracking e.g. to improve
  - Durability (e.g. parking garages)
  - Water tightness (water reservoirs)
  - Air tightness (reactor buildings in nuclear power plants)
  - Control of deflections
  - Light weight
  - Aesthetics
- To prevent cracking during execution (e.g. piles)
- To reduce false work and moulds on site (solid planks, segmental bridges)
- To facilitate production (e.g. hollow core slabs)

Analysis of pretensioned beams

(The losses of prestress are first ignored)

Stage I: **The tendon** is pulled with force P. The stress of the steel is

$$\sigma_{p0} = \frac{P}{A_p} \quad \text{(initial prestress)} \qquad n = \frac{E_p}{E_c}$$

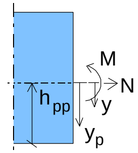


Fig. 24.

Stage II: (Hardened concrete): The tendon is released i.e. at the ends of the tendon, the counterforce of P is applied. This counterforce will load **the whole cross-section**. The stresses are are calculated from

$$\sigma_{c,P} = \frac{-P}{A_m} + \frac{-P y_P}{I_m} y \quad \text{(concrete)}$$

$$\sigma_{p,P} = \sigma_{p0} + n\left(\frac{-P}{A_m} + \frac{-P y_P}{I_m} y_p\right) \quad \text{(steel)}$$

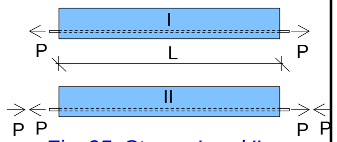


Fig. 25. Stages I and II.

A_p and E_p are the cross-sectional area and elasticity modulus of the tendon. E_c is the elasticity modulus of the concrete, A_m ja I_m are the transformed area and second moment of area of the cross-section, steel included.

When considering the stress state of a member, the release of the prestressing force P is equivalent to applying an outer normal force equal to P at the ends of the prestressed member.

Stage III: The stress of the concrete ($\sigma_{c,g+q}$) and steel ($\sigma_{p,g+q}$) due to the self weight g and outer load q are

$$\sigma_{c,g+q} = \frac{M_{g+q}}{I_m} y \qquad \sigma_{p,g+q} = n \frac{M_{g+q}}{I_m} y_p$$

M_{g+q} is the sum of M_g and M_q. M_g and M_q are the bending moments due to g and q, respectively. The stresses of stage III are superimposed to the previous stresses.

Above, the stresses due to different actions have been superimposed. The principle of superposition is also applied when there are several tendons. The effect of all tendons is the sum of the effects of each individual tendon.

Several tendons: P_k, $A_{p,k}$, $E_{p,k}$, $y_{p,k}$, $n_{p,k}$ as above, but refer to tendon k. Notation:

$$\sigma_{p0,k} = \frac{P_k}{A_{p,k}} \qquad n_k = \frac{E_{p,k}}{E_c}$$

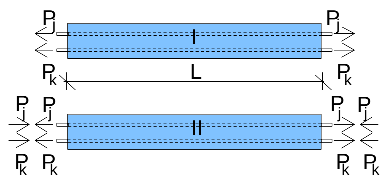


Fig. 26.

$$\sigma_{c,P} = \frac{-\sum_j P_j}{A_m} + \frac{-\sum_j P_j y_{P,j}}{I_m} y \quad \text{(concrete)}$$

$$\sigma_{p,k,P} = \sigma_{p0,k} + n_k\left(\frac{-\sum_j P_j}{A_m} + \frac{-\sum_j P_j y_{p,j}}{I_m} y_{p,k}\right) \quad \text{(steel)}$$

Comments

1. The losses of prestress are taken into account by reducing the prestressing force in the expressions above by a reduction factor β which corresponds to the losses (creep, shrinkage, relaxation). In other words, P is replaced by βP, $\beta < 1,0$.

2. P or P_k are the prestressing forces **before** the release. During the release, the stress in the tendon is reduced due to an elastic strain, but this is neither a loss nor is it allowed to reduce the prestressing force in the expressions above. Instead, the expressions can be used to solve the elastic strain due to the release.

3. In the British and American textbooks and practice, the gross cross-sections are often used istead of transforced sections. The gross-section is obtained by replacing all steel with concrete, see Fig. 27. In ordinary members the error due to this approximation is small (the inner forces are not in perfect equilibrium) and usually on the safe side.

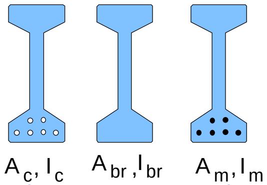


Fig. 27. Net, gross and transformed section.

4. The gross values A_{br} and I_{br} are suitable to the preliminary design in which the exact amount and location of the tendons is not known.

5. The magnitude of the prestress does not affect the stiffness.

6. The expressions above are suitable to calculation of stresses and cracking moment.

7. The expressions above are suitable to straight tendons only and statically determined structures.

8. The prestressing force at the end of the beam is equal to 0 and increases to the full prestressing force within a certain length called transfer length which depends on the bond properties of the tendon, concrete and vertical position of the tendon. This is taken into account when calculating the stresses next to the beam end. To prevent excessive cracking of the beam end, it is sometimes necessary to debond some tendons at their ends. When calculating the deflections and the stresses outside the transfer zone, full prestress is assumed for the whole length of the tendon.

Deflections

The deflections in the SLS (serviceability limit state) are calculated as for a composite beam loaded by an eccentric, compressive force P. The second derivative of deflection w is

$$w'' = -\kappa = -\frac{M}{(EI)} = -\frac{M}{E_0 I_m}$$

In particular, the second derivative of w_p due to P is

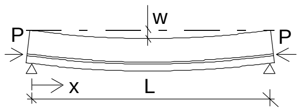
$$w_P'' = -\kappa = -\frac{M}{E_0 I_m} = -\frac{-Py_p}{E_0 I_m} = \frac{Py_p}{E_0 I_m} \Rightarrow w_P = \frac{1}{2}\frac{Py_p}{E_0 I_m}x^2 + Cx + D$$

If (EI) is constant

E.g., for the beam in Fig. 20

$w_p(0) = w_p(L) = 0 \Rightarrow D = 0$ and

$$C = -\frac{1}{2}\frac{Py_p}{E_0 I_m}L \Rightarrow w_P = \frac{1}{2}\frac{Py_p}{E_0 I_m}(x^2 - Lx)$$



In the mid-span: $w_P = -\frac{1}{8}\frac{Py_p}{E_0 I_m}L^2 = \frac{1}{8}\kappa L^2$

Fig. 20.

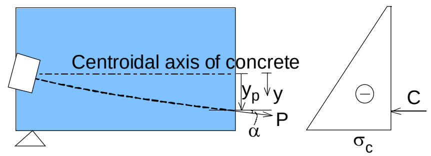
(Note: Mohr's analogy)

Analysis of post-tensioned beams

(Losses of prestress are assumed to be = 0)

Consider first the effect of the prestress on a **free body diagram** comprising one end of a **simply supported beam** after post-tensioning but before grouting, see Fig. 28. The prestressing force is = P.

Fig. 21.



Outer axial forces = 0 $\Rightarrow$ compression resultant of concrete $C = P\cos\alpha \approx P$

Outer moment = 0 $\Rightarrow$ C & P act along the same axis $\Rightarrow$

$$\sigma_c = \frac{-C}{A_c} + \frac{-Cy_P}{I_c}y = \frac{-P}{A_c} + \frac{-Py_P}{I_c}y$$

$(A_c < A_{br} < A_m$ and $I_c < I_{br} < I_m)$

Note 1: σ_c depends only on P and its line of action.

Huom 2: The result **is not in force** for statically undetermined beams .

A more general solution is obtained in another way. The tendon is considered a free body subjected to forces generated by the concrete. These forces are first determined. Thereafter, the concrete is regarded as a free body subjected to forces which are equal but opposite to the forces on the tendon (Newton's 3rd law). It is first assumed that the forces parallel to the tendon vanish excepth the anchoring forces.

A straight tendon is loaded by the anchor forces at the ends only.

When the direction of the tendon changes by an agle θ, see Fig. 22, a force $2P\sin(\theta/2)$ is needed to balance the axial tendon forces.

For small values of θ

$2P\sin(\theta/2) \approx 2P(\theta/2)=P\theta$

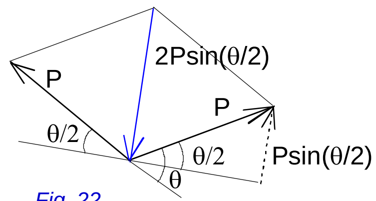


Fig. 22.

For a *curved tendon,* the symbols shown in Fig. 23 are used. The aim is to determine the force distribution $q(x)$ perpendicular to the tendon.

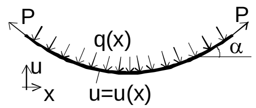


Fig. 23. Curved tendon.

There is a force = P at both ends of the tendon (anchor force). Due to the assumption of no friction, there are only transverse forces. Their distribution is given by $q(x)$.

Special case 1. A circular arc, uniformly distributed q. Using the symbols shown in Fig. 24, the horizontal forces are in equilibrium if

$P = \int_0^{\pi/2} qr\sin\varphi d\varphi = \ldots = qr$, from which

$q = \dfrac{P}{r}$

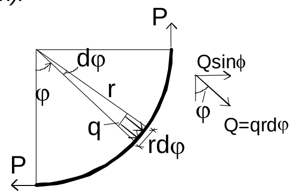


Fig. 24. Circular tendon.

Fig. 25 shows a free body diagram of a cut of an arbitrary curved arc $u=u(x)$.

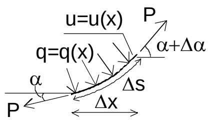


Fig. 25.

$\Delta x \approx \cos\alpha\Delta s \qquad \dfrac{du}{dx} = \tan\alpha \qquad \dfrac{d^2u}{dx^2} = \dfrac{1}{\cos^2\alpha}\dfrac{d\alpha}{dx}$

$$\Delta\alpha \approx \frac{d\alpha}{dx}\Delta x = \frac{d^2u}{dx^2}\cos^3\alpha\Delta s$$

Horizontal component of q is $Q_{hor} = \int_\alpha^{\alpha+\Delta\alpha} q\sin(\alpha)ds$

$q\sin(\alpha)\Delta s < \int_\alpha^{\alpha+\Delta\alpha} q\sin(\alpha)ds < q\sin(\alpha+\Delta\alpha)\Delta s \Rightarrow Q_{hor} = q\sin(\alpha+\xi\Delta\alpha)\Delta s$ where $0<\xi<1$

Horizontal equilibrium $\Rightarrow P\cos\alpha = q\Delta s\sin(\alpha+\xi\Delta\alpha) + P\cos(\alpha+\Delta\alpha)$

$$q\Delta s\sin[\alpha+\xi\Delta\alpha] = -P[\cos(\alpha+\Delta a)-\cos\alpha]$$

$$\Delta\alpha\to 0:\quad q\cancel{\Delta s}\underset{\downarrow\ \sin\alpha}{\sin[\alpha+\xi\Delta\alpha]} = -P\underset{\downarrow\ -\sin\alpha}{\left[\frac{\cos(\alpha+\Delta a)-\cos\alpha}{\Delta\alpha}\right]}\overbrace{\cos^3\alpha\frac{d^2u}{dx^2}\cancel{\Delta s}}^{=\Delta\alpha}$$

Result: $\underline{q = P\cos^3\alpha\dfrac{d^2u}{dx^2}}$ For small values of α: $q \approx P\dfrac{d^2u}{dx^2}$

The same result is obtained from vertical equilibrium, when $\sin\alpha = 0$.

Example. Circular arc in Fig. 26:

$$q = P\cos^3\alpha\frac{d^2u}{dx^2} = \ldots$$

$$= P\cos^3\alpha\left(\frac{1}{\sqrt{r^2-x^2}} + \frac{x^2}{\left(\sqrt{r^2-x^2}\right)^3}\right)$$

$$= P\cos^3\alpha\left(\frac{1}{r\cos\alpha} + \frac{r^2\sin^2\alpha}{(r\cos\alpha)^3}\right) = \frac{P}{r}$$

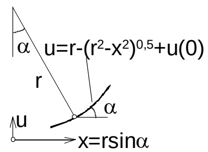


Fig. 26.

Special case 2. Parabola $u=ax^2+bx+c$

$$q = P\cos^3\alpha\frac{d^2u}{dx^2} = 2Pa\cos^3\alpha$$

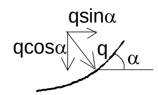


Fig. 27.

The vertical and horizontal components of q, i.e. q_v and q_h (Fig. 27) are

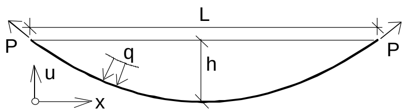
$q_v = 2Pa\cos^4\alpha \qquad q_h = 2Pa\cos^3\alpha\sin\alpha$

For small values of α: $\underline{q_v \approx q \approx 2Pa}$ $\quad q_h \approx 0$

The 2nd derivatives of parabolas $u=ax^2+bx+c$ ja $v=u+(kx+t)$ are the same. Consequently, a linear change in a parabolic tendon geometry results in a slight change of q due to the term $\cos^3\alpha$ and a slight change in the direction of q. Normally these changes are ignored in the design because in ordinary structures $\cos\alpha$ equals 1,0 with a great precision. A linear change in the tendon geometry affects the support reactions which may have consequences in the design of the supporting structures.

Fig. 28 presents the parabola

$$u = \frac{4h}{L^2}\left(x - \frac{L}{2}\right)^2$$



$$\underline{q} = 2 \cdot P \frac{4h}{L^2}\cos^3\alpha = \frac{8Ph}{L^2}\cos^3\alpha \approx \underline{\frac{8Ph}{L^2}}$$

Fig. 28.

If the ends of a parabolic arc are not at the same level (Fig. 36), tangent T, parallel to line S connecting the end of the arc, is drawn. If the vertical distance of S and T is h, the previous formula still holds.

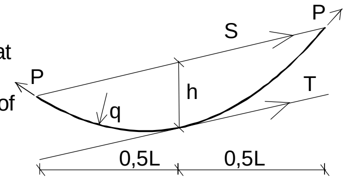


Fig. 29.

Justification: Perform a linear change to the arc, S and T in such a way that the right end is lowered and the left end is kept where it is, until S and T are horizontal. h remains the same. The second derivative of the curve also remains the same and q does not change.

Example. Beam in Fig. 37. L=20 m, h = 0,75 m, P = 1 MN.

u=(0,0075 m^{-1})(x-L/2)^2-0,025x+0,25m

$q = u'' = 2P$(0,0075 m^{-1})$\cos^3\alpha$ = (15 kN/m)$\cos^3\alpha$

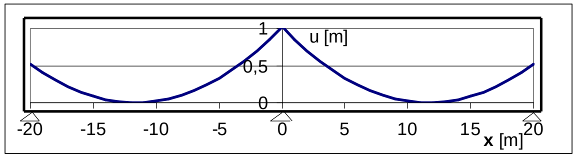


Fig. 30.

Both q and its horizontal and vertical components q_h and q_v are depicted in Fig. 31. Although $|\alpha_{max}|$ = 9,9°, i.e. a considerable angle, (the depth of the beam is > L/20), q and q_v are almost equal. Close to the supports, q_v is clearly smaller than in the middle of the span but this is not significant when considering the deflections and stresses. The loads near to the supports have a minor effect on the deflections and bending stresses when compared with the effects of the loads in the middle of the span.

From Fig. 31 it can be calculated that at a distance of 9 m from the beam end there is acting a horizontal force H = 0,5x(9m)x(1,8 kN/m)= 8 kN due to q_h. This is 0,8% of P. Elsewhere the effect of q_h is still smaller. Hence, it can totally be ignored.

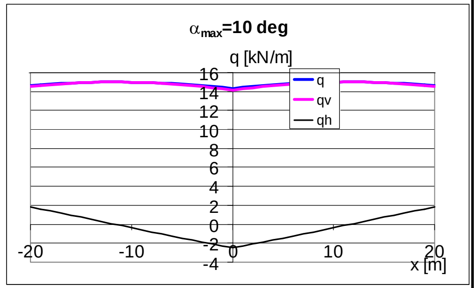


Fig. 31.

Table 1 ja Figs 32 – 33 present, what happens when the beam in Fig. 31 is made deeper and the curvature of the tendons is increased . When α_{max} = 20°, approximation of q with $q_v = q_{max}$ results in an error both in the moment distribution and horizontal force. In addition, the approximation overestimates the advantage of the prestressing force (is on the unsafe side)

Table 1.

α_{max}	h	$q_{v,min}/q_{max}$	H/P
deg	mm		
10	0,75	0,94	0,008
15	1,22	0,87	0,022
20	1,70	0,78	0,041

Fig. 32.

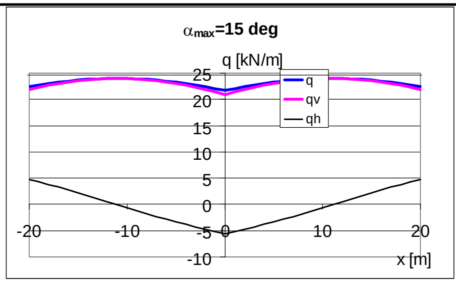
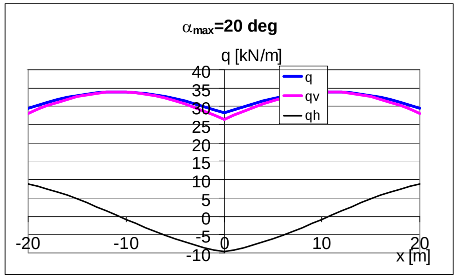


Fig. 33.



Conclusion: If α is the angle between a parabolic curve and a line parallel to the beam and ja -10° < α < +10°, the transverse forces *q* may be assumed perpendicular to the beam and

$$q = 2 \cdot Pa = \frac{8Ph}{L^2}$$

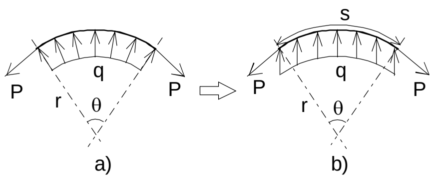
P is the prestressing force, *a* the coefficient of the 2nd degree terms in the expression for the parabola, *h* the depth of the parabola (see Fig. 29) and *L* the length of the parabola's projection on the centroidal axis of the beams. 10° is no exact limit. Higher curvatures are acceptable in case measures are taken to eliminate the possible negative effects of the inaccurate approximation.

When a parabolic arc is short when compared with its radius of curvature, it can be handled as a circle. In such a case the equation for a circle $x^2+u^2=r^2$ can be written in the form

$$u = r\sqrt{1-(x/r)^2} = r\left[1-\frac{1}{2}\left(\frac{x}{r}\right)^2+\frac{1}{2\cdot 4}\left(\frac{x}{r}\right)^4-+...\right] \approx r\left[1-\frac{1}{2}\left(\frac{x}{r}\right)^2\right]$$

In other words, a circle can be approximated by a parabola and vice versa.

When transverse forces *q* in Fig 34.a are replaced by vertical forces $q_v \approx q$ as in Fig. 34.b, the resultant of the forces *q* is



$$Q \approx q_v s \approx qs = \frac{P}{r} r\theta = P\theta$$

Fig. 34.

θ is the change in the angle between the ends of the arc.

A short parabolic arc is a good approximation for a short circular arc. So, the result above is also valid for a short parabolic arc. In fact, it follows from the equilibrium of forces that the resultant *Q* is equal to $P\theta$ for a small θ (see Fig. 22) independently of the shape of the curve.

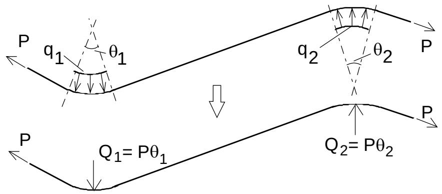


Fig. 35.

Loads on concrete due to shallow tendon geometry

Above, the loads needed to maintain the ***tendon*** geometry and tension, have been presented. For this purpose, free body diagrams for the tendon have been presented. Vice versa, the ***concrete*** in a beam can be regarded as a free body loaded by the counterforces of the forces acting on the tendons. The following rules follow when

- *P* is the tendon force
- α is the inclination angle of the tendon and θ the change in angle α,
- *L* is the length of the horizontal projection of te tendon
- *h* is the height of a parabola according to Fig. 29.

1. A horizontal force $P\cos\alpha$ and a vertical force $P\sin\alpha$ will develop due to the anchors
2. Straight tendons create no transverse force between the anchors.
3. A change θ in the inclination creates a force $Q = P\theta$ which can be considered perpendicular to the centroidal axis of the beam.
4. A parabolic arc of the form $u=ax^2+bx+c$ creates a transverse force $q = 2Pa = 8Ph/L^2$ per length unit.
5. A circular arc which is short in relation to the radius of curvature *r*, creates a transverse force $q = P/r$ per unit length.

Example. For the concrete component of the beam in Fig. 36.a, the free body diagram in Fig. 36.b and mechanical model in Fig. 36.c are obtained.

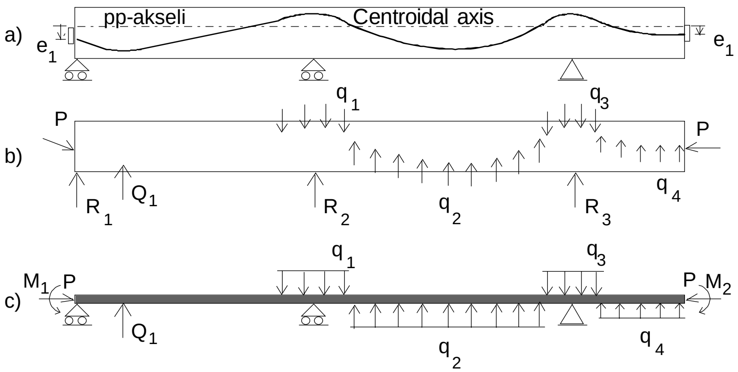


Fig. 36. a) Beam. b) Concrete as free body. c) Mechanical model.

The stresses and strains of the concrete due to the prestressing force may now be calculated as for any continuous beam subjected to transverse and longitudinal loads as well as couples at the ends.

The stresses and strains due to other loads are calculated separately and superimposed to the stresses and strains due the prestressing force.

Note 1. When loaded by prestressing force and self weight (all loads for unbonded tendons), it would be most accurate to use the net concrete section properties (A_c, I_c). For loads applied after grouting, the transformed section properties would be the best (A_m, I_m). In English speaking countries the gross section properties (A_{br}, I_{br}) are the most likely to be used in all cases. This simplifies the calculations. Furthermore, the error due to this simplification is quite small and normally results in a conservative design. The net section properties suit particularly well for the preliminary design.

Note 2. In the preliminary design, the negative curvatures at the intermediate supports may be ignored. The tendons may be modelled as those with a finite change in inclination at the supports, see Fig. 37.

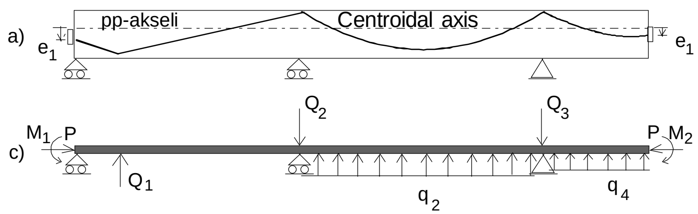


Fig. 37. Simplification of Fig. 36 for preliminary design.

Example. (Lin@Burns) Consider the beam shown in Fig. 38. Calculate the deflections due to the prestressing force P and self weight g immediately after prestressing. Data:

E_c = 28 GPa (elasticity modulus of concrete)
σ_{p0} = 965 MPa (initial prestress)
A_p = 780 mm² (cross-sectional area of tendon)
g = 3,25 kN/m (self weight)

We get:

$$I = \frac{300 \cdot 450^3}{12} = 2{,}278 \cdot 10^8\,mm^4$$

$$P = 965 \cdot 780N = 752{,}7kN$$

$$P = (965MPa)\cdot(780mm^2) = 752{,}7kN$$

The height of the parabola

$$h = 150\,mm$$

$$M_0 = (752{,}7kN)\cdot(25\,mm) = 18{,}82kNm$$

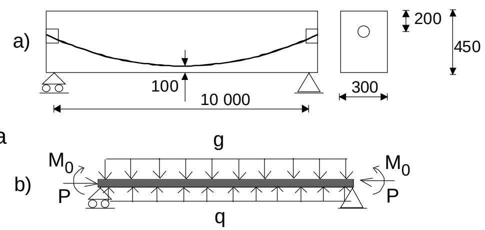


Fig. 38. a) Beam. b) Mechanical model.

Uniform. load due to P: $q = \frac{8Ph}{L^2} = \frac{8 \cdot 752{,}7kN \cdot 150\,mm}{(10\,m)^2} = 9{,}03kN/m$

Deflection due to q and g:

$$w_{q+g} = \frac{5}{384}\cdot\frac{(g-q)L^4}{EI} = -\frac{5\cdot(3{,}25kN/m - 9{,}03kN/m)\cdot(10m)^4}{384\cdot(28GPa)\cdot(2{,}278\cdot10^8mm^4)} = -11{,}8\,mm$$

Deflection due to M_0: $w_{M_0} = \frac{1}{8}\cdot\frac{M_0L^2}{EI} = \frac{(18{,}82kN/m)(10m)^2}{8\cdot(28GPa)\cdot(2{,}278\cdot10^8mm^4)} = 3{,}7\,mm$

Net deflection: w = -11,8 mm + 3,7 mm = - 8,1 mm

Load balancing (Lin)

If the prestressing force *P* and the tendon geometry are properly chosen, the transverse forces due to *P* balance the transverse outer loads. Similarly, the outer couples (point moments) at the ends of the beam may sometimes be balanced with countermoments created by eccentric anchors. This is called *load balancing.*

In perfect load balancing the beam remains straight and it is only subjected to axial compression.

Example. In the beam of Fig. 38 no uniformly distributed load can be perfectly balanced because no such load can eliminate the effect of the end moments.

Example. In the beam of Fig. 39 , the end moments are equal to 0. Find the uniformly distributed load *g* balanced by the tendon.

Fig. 39.a ⇒

$h = 125\ mm$

$q = 8Ph/L^2$

Fig. 39.b ⇒

$P\sin(\alpha_0) = 0{,}5qL$

Fig. 39.c ⇒ when $g = q$ all vertical forces are balanced:

$q = 8Ph/L^2$

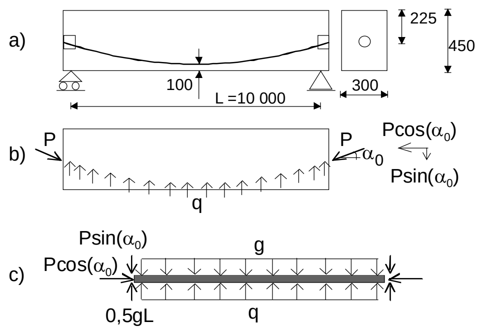


Fig. 39. a) Beam. b) Actions, due to P, on the concrete. c) All actions on the concrete.

$$g = q = \frac{8\cdot(752{,}7kN)\cdot(125mm)}{(10m)^2} = 7{,}53kN/m$$

This exceeds considerably the self weight 3,25 kN/m.

All previous rules concerning the interpretation of the transverse forces due to *P* as transverse outer loads can be inverted in such a way that *if a a transverse load distribution is needed to maintain a certain tendon force P and tendon geometry, the same load distribution is balanced by the same P and tendon geometry*. For example:

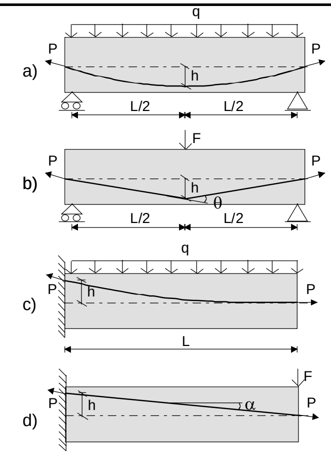
Point load *F* (Fig. 40.b): *P* and change of inclination θ are chosen in such a way that $P\theta = F$

Uniformly distributed load *p* (Fig. 40.a): *P ja* height of parabola *h* are chosen in such a way that $8Ph/L^2 = p$

End moment *M*: *P* and vertical position of the anchor y_a are chosen in such a way that

$Py_a = M$

Vertical load *F* at the end of cantilever (Fig. 40.d): a straight tendon with such a *P* and α are chosen that, the vertical component $P\sin\alpha$ of anchor force balances the effects of *F*.



Condition for load balancing:

a) $q = \frac{8\cdot Ph}{L^2}$

b) $F = P\theta = \frac{4Ph}{L}$

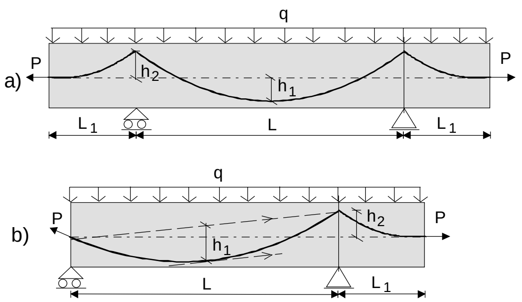
c) $q = \frac{2Ph}{L^2}$

d) $F = P\sin\theta \approx P\tan\theta = \frac{Ph}{L}$

Fig. 40. The dot-and-dash line tells the position of the centroidal axis.

Note: When the vertical support is missing, the vertical component of the anchor force must not be ignored!

Conditions for load balancing:



$$q = \frac{8 \cdot P(h_1 + h_2)}{L^2} = \frac{2 \cdot Ph_2}{L_1^2}$$

$$q = \frac{8 \cdot Ph_1}{L^2} = \frac{2 \cdot Ph_2}{L_1^2}$$

Fig. 41.

Fig, 41 shows, that even in simple cantilevered beams there may be additional conditions which make the balancing of a uniformly distributed load difficult. E.g. in Fig. 41.a the effective use of a rectangle beam would mean that $h_1 = h_2 \Rightarrow L_1 = L/8^{0,5}$

A continuous beam shown in Fig. 42.a could be load balanced with the black tendon anchored at the centroidal axis. This would mean that a half of the beam depth would be useless. When the tendon geometry is linearly transformed span by span as indicated with the red line, the depth of the beam can be reduced by 40% while the load balancing remains unchanged (Fig. 42.b).

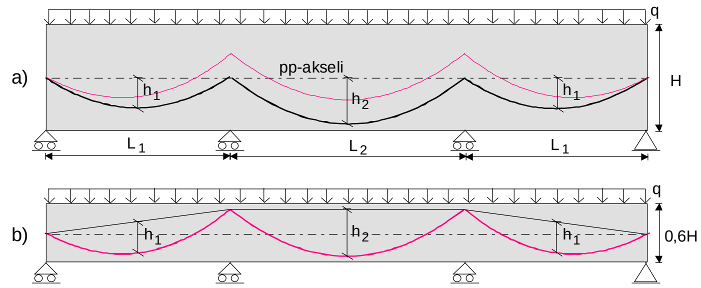


Fig. 42. Continuous beam, short outer spans.

Fig. 43 illustrates a beam with equal spans, balanced for a uniformly distributed load. When the anchors are placed at the centroidal axis to eliminate end moments, the concrete cover at the bottom of the middle span becomes very thick and the effective depth smaller than in the outer spans. This effect is even more pronounced if the mid-span is shorter than the outer spans.

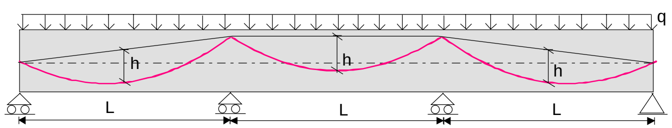


Fig. 43. Continuous beam with equal spans.

In general, only the self weight and a part of the imposed load are balanced. Only such loads can be regarded as self weight which are active when the prestressing is applied. Load balancing for major loads not acting when prestressing may damage the structure during the prestressing.

The principles of load balancing can also be applied as follows:

1. The external loads corresponding to the transverse tendon forces are determined
2. From the real external forces, those mentioned above are subtracted
3. Only the load difference is regarded as a load in the calculations

<u>Example.</u> Self weight of beam is 10 kN/m and imposed load 15 kN/m.

If the tendon force balances a tranverse load = 11 kN/m and the anchors are at the centroidal axis, calculate the bending moment and deflections

- due to self weight and prestressing force for the load
  10 -11 kN/m = -1 kN/m (upwards)
- due to self weight, prestressing force and imposed load for the load
  10+15-11 kNm = 14 kN/m (downwards).

When calculating the stresses, those due to the bending moment are superimposed to the axial stresses *–P/A* due to the prestressing force.

Primary and secondary moment due to prestressing force

As stated previously, in a statically determined beam, the moment due to the prestressing force P is $M_P = -Py_P$ where y_P is the y-coordinate of the centroidal axis of the tendon.

In a statically nondetermined beam, the deflections due to P cannot take place free because there are extra restraints. Therefore, P causes support reactions which must be taken into account when calculating M_P. Formally, the expression $M_P = M_{P1} + M_{P2}$ can be written where

$M_{P1} = -Py_P$ is the *primary moment* (due to P)

$M_{P2} = M_P - M_{P1}$ is the *secondary moment* (due to P).

Fig. 44 illustrates the concepts. The beam is first detached from the intermediate support. It deflects as shown in Fig. 44.b and the moment distribution is the one shown in Fig. 44.e. F denotes the transverse point forces due to P. Calculate the deflection w_1 at the intermediate support due to the primary moment. The support reaction due to P is denoted by R. Calculate the deflection w_2 due to R. Solve R from equation $w_1 + w_2 = 0$ and superimpose the primary moment and the secondary moment due to R. The result is shown in Fig. 44.f.

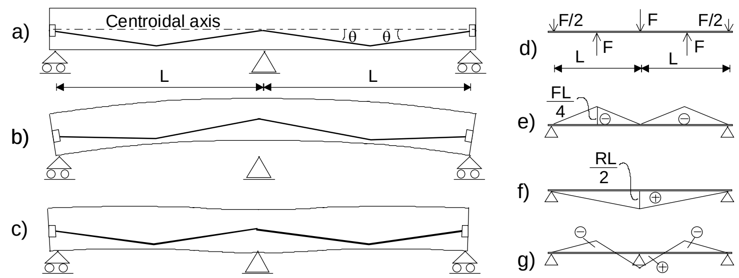


Fig. 44. a) Beam. b) Effect of primary moment. c) Effect of total moment. d) Transverse forces due to P. e) Primary moment. f) Secondary moment. g) Total moment.

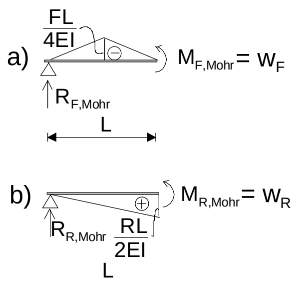
Applying Mohr's analogy (see the next page):

$$w_F = -\frac{FL^3}{16EI} \qquad w_R = \frac{RL^3}{6EI}$$

$$w_F + w_R = -\frac{FL^3}{16EI} + \frac{RL^3}{6EI} = 0 \quad \Rightarrow \quad R = \frac{3F}{8}$$

Intermediate support: $M_P = 0 + \frac{1}{2}\cdot\frac{3F}{8}L = \frac{3F}{16}L$

Span: $M_P = \frac{1}{2}\cdot\frac{3F}{8}\cdot\frac{L}{2} - \frac{FL}{4} = \frac{-5FL}{32}$



$$R_{F,Mohr} = 0{,}5\cdot\frac{-FL}{4EI}L = \frac{-FL^2}{8EI}$$

$$w_F = \frac{-FL^2}{8EI}\cdot\frac{L}{2} = -\frac{FL^3}{16EI}$$

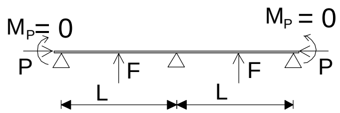
$$R_{R,Mohr} = 0{,}5\cdot\frac{RL}{2EI}L = \frac{RL^2}{4EI}$$

$$w_R = \frac{RL^2}{4EI}\cdot\frac{2}{3}L = \frac{FL^3}{6EI}$$

Fig. 45. Deflection of mid-span.
a) From primary moment. b) From secondary moment.

Easier: Determine the forces on the concrete (Fig 46). E.g. Rakentajain kalenteri (or some other handook) gives :

Intermediate support: $M_P = 0{,}188PL$
Mid-span: $M_P = -0{,}156PL$



Kuva 46. Effect of P on the concrete.

About losses of prestress

In this course, the losses of prestress mean
1. time-dependent redistribution of stresses in a prestressed cross-section or between different parts of a prestressed structure or
2. effects occurring during tensioning of tendons which result in tendon forces smaller than the initial prestressing force P_0.

It is typical that losses of prestress develop although the support conditions and loading were unchanged. The losses of prestress are caused e.g. by
- creep and shrinkage of concrete
- creep and relaxation of steel
- friction between tendon and duct in post-tensioned tendons
- slip of tendons during locking at anchors.

According to some researchers, we should not talk about losses but redistribution of stresses. However, this is no good concept for losses due to friction and slip during locking.

The calculation methods presented before give misleading results if the lossses of prestress are ignored. It is common to use the formulae in such a way that, the initial prestressing force $P = P_0$ (initial prestresss $\sigma_p = \sigma_{p0}$) is replaced by the reduced force $\beta_L P_0$ (reduced initial prestress $\beta_L \sigma_{p0}$). It is a common practice to ***first*** evaluate the losses which are then used to calculate the stresses and strains of the concrete and steel. However, it is not always possible to avoid iteration.

The difference $(1-\beta_L)$ reflects the relative losses of prestress. The long-term losses are typically of the order of 20% for pretensioned tendons. The losses for post-tensioned tendons may be essentially greater particularly in cylindrical shells or domes.

The elastic deformation due to the release of the prestressing force (pretensioning) or due to the tensioning of the neighbouring tendons (post-tensioning) is regardes as a loss of prestress by some textbooks and designers. Let's consider this in the view of a few examples.

Example. I. Pretensioned beam in Fig. 47. Previously (p. 22) it has been stated, that with the initial prestressing force P_0

the stress of the concrete ($\sigma_{c,P}$) and steel ($\sigma_{p,P}$) are obtained from

$$\sigma_{c,P} = \frac{-P_0}{A_m} + \frac{-P_0 y_P}{I_m} y \quad \text{(concrete)} \qquad n = \frac{E_p}{E_c}$$

$$\sigma_{p0} = \frac{P_0}{A_p} \quad \Bigg| \quad \sigma_{p,P} = \sigma_{p0} + n\left(\frac{-P_0}{A_m} + \frac{-P_0 y_P}{I_m} y_p\right) \quad \text{(steel)}$$

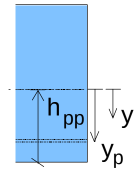


Fig. 47.

The stresses are explicit functions of P_0 and the concept of elastic loss is useless.

So called "elastic losses" are given by $\sigma_{p,P} - \sigma_{p0}$, which is equal to the change in steel stress due to an eccentric normal force equal to P_0.

If A_m and I_m in the expressions above are replaced by net values A_c and I_c or gross values A_{br} and $I_{br\,t}$, the elastic losses should be taken into account by reducing P according to the elastic deformation. This is what some engineers do, but there is not much sense in doing so because the easiest way to calculate the deformations is by using the transformed section properties A_m and I_m .

In the same way, when the net or gross values are used to evaluate the effects of the external loads, one should take into account the effects of the external loads on the tendon force. This is, however, never done. Again, it would be easier to use the transformed values A_m and I_m from the very beginning, but using them would mean that the benefits of the simplification would disappear. To conclude, *if A_m and I_m are replaced by A_c and I_c or A_{br} and I_{br} to simplify the calculations, it is preferable to forget the elastic losses an accept the error due to this simplification.*

Conclusions for a pretensioned beam:
1. The stress in the tendons changes due to the release of the prestressing force, but this elastic deformation does not mean loss of prestress.
2. To calculate the stresses accurately, use the transformed section characteristics.
3. To calculate the stresses less accurately but on the safe side, use the net or gross sectional characteristics.The elastic deformation is not regarded as a loss of prestress. The error du to this simplification is accepted.

Example. II. Post-tensioned beam.

Case 1: All tendons are tensioned simultaneously.

The elastic deformation takes place before the force in the tendons is measured. Therefore, it does not mean a loss of prestress.

To calculate the effects of the prestressing force accurately, use the net values of sectional characteristics, but the gross values also give results that are fairly accurate. .

Case 2: Tensioning in stages.

If the tendons are tensioned gradually in such a way that the increase in tendon forces balance the increase in self weight and the stresses in the beam are mainly axial compression, the change in the tendon force may be calculated assuming that the beam is a composite member, i.e. a compressed strut, which comprises the concrete, the previously tensioned tendons and the grouted ducts (if such ducts exist). At each stage, the effective cross-section at that stage is used.

The transformed sectional characteristics of this cross-section, are used when calculating the additional effects due to the tensioning at this stage

Fig. 48 illustrates a case in which four identical ltendons are post-tensioned in four stages. The black columns represent the tendon force P due to the tensioning (= P_0), the red ones (=P^i) the final value afer elastic deformations. The tensioning reduces the tendon force only in the previously tensioned tendons.

The response of the structure is determined by the average tendon force. If there are n stages, it is enough to calculate the change in tendon force in tendon number $n/2$ due to the tensioning of tendon number $n/2+1$. This change, multiplied by $n(n-1)/2$, is the change in the total tendon force.

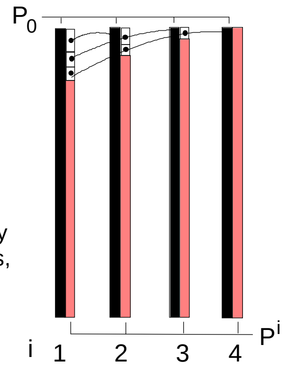


Fig. 48.

This rule is applied to the case shown in Fig. 48 (n = 4). The tensioning of the 3rd tendon reduces the force in the 2nd tendon by one white box. Consequently, the total change in tendon force is 4x(4-1)/2 = 6 white boxes. This can also be seen in Fig. 48. The white boxes become smaller when the tensioning propagates because the axial stiffness increases with increasing number of effective tendons. The error due to the assumption that all white boxes are equal is small and can be ignored.

How to calculate the effect of one tensioning stage to the tendons tensioned in the previous stages? The transformed section characteristics can be used, but applying the gross characteristics does not result in major errors, either.

The final target is to find out the average prestressing force, the anchor forces of which, together with the transverse forces, affect the concrete free body. It would be most logical to use the net sections to calculate the effects of the tendon force, but the gain in accuracy due to this choice is small when compared with the gain in simplicity in calculations when the gross sections are used. Therefore, it is common to use the gross sections when the demands for the accuracy are not too tight.

Conclusions for post-tensioned beams:

1. It is a matter of taste, whether the elastic deformation of a tendon due to tensioning of other tendons is regarded as a loss of prestress. In any case, this deformation has to be taken into account when calculating the effects of the prestressing force.
2. When calculating the effects of the prestressing, the (net or) gross sectional characteristics are used for the effects of tendons tensioned in the first stage, and approximatively also for effects of tendons tensioned in the later stages. For later stages it would be accurate to take into account the tendons and possible grouting of ducts in the previous stages. For the external loads the gross section can be applied to get an approximation of the structural response. If their are high demands for the accuracy, transformed sections should be applied.
3. If the beam remains straight or almost straight during the tensioning (load-balancing), the elastic deformation may be evaluated assuming that there is no bending.

General remark:
In the ultimate limit state of bending the losses of prestress play a minor role.

Total strain of concrete

The total strain of concrete at time t can be given in the form of

$$\varepsilon_c(t) = \varepsilon_{ce}(t) + \varepsilon_{cs}(t) + \varepsilon_{cc}(t) + \varepsilon_{c\Delta T}(t)$$

Here is

$\varepsilon_{ce}(t)$ elastic strain ($e \leftrightarrow$ elastic),

$\varepsilon_{cs}(t)$ shrinkage ($s \leftrightarrow$ shrinkage),

$\varepsilon_{cc}(t)$ creep ($c \leftrightarrow$ creep),

$\varepsilon_{c\Delta T}(t) = \alpha_T \Delta T$, strain due to temperature change Δt,

α_T temperature coefficient.

The shrinkage and temperature change affect the stresses of the concrete only if they cannot take place freely. Restraints may be provided by supports, reinforcement etc.

The temperature changes are not considered in this course, but the calculation methods are similar to those used for the shrinkage.

Shrinkage (Fig. 49)

Chemical reactions in the concrete during hardening result in volume change. In a beam, a longitudinal deformation ε_{ca} (autogenous shrinkage) is observed.

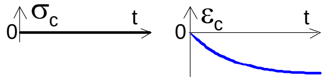


Fig. 49. Shrinkage.

Another component of shrinkage is drying shrinkage ε_{cd} . The total shrinkage of the concrete is

$$\varepsilon_{cs}(t)=\varepsilon_{ca}(t)+\varepsilon_{cd}(t)$$

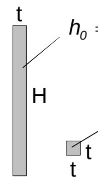
The rate of shrinkage reduces with time. Both the rate of shrinkage and the long-term shrinkage depend on the concrete itself, ambient relative humidity, geometry of the member etc.

EC2 (EN 1992-1-1):

$$\varepsilon_{cd}(t,t_s)=k_h\varepsilon_{cd,0}\beta_{ds}(t,t_s)=k_h\varepsilon_{cd,0}\frac{t-t_s}{(t-t_s)+0{,}04\sqrt{h_0^3}} \qquad [t],[t_s] = 1\text{ d}$$

Here is

k_h coefficient, depends on parameter h_0 (mm), tabulated data,
$\varepsilon_{cd,0}$ final value of drying shrinkage, expressions given in various codes,
t_s age of concrete when hardening begins (or curing ends),
h_0 notional thickness = $2A_c/u$, where A_c is the area of concrete section and ja u the perimeter exposed to drying *(see Fig. 50).*



$$h_0=\frac{2tH}{2(t+H)}\approx t$$

$$h_0=\frac{2t^2}{4t}=\frac{t}{2}$$

Fig. 50.

EN 1992-1-1: autogenous shrinkage is obtained from

$$\varepsilon_{ca}(t)=\varepsilon_{ca}(\infty)\beta_{as}(t)$$
$$=\varepsilon_{ca}(\infty)\left[1-\exp\left(-0{,}2t^{0,5}\right)\right]$$

$\varepsilon_{ca}(\infty)$ is the final value of autogenous shrinkage, calculated from

$$\varepsilon_{ca}(\infty)=2{,}5\cdot10^{-6}MPa^{-1}\left(f_{ck}-10MPa\right)$$

EC 1992-1-1. $\varepsilon_{cd,0}$ for concrete with *N*-class cement

	RH (°/°°)					
Betoni	20	40	60	80	90	100
C20/25	0,62	0,58	0,49	0,30	0,17	0,00
C40/45	0,48	0,46	0,38	0,24	0,13	0,00
C60/75	0,38	0,36	0,30	0,19	0,10	0,00
C80/95	0,30	0,28	0,24	0,15	0,08	0,00
C90/105	0,27	0,25	0,21	0,13	0,07	0,00

E.g. Notation C40/45:

Cylinder strength = 40 MPa
Cube strength = 45 MPa

EN 1992-1-1. k_h - h_0 -relationship.

h_0 mm	k_h
100	1,00
200	0,85
300	0,75
≥ 500	0,70

Example. Beam *hxb* = 600x300 mm² ⇒ h_0 = 200 mm and k_h = 0,85
Concrete C50/60 ⇒ f_{ck} = 50 MPa ⇒ $\varepsilon_{ca}(\infty)$ = 0,0001
Cement *N* ja relative humidity 50 % ⇒ $\varepsilon_{cd,0}$ = 0,00037
Curing for 2 days, t_s = 2 d.

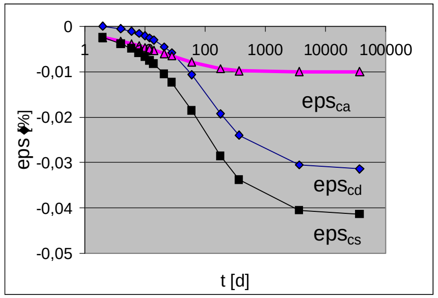


Fig. 51.

We obtain (Fig. 51):

$\varepsilon_{cd}(\infty)$ = 0,00032
$\varepsilon_{cs}(\infty)$ = $\varepsilon_{ca}(\infty)+\varepsilon_{cd}(\infty)$
= 0,0001+0,00032
= 0,00042

This corresponds to loss of prestress 81 MPa (6 %), if E_p = 195 GPa and initial prestress σ_{p0} = 1300 MPa

Creep of concrete

When concrete is loaded at time t_0, the instantaneous stress $\sigma_c(t_0)$ corresponds to elastic strain

$$\varepsilon_{ce}(t_0) = \frac{\sigma_c(t_0)}{E_c(t_0)}$$

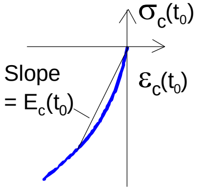


Fig. 52.

Elasticity modulus $E_c(t_0)$ (Fig. 52) depends on the stress but it is assumed to be constant in calculations.

Let's assume that the concrete can freely deform in constant temperature. When the stress ($\neq 0$) is kept constant until $t > t_0$, a deformation

$$\varepsilon_{cc}(t) = \varepsilon_c(t) - \varepsilon_{ce}(t) - \varepsilon_{cs}(t) \neq 0.$$

It is called creep . Let's denote $\varepsilon_{cc}(t) = \varphi(t, t_0)\varepsilon_{ce}(t_0)$. Here $\varphi(t,t_0)$ is the creep coefficient which, among many other things, depends both on the age of the concrete and time of loading. It is the ratio of the creep to the elastic deformation $\varepsilon_{ce}(t_0)$, see Fig. 53.

The strain of concret at time t is

$$\varepsilon_c(t) = \varepsilon_{ce}(t_0) + \varepsilon_{cc}(t) + \varepsilon_{cs}(t) = \sigma_{ce}(t_0)\frac{[1 + \varphi(t, t_0)]}{E_c(t_0)} + \varepsilon_{cs}(t)$$

If φ is known, the creep and elastic deformation can be analysed as a deformation of an elastic material with elasticity modulus $E_c/(1+\varphi)$.

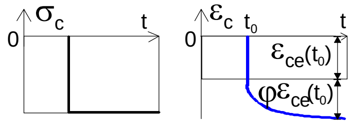


Fig. 53. Elastic strain ε_{ce} and and creep $\varphi\varepsilon_{ce}$.

The formulae presented before are useful, if φ is independent of the stress. In such a case the creep is called linear.

The assumption of linearity is accurate enough if the long-term stress in the concrete is less than half of the strength at loading. The limit given in EC2 is 45% of the mean compression strength at loading age. This is often the case in prestressed structures.

EC2-1-1, General rules and rules for buildings, informative annex, gives complicated formulae for the development of the creep. EC2-2-1, Bridges, gives different formulae. The creep resulting from these formulae differ considerably even for the same concrete.

Because the creep formulae are presented in informative annexes, it is possible to apply other creep formulae as well, e.g. those of the Finnish B4 or CEB-FIP Model Code 1990 (MC-90).

When the concrete is loaded by compressive normal stress σ_c at time t_0, the creep of concrete at time $t = \infty$ is

$$\varepsilon_{cc}(\infty, t_0) = \varphi(\infty, t_0)\frac{\sigma_c}{E_c}$$

EC2 gives a simple graphical method for determining the final value $\varphi(\infty,t_0)$, see Figs 54 and 55. Here ∞ corresponds to roughly 70 years.

The final value of creep acording to EC2-1-1:

t_0	age at loading (d)	E.g.	t_0 = 4 d
S, N, R	types of cement		cement *R*
h_0	notional thickness		h_0 = *200 mm*

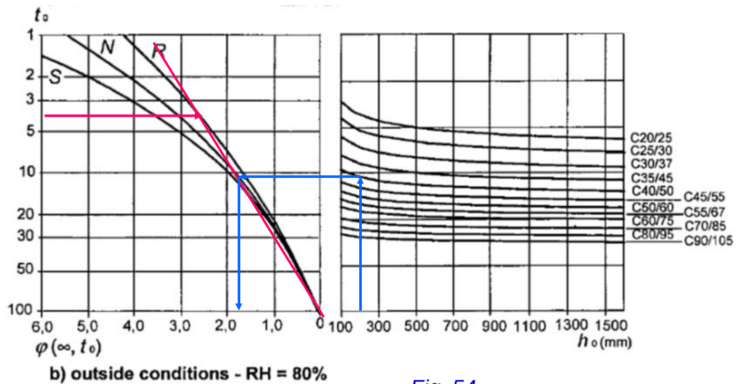


Fig. 54.

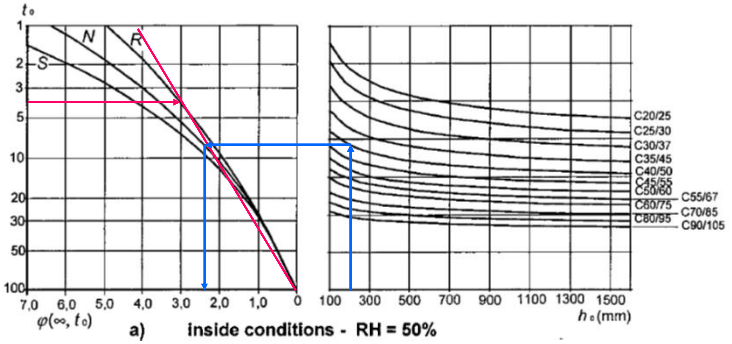
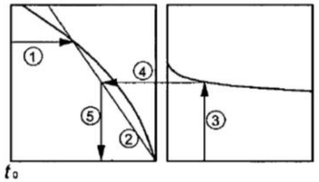


Fig. 55.

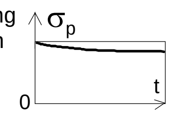


Note:
- intersection point between lines 4 and 5 can also be above point 1
- for t_0 > 100 it is sufficiently accurate to assume t_0 = 100 (and use the tangent line)

Fig. 56.

Relaxation of prestressing steel

The gradual reduction of the stress during constant strain is called relaxation, when the ambient temperature and other conditions of the environment are kept constant. The relaxation of the prestressing steel is important because it reduces the beneficial effects of the initial prestress.



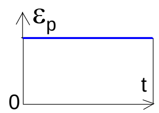


Fig. 57. Relaxation.

The relaxation increases rapidly when the stress is close to the yield . It is given as ρ_{1000} which is the loss of prestress [in %] in a 1000 h test in which the initial prestress is 70% of the measured ultimate strength of the steel.

EC2 divides the prestressing steel in three classes based on the relaxation:

	ρ_{1000}
Class 1	$\leq$ 8 %
Class 2	$\leq$ 2,5 %
Class 3	$\leq$ 4 %

Wires and strands belong to Class 1 or Class 2, bars to Class 3. For each Class, there is an expression from which the relaxation $\Delta\sigma_{pr} = \Delta\sigma_{pr}\,(t, \sigma_{pi})$ can be calculated. σ_{pi} is the initial prestress.

Class 1 $$\frac{\Delta\sigma_{pr}}{\sigma_{pi}} = 5,39\cdot10^{-5}\,\rho_{1000}e^{6,7\mu}\left(\frac{t}{1000\,h}\right)^{0,75(1-\mu)}$$

Class 2 $$\frac{\Delta\sigma_{pr}}{\sigma_{pi}} = 0,66\cdot10^{-5}\,\rho_{1000}e^{9,1\mu}\cdot\left(\frac{t}{1000\,h}\right)^{0,75(1-\mu)}$$

Class 3 $$\frac{\Delta\sigma_{pr}}{\sigma_{pi}} = 1,98\cdot10^{-5}\,\rho_{1000}e^{8\mu}\left(\frac{t}{1000\,h}\right)^{0,75(1-\mu)}$$

Obs. If ρ_{1000} = k %, replace ρ_{1000} with k in the expression.

$\mu = \sigma_{pi}/f_{pk}$, where f_{pk} is the nominal (failure) strength of the steel (should be *characteristic* strength but such a strength is normally not determined).

Example. Steel St 1640/1860 belongs to Class 2. The nominal strength f_{pk} is 1860 and the actual strength (f_p) 1930 MPa. The relaxation test is carried out with initial stress 0,70x 1930 = 1351 MPa and ρ_{1000} is obtained. The steel is tensioned to 1351 MPa $\Rightarrow \mu$ = 1351/1860 = 0,726 $\Rightarrow$ in 1000 h:

$$\frac{\Delta\sigma_{pr}}{1351MPa} = 0,66\cdot10^{-5}\,\rho_{1000}e^{9,1\cdot0,726} = 0,00488\,\rho_{1000}$$ ↯ because

the relaxation test gave $$\frac{\Delta\sigma_{pr}}{1351MPa} = 0,01\rho_{1000}$$

The relaxation formulae of EC2 are easy to use but clearly incorrect. Fig. 62 ilustrates the increase of relaxation with time when ρ_{1000} = 1,3 %. In 57 years, the relaxation is 3,5 times the 1000 h relaxation. This ratio is of the same order as that obtained using other methods. It is likely that the relaxation formulae of EC2 can be used in case they are calibrated in such a way that they give the 1000 h relaxation ρ_{1000} at t = 1000 h.

It would also be more consistent to replace 10^{-5} by 10^{-3}. Then ρ_{1000} could be used as such, e.g. as 0,013 or as 1,3%.

The relaxation is very sensitive to elevated temperature. E.g. heat treatment enhances the relaxation. Instructions for this are given e.g. in EC2.

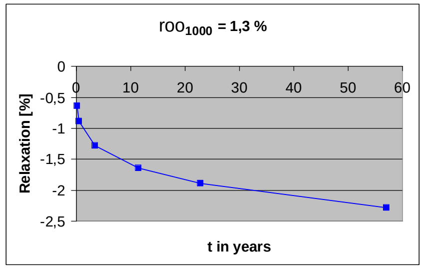


Fig. 58. Relaxation vs. time in Class 2 according to EC2.

Frictional and anchorage losses in posttensioned structures

Without friction between the tendon and duct the tendon force during tensioning would be constant between the active and passive anchor. Due to the friction, the situation shown in Fig. 59 results. A small change in angle = $d\varphi$ (Fig. 59.b) causes a transverse force $q = P/r$ per unit length. The corresponding friction force is $\mu q = \mu P/r$ and the change in P in interval ds = r$d\varphi$ is $dP = -\mu(P/r)rd\varphi = -\mu P d\varphi$. We obtain ($C$ is integration constant)

$$dP / d\varphi = -\mu P \;\Rightarrow\; \ln P = -(\mu\varphi + C) \;\Rightarrow\; P = e^{-(\mu\varphi + C)} = e^{-\mu\varphi}e^{-C}$$

$P(0) = P_0 \quad \Rightarrow$

$$\underline{P = P_0 e^{-\mu\varphi}}$$

or

$$\underline{P = P_0 e^{-\mu\Delta\theta}}$$

where $\Delta\theta$ is the absolute value of the change in inclination angle or

$$\Delta\theta = |\theta_2 - \theta_1|$$

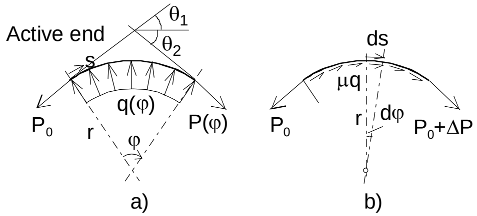


Fig. 59. Circular tendon. a) Transverse force. b) Friction force.

A corresponding relatiohship can be deduced for an arch of parabola $u = ax^2 + bx + c$ using the close relationship of a parabola and circle in case the curvature is small in comparison with the arc length. At x_1 and x_2 (Fig. 60) the inclination angles are θ_1 ja θ_2. Approximatively

$$\theta_i \approx \tan(\theta_i) = u'(x_i) = 2ax_i + b$$

$$\underline{\Delta\theta} = |\theta_2 - \theta_1| \approx |2a(x_2 - x_1)| = \underline{2|a|\Delta x}$$

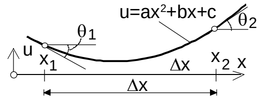


Fig. 60.

In the case of Fig. 61, the change in angle is

$$\underline{\Delta\theta} \approx 2a\Delta x = 2\frac{4h}{\Delta x^2}\Delta x = \underline{\frac{8h}{\Delta x}}$$

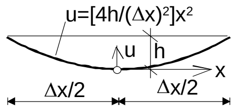


Fig. 61.

Denoting $\theta = \Sigma\Delta\theta_i$, i.e. the sum of changes in inclination angles $\Delta\theta_i$ (absolute values) within distance x from the active anchor, the reduced force at distance x equals

$$P = P_0 e^{-\mu\theta}$$

The ducts are pointwise supported and deflect between the supports during concreting and before it. For this reason, the ducts aimed to be straight are never perfectly straight and friction between the tendon and duct is generated. This *length effect* is taken into account in such a way that friction force due to it is assumed to be proportional to the distance x from the active anchor. In more detail, the change dP in the tensioning force within a distance dx is proportional to the dx and P or

$dP = -KPdx$, where K is the proportionality coefficient. In general, notation $K = \mu k$, is adopted. Here k is called aaltoisuusluku per unit length. As for the angle change, we obtain

$$P_x = P_0 e^{-\mu k x}$$

The losses due to the angle change (ΔP_θ) and length effect (ΔP_x) are combined in such a way that the tendon force, reduced by the angle change, is still reduced by the length effect. This results in

$$P = \left(P_0 e^{-\mu\theta}\right)e^{-\mu k x} = P_0 e^{-\mu(\theta + kx)}$$

$$\Delta P = P - P_0 = P_0(e^{-\mu(\theta + kx)} - 1)$$

μ and k depend on the post-tensioning system. They have been specified for each manufacturer. See typical values overleaf.

Typical values of μ and k. See Leskelä (BY210) .

(You may skip this page)

Jännetyyppi	Suojaputki	μ	k
Päällystämätön lanka tai suurihalkaisijaiset punokset	Kirkas taipuisa peltiputki	0,30	0,0066
	Galvanoitu taipuisa peltiputki	0,25	0,0049
	Galvanoitu jäykkä peltiputki	0,25	0,0007
	Paperi- tai muovipäällyste	0,05	0,0049
Päällystämätön seitsemänsäikeinen punos	Kirkas taipuisa peltiputki	0,30	0,0066
	Galvanoitu taipuisa peltiputki	0,25	0,0049
	Galvanoitu jäykkä peltiputki	0,25	0,0007
	Paperi- tai muovipäällyste	0,08	0,0046
Kirkkaat terästangot	Kirkas taipuisa peltiputki	0,20	0,0010
	Galvanoitu taipuisa peltiputki	0,15	0,0007
	Galvanoitu jäykkä peltiputki	0,15	0,0007
	Paperi- tai muovipäällyste	0,05	0,0007

When the tendon is locked and the actuator force is removed, there is a small slip between the tendon and the anchor. Due to this, the tendon force at the active end reduces, see Fig. 62. This is called *locking or anchorage loss*. It is a local effect in a curved tendon. Due to the friction, the locking loss gradually fades as the distance to the anchor increases. The magnitude of the locking loss is specific for each post-tensioning system. The anchorage loss can be reduced by retensioning and shimming in certain systems.

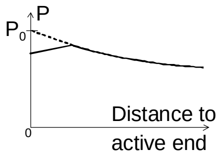


Fig. 62.

About calculation of losses

Above, the losses have been treated as effects which are independent of each other. The loads have been simplified. In the real world, the ambient temperature and humidity, the time dependence of the loads and the actual material properties are unknown, the reinforcement and boundary conditions affect the deformations etc. Therefore, it does not pay to apply too sophisticated methods but use simplified calculation rules. What is an acceptable simplification, varies from case to case.

On p. 70 it has been stated that the sum of the elastic strain and creep statrted at time t_0 is

$$\varepsilon_c(t) = \varepsilon_{ce}(t_0) + \varphi(t,t_0)\varepsilon_{ce}(t_0) = \varepsilon_{ce}(t_0)[1+\varphi(t,t_0)] = \sigma_c(t_0)\frac{[1+\varphi(t,t_0)]}{E_c(t_0)}$$

Linearity of creep $\Rightarrow$ the strain due to stress change $\Delta\sigma_c(\tau)$ at $\tau > t_0$ can be superimposed to the initial strain.

$$\varepsilon_c(t) = \sigma_c(t_0)\frac{[1+\varphi(t,t_0)]}{E_c(t_0)} + \Delta\sigma_c(\tau)\frac{[1+\varphi(t,\tau)]}{E_c(\tau)}$$

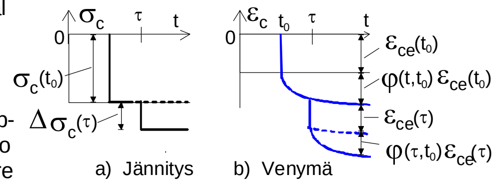


Fig. 63.

If the stress changes step-by-step, the strains due to the individual changes are superimposed and we get

$$\varepsilon_c(t) = \sigma_c(t_0)\frac{[1+\varphi(t,t_0)]}{E_c(t_0)} + \sum_i \frac{[1+\varphi(t,\tau_i)]}{E_c(\tau_i)}\Delta\sigma_c(\tau_i)$$

If the stress varies continuously, the sum is replaced by an integral. When the shrinkage is added, the total strain is obtained:

$$\varepsilon_c(t) = \sigma_c(t_0)\frac{[1+\varphi(t,t_0)]}{E_c(t_0)} + \int_{\sigma_c(t_0)}^{\sigma_c(t)} \frac{[1+\varphi(t,\tau)]}{E_c(\tau)} d\sigma_c(\tau) + \varepsilon_{cs}(t)$$

Here $d\sigma_c(t)$ denotes a small change in concrete stress at time τ ($t_0 < \tau < t$). The integration can be replaced by addition, in which the stress varies stepwise. Another and simpler way is to think that the sum of stress changes in time interval (t_0,t) takes place at t_0, but because actually a part of the change takes place later, the creep coefficient is reduced by a factor χ. In this way we obtain

$$\varepsilon_c(t) = \sigma_c(t_0)\frac{[1+\varphi(t,t_0)]}{E_c(t_0)} + \frac{[1+\chi\varphi(t,t_0)]}{E_c(t_0)}\Delta\sigma_c(t) + \varepsilon_{cs}(t)$$

$\dfrac{E_c(t_0)}{1+\chi\varphi(t,t_0)}$ is the *age-adjusted elasticity modulus* of the concrete.

The simplification follows from the fact that, χ does not vary too much, and e.g. EC2 gives a constant value $\chi = 0{,}8$.

Consider next the effect of simultaneous creep, shrinkage and relaxation on the losses of prestress in posttensioned tendons. Notation:

$\Delta\sigma_{c,QP}$ change of concrete stress due to prestressing force P and external forces Q (= stress) at time t_0

$\Delta\sigma_{p,c+s+r}$ change of tendon stress due to creep, shrinkage and relaxation during period (t_0,t), $A_P\Delta\sigma_{p,c+s+r} = \Delta P_{c+s+r}$ represents the corresponding change in tendon force

$\Delta\sigma_{pr}$ relaxation during period (t_0,t), corresponding to initial prestress

$\overline{\Delta\sigma_{pr}}$ actual relaxation during period (t_0,t). Estimate: $\overline{\Delta\sigma_{pr}} = 0{,}8\Delta\sigma_{pr}$

The change in concrete compression resultant relaxation during period (t_0,t):

$$\Delta C = -\Delta P_{c+s+r} = -A_P\Delta\sigma_{p,c+s+r}$$

The change of concrete stress on the centroidal axis of the tendons is

$$\Delta\sigma_{c,c+s+r} = \left(\frac{1}{A_c} + \frac{1}{I_c}y_p^2\right)\Delta C = \left(\frac{1}{A_c} + \frac{1}{I_c}y_p^2\right)A_p\left(-\Delta\sigma_{p,c+s+r}\right)$$

Set next the change of steel strain at the centroid of the tendons = the change in concrete strain at the same level in time interval (t_0,t). Take into account that the change in the concrete compressive force is equal but opposite to ΔP_{c+s+r} , the change of tendon force due to the creep, shrinkage and relaxation. $\Delta\bar{\sigma}_{pr}$, the stress change due to the relaxation, does not change the strain of the tendon.

$$\frac{\Delta\sigma_{p,c+s+r} - \Delta\bar{\sigma}_{pr}}{E_p} = \varepsilon_{cs} + \frac{\varphi(t,t_0)}{E_c(t_0)}\sigma_{c.QP} + \frac{[1+\chi\varphi(t,t_0)]}{E_c(t_0)}\left(\frac{1}{A_c}+\frac{1}{I_c}y_p^2\right)A_p\left(-\Delta\sigma_{p,c+s+r}\right)$$

Solve $\Delta\sigma_{c+s+r}$:

$$\Delta\sigma_{p,c+s+r} = \frac{\varepsilon_{cs}E_p + \Delta\bar{\sigma}_{pr} + \frac{E_p}{E_c(t_0)}\varphi(t,t_0)\sigma_{c.QP}}{1+\frac{E_p}{E_c(t_0)}\frac{A_p}{A_c}\left(1+\frac{A_c}{I_c}y_p^2\right)[1+\chi\varphi(t,t_0)]}$$

Estimate that $\Delta\bar{\sigma}_r = 0{,}8\Delta\sigma_{pr}$ ja $\chi = 0{,}8$. Denote $E_c(t_0) = E_{cm}$. Eq. (5.46) of EC2 is obtained:

$$\Delta P_{c+s+r} = A_p\Delta\sigma_{p,c+s+r} = A_p\frac{\varepsilon_{cs}E_p + 0{,}8\Delta\sigma_{pr} + \frac{E_p}{E_{cm}}\varphi(t,t_0)\sigma_{c.QP}}{1+\frac{E_p}{E_{cm}}\frac{A_p}{A_c}\left(1+\frac{A_c}{I_c}y_p^2\right)[1+0{,}8\varphi(t,t_0)]}$$

The symbols have the following meaning:

- ε_{cs} = $\varepsilon_{cs}(t,t_0)$ shrinkage in time interval (t,t_0) ,
- $\Delta\sigma_r$ relaxation loss assuming constant strain in time interval (t,t_0),
- $\sigma_{c,QP}$ stress of concrete due to prestressing, permanent loads and long-term share of imposed loads at the centroid of the tendons at time t_0,
- A_p Cross-sectional area of tendons,
- A_c net (or gross) area of concrete section,
- I_c second moment of area of concrete section (netor gross value),
- y_p y-coordinate of centroid of tendons (origin at centroid of concrete),
- E_p elasticity modulus of prestressed steel,
- E_{cm} = $E_c(t_0)$, elasticity modulus of concrete, see Fig. 56,
- t_0 age of concrete at first loading,
- $\varphi(t,t_0)$ creep coefficient.

The formula suits best for post-tensioned structures in which the shrinkage before the tensioning at t_0 does not affect. This is why it is enouh to consider the shrinkage $\varepsilon_{cs} = \varepsilon_{cs}(t,t_0)$:n and the shrinkage $\varepsilon_{cs}(0,t_0)$ can be ignored. In the same way, there is no relaxation before t_0.

For pretensioned tendons the shrinkage and relaxation in time interval $(0,t_0)$ must be added to the losses.

The formula of EC2 is most accurate when all long-term loads (tendon force, self weight and long-term imposed loads) start simultaneously. This is the case e.g. when the loads during execution are replaced by service loads of the same magnitude.

If this is not the case, but an additional load Q_k is applied at time $t_k > t_0$, the change due to the creep caused by this load has to be taken into account. In other words,

$$\left(\Delta P_{c+s+r}\right)_k = A_p\frac{\frac{E_p}{E_{cm}}\varphi(t,t_k)\Delta\sigma_{c.Qk}}{1+\frac{E_p}{E_{cm}}\frac{A_p}{A_c}\left(1+\frac{A_c}{I_c}y_p^2\right)[1+0{,}8\varphi(t,t_k)]}$$

has to be added to ΔP_{c+s+r} calculated above.

Evaluate the importance of the denominator in the formula of EC2 for the hollow core slab shown in Fig. 64 with almost maximum prestress. A_p = 10x93 mm^2, E_{cm} = 35 GPa, E_p = 190 GPa, A_c = 173000 mm^2, I_c = 1,74x109 mm^4, $\varphi(\infty,t_0)$ = 3. Result: Denominator = 1,10.

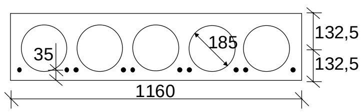


Fig. 64.

In other words: If the denominator is replaced by 1,0, the final losses are overestimated at most by 10% and the losses at younger ages less. Therefore, it is common to calculate an approximative value for the creep, shrinkage and relaxation losses from

$$\Delta\sigma_{p,c+s+r} = E_p\varepsilon_{cs}(t,t_0) + 0{,}8\Delta\sigma_{pr} + E_p\varphi(t,t_0)\frac{\sigma_{c.QP}}{E_{cm}}$$

The effect of an optional load Q_k, applied at a later stage t_k

$$\left(\Delta\sigma_{p,c+s+r}\right)_k = E_p\varphi(t,t_k)\frac{\Delta\sigma_{c.Qk}}{E_{cm}}$$

is added. These simplifications are less accurate, if the amount of non-prestressed steel is considerable.

If the centroids of the prestressed and reinforced reinforcement coincide, the following expression is obtained (not deduced, see e.g. *Ghali & Favre: Concrete structures: Stresses and deformations*. Chapman & Hall. p. 56 - 57)

$$\Delta P_{c+s+r}=\frac{A_p}{A_{st}}\left[\frac{A_{st}\varepsilon_{cs}E_{st}+0{,}8\Delta\sigma_{pr}A_p+A_{st}\dfrac{E_{st}}{E_{cm}}\varphi(t,t_0)\sigma_{c.QP}}{1+\dfrac{E_{st}}{E_{cm}}\dfrac{A_{st}}{A_c}\left(1+\dfrac{A_c}{I_c}y_p^2\right)\left[1+0{,}8\varphi(t,t_0)\right]}+0{,}8\Delta\sigma_{pr}A_{ns}\right]$$

where

A_{ns} is the cross-sectional area of the non-prestressed steel

A_{st} $= A_{ns}+A_p$

E_{st} is a ”compromise” value of easticity modulus between reinforcing and prestressing steel, e.g. 195 GPa or 200 GPa.

Load Q_k, applied at a later stage t_k, gives rise to a change

$$(\Delta P_{c+s+r})_k=\frac{A_p\dfrac{E_{st}}{E_{cm}}\varphi(t,t_k)\Delta\sigma_{c.Qk}}{1+\dfrac{E_{st}}{E_{cm}}\dfrac{A_{st}}{A_c}\left(1+\dfrac{A_c}{I_c}y_p^2\right)\left[1+0{,}8\varphi(t,t_k)\right]}$$

If the slab in Fig. 63 is provided with additional non-prestressed reinforcement, the amount of which is $A_{np} = A_p$, the denominator in the brackets equals 1,20. If this is replaced by 1,0, the reduced tendon force is

$$\Delta P_{c+s+r}=A_p\left(\varepsilon_{cs}E_{st}+0{,}8\Delta\sigma_{pr}A_p+\frac{E_{st}}{E_{cm}}\varphi(t,t_0)\sigma_{c.QP}\right)$$

In this case, the resulting error in ΔP is < 20 %.

Stresses and strains in a statically determined beam taking into account the losses

When there is steel in a concrete section, the location of the centroidal axis varies with increasing creep. Keeping the reference axis constant facilitates the calculation of the geometric cross-sectional characteristics. In Fig. 64, the origin of y-coordinate is on an arbitrary reference axis, the distance of which from the bottom fibre is h_{ref}. The axial force N and bending moment M are also calculated with respect to this reference axis.

Using the axial strain ε_0 at the reference axis and the curvature κ, the axial strain ε and stress σ_k can be expressed as

$\varepsilon=\varepsilon_0+\kappa y \qquad \sigma_k=E_k(\varepsilon_0+\kappa y)$

From these

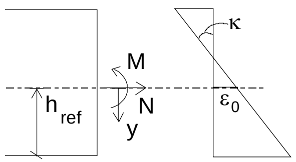


Fig. 65.

$$N=\sum_k\int_{A_k}E_k(\varepsilon_0+\kappa y)dA=\varepsilon_0\sum_k E_k\int_{A_k}dA+\kappa\sum_k E_k\int_{A_k}ydA=\varepsilon_0(EA)+\kappa(ES)$$

$$M=\sum_k\int_{A_k}E_k y(\varepsilon_0+\kappa y)dA=\varepsilon_0\sum_k E_k\int_{A_k}ydA+\kappa\sum_k E_k\int_{A_k}y^2dA=\varepsilon_0(ES)+\kappa(EI)$$

(ES) = 0 only when the reference axis and the centroidal axis coincide. Such a choice is not useful because the location of the centroidal axis varies with the creep.

Using the transformed cross-sectional characteristics ($E_0 = E_c$, in a composite beam E_0 is the E-modulus of some concrete part), and adopting matrix notation, the previous equations can be expressed in the form

$$\begin{Bmatrix}N\\M\end{Bmatrix}=E_c\begin{bmatrix}A_m & S_m\\ S_m & I_m\end{bmatrix}\begin{Bmatrix}\varepsilon_0\\ \kappa\end{Bmatrix}=E_c\hat{C}\begin{Bmatrix}\varepsilon_0\\ \kappa\end{Bmatrix}$$

$$\begin{Bmatrix}\varepsilon_0\\ \kappa\end{Bmatrix}=\frac{1}{E_c\left(A_mI_m-S_m^2\right)}\begin{bmatrix}I_m & -S_m\\ -S_m & A_m\end{bmatrix}\begin{Bmatrix}N\\M\end{Bmatrix}=\frac{1}{E_c}\hat{C}^{-1}\begin{Bmatrix}N\\M\end{Bmatrix}$$

$$A_m=\sum_k n_kA_k$$

$$S_m=\sum_k n_k\int_{A_k}ydA=\sum_k n_kS_k$$

$$I_m=\sum_k n_k\int_{A_k}y^2dA=\sum_k n_kI_k+n_kA_ky_{pp,k}^2$$

Here $n_k = E_k/E_c$, $y_{pp,k}$ is the y-coordinate of the centroid of part k *and* $S_k = y_{pp,k}A_k$.

Stress and strain at time t_0 immediately after prestressing

Combine first the outer normal force N and moment M with the actions due to the tendon forces P_j to get the load vector

$$\begin{Bmatrix}N_{ekv}\\M_{ekv}\end{Bmatrix}=\begin{Bmatrix}N-\sum P_j\\ M-\sum P_jy_{p,j}\end{Bmatrix}$$

$y_{p,j}$ is the y-coordinate of tendon j

We obtain

$$\begin{Bmatrix}\varepsilon_0(t_0)\\ \kappa(t_0)\end{Bmatrix} = \frac{1}{E_c(t_0)(A_m I_m - S_m^2)}\begin{bmatrix} I_m & -S_m \\ -S_m & A_m\end{bmatrix}\begin{Bmatrix}N_{ekv}\\ M_{ekv}\end{Bmatrix}$$

If the reference axis and the centroidal axis coincide, $S_m = 0$, and the well-known expression follows:

$$\begin{Bmatrix}\varepsilon_0(t_0)\\ \kappa(t_0)\end{Bmatrix} = \frac{1}{E_c(t_0)}\begin{Bmatrix}\dfrac{N_{ekv}}{A_m}\\ \dfrac{M_{ekv}}{I_m}\end{Bmatrix}$$

The parts of the section to be included in A_m ja I_m.depend on the situation. E.g. for a post-tensioned beam, the strains due to the tensioning should be calculated without prestressed steel and grout inside the ducts, but the reinforcing (non-prestressed) steel is included. In pretensioned beams the prestressing steel should be included from the very beginning.

The strain ε_c and stress σ_c of the concrete depend on y as follows:

$$\varepsilon_c(t_0) = \varepsilon_0(t_0) + \kappa(t_0)y$$

$$\sigma_c(t_0) = E_c(t_0)[\varepsilon_0(t_0) + \kappa(t_0)y]$$

The corresponding stresses of the steel are:

Non-prestressed: $\sigma_s(t_0) = E_s[\varepsilon_0(t_0) + \kappa(t_0)y]$

Pretensioned (initial prestress = σ_{p0}): $\sigma_p(t_0) = \sigma_{p0} + E_p[\varepsilon_0(t_0) + \kappa(t_0)y]$

Post-tensioned (initial prestress = σ_{p0}): $\sigma_p(t_0) = \sigma_{p0}$

Note. Neither the relaxation loss of a pretensioned tendon nor the friction or locking loss of post-tensioned tendon are included above. They must be subtracted from the initial prestress σ_{po}.

Changes in strain and stress in time interval (t_0,t)

Assume that $\varepsilon_0(t_0)$ ja $\kappa(t_0)$ are known. Assume also that the strain increment and curvature increment in time interval (t_0,t) are first prevented by a normal force ΔN and moment ΔM placed at the reference axis. Next, the composite section is loaded by $-\Delta N$ ja $-\Delta M$, which balances the effects of ΔN and moment ΔM. It follows that

$$\begin{Bmatrix}\Delta\varepsilon_0\\ \Delta\kappa\end{Bmatrix} = \frac{1}{E_c'(A_m I_m - S_m^2)}\begin{bmatrix} I_m & -S_m \\ -S_m & A_m\end{bmatrix}\begin{Bmatrix}-\Delta N\\ -\Delta M\end{Bmatrix}$$

Here E_c' is the age-adjusted elasticity modulus or $E_c' = \dfrac{E_c(t_0)}{1+\chi\varphi(t,t_0)}$

All cross-sectional characteristics, i.e. A_m, S_m ja I_m are calculated using E_c' as the reference modulus. Notation:

$\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{shr}$ prevents the shrinkage $\quad\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{cre}$ prevents the creep $\quad\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{rel}$ prevents the relaxation $\Rightarrow$

$$\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix} = \begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{shr} + \begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{cre} + \begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{rel}$$

If the creep could develop freely in time interval (t_0,t), ε_0 ja κ would change by $\varphi(t,t_0)\varepsilon_0(t_0)$ ja $\varphi(t,t_0)\kappa(t_0)$. The force and moment to prevent this are solved from equation

$$\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{cre} = -\sum_{k=1}^{n}\left\{E_c'\varphi\begin{bmatrix}A_c & S_c\\ S_c & I_c\end{bmatrix}\begin{Bmatrix}\varepsilon_0(t_0)\\ \kappa(t_0)\end{Bmatrix}\right\}_k$$

Subscript k refers to part k of the section and n is the total number of the parts. E_c' ja φ may be different in different parts.

The cross-sectional characteristics are calculated for the net concrete section with respect to the reference axis. E_c' ja φ are the age-adjusted elasticity modulus and creep coefficient for part k, respectively:

$$E_c' = \frac{[E_c(t_0)]_k}{1+\chi[\varphi(t,t_0)]_k} \qquad \varphi = [\varphi(t,t_0)]_k$$

Force and moment preventing the shrinkage: $\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{shr} = -\sum_{k=1}^{n}\left\{E_c'\varepsilon_{cs}\begin{Bmatrix}A_c\\ S_c\end{Bmatrix}\right\}_k$

ε_{cs} is the free shrinkage of part k .

Force and moment preventing the relaxation: $\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{rel} = \sum_{q=1}^{m}\begin{Bmatrix}A_p\Delta\bar{\sigma}_{pr}\\ A_p y_p\Delta\bar{\sigma}_{pr}\end{Bmatrix}_q$

The addition is extended over all tendon layers q, the number of which is m.

$\Delta\bar{\sigma}_{pr}$ is the reduced relaxation, A_p the cross-sectional area and y_p the y-coordinate in layer q. The reduced relaxation is calculated from

$$\Delta\bar{\sigma}_{pr} = \chi_r\Delta\sigma_{pr}$$

$\Delta\sigma_{pr}$ is the relaxation correponding to the initial prestress and χ_r reduction coefficient, the value of which may be taken as 0,8. More specific values can be found in the textbook of Ghali & Favre.

The strain increment in time interval (t_0,t) is $\Delta\varepsilon = \Delta\varepsilon_0 + y\Delta\kappa$. The total strain is obtained from

$$\varepsilon(t) = \varepsilon(t_0) + \Delta\varepsilon_0 + y\left[\kappa(t_0) + \Delta\kappa\right]$$

the stress increment in time interval (t_0,t) is

$$\Delta\sigma_c = \sigma_{restrained} + E'_{ck}(t,t_0)(\Delta\varepsilon_0 + y\Delta\kappa)$$ concrete, part k

$$\sigma_{restrained} = -E'_{ck}(t,t_0)\left[\varphi(t,t_0)\varepsilon_c(t,t_0) + \varepsilon_{cs}\right]$$ stress which prevents the creep and shrinkage

$$\Delta\sigma_s = E_s(\Delta\varepsilon_0 + y_s\Delta\kappa)$$ non-prestressed steel, vertical position y_s

$$\Delta\sigma_p = \Delta\bar{\sigma}_{pr} + E_p(\Delta\varepsilon_0 + y_p\Delta\kappa)$$ prestressing steel, vertical position y_p

Simplification for a simple non-composite beam

Use the gross values of the cross-sectional characteristics $\Rightarrow$ the reducing effect of the steel on the long-term deformations is ignored.

Effect of the creep in time interval (t_0,t):

$$\begin{Bmatrix}\Delta\varepsilon_0\\ \Delta\kappa\end{Bmatrix}_{cre} = \frac{1}{E'_c(AI-S_m^2)}\begin{bmatrix}I & -S\\ -S & A\end{bmatrix}\begin{Bmatrix}-\Delta N\\ -\Delta M\end{Bmatrix} = \frac{1}{E'_c(AI-S^2)}\begin{bmatrix}I & -S\\ -S & A\end{bmatrix}\begin{bmatrix}A & S\\ S & I\end{bmatrix}E'_c\varphi\begin{Bmatrix}\varepsilon_0(t_0)\\ \kappa(t_0)\end{Bmatrix}$$

$$= \varphi\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\varepsilon_0(t_0)\\ \kappa(t_0)\end{Bmatrix} = \varphi\begin{Bmatrix}\varepsilon_0(t_0)\\ \kappa(t_0)\end{Bmatrix}$$

Effect of the shrinkage:

$$\begin{Bmatrix}\Delta\varepsilon_0\\ \Delta\kappa\end{Bmatrix}_{shr} = \frac{1}{E'_c(AI-S_m^2)}\begin{bmatrix}I & -S\\ -S & A\end{bmatrix}\begin{Bmatrix}-\Delta N\\ -\Delta M\end{Bmatrix} = \frac{1}{E'_c(AI-S^2)}\begin{bmatrix}I & -S\\ -S & A\end{bmatrix}E'_c\varepsilon_{cs}\begin{Bmatrix}A\\ S\end{Bmatrix} = \varepsilon_{cs}\begin{Bmatrix}1\\ 0\end{Bmatrix}$$

Effect of the relaxation (gross values $\Rightarrow$ the section characteristics are time independent $\Rightarrow$ reference axis = centroidal axis is a reasonable choice $\Rightarrow$ $S = 0 \Rightarrow$ simplified expressions):

$$\begin{Bmatrix}\Delta\varepsilon_0\\ \Delta\kappa\end{Bmatrix}_{rel} = \frac{-A_p\Delta\bar{\sigma}_{pr}}{E'_c(AI)}\begin{bmatrix}I & 0\\ 0 & A\end{bmatrix}\begin{Bmatrix}1\\ y_p\end{Bmatrix} = \frac{-A_p\Delta\bar{\sigma}_{pr}}{E'_c(AI)}\begin{Bmatrix}I\\ Ay_p\end{Bmatrix} = -\frac{A_p}{E'_c}\Delta\bar{\sigma}_{pr}\begin{Bmatrix}1/A\\ y_p/I\end{Bmatrix}$$

With these assumptions (the steel is ignored, statically determined beam) the creep and the shrinkage do not change directly the stress of the concrete because the forces preventing them and their counterforces act on the same gross cross-section.

The relaxation changes the tendon force by $A_p\Delta\bar{\sigma}_{pr}$, which results in a concrete stress increment

$$\Delta\sigma_c = -A_p\Delta\bar{\sigma}_{pr}\left(\frac{1}{A} + \frac{y_p}{I}y\right)$$

In the same way, the creep and shrinkage affect the concrete stresses indirectly by changing the prestress in te tendon by

$$\Delta\sigma_{p,shr} = +E_p\Delta\varepsilon_{cs} \qquad \Delta\sigma_{p,cre} = E_p(\Delta\varepsilon_{0,cre} + y_p\Delta\kappa_{cre}) = E_p\varphi\frac{\varepsilon_c(t,t_0)}{E_c(t_0)}$$

The increment of the tendon force due to the creep, shrinkage and relaxation is

$$\Delta P_{c+s+r} = A_p\Delta\sigma_p = A_p(\Delta\sigma_{p,cre} + \Delta\sigma_{p,shr} + \Delta\bar{\sigma}_{pr})$$

Hence, the creep, shrinkage and relaxation give rise to a concrete stress increment

$$\Delta\sigma_c = -A_p\Delta\sigma_p\left(\frac{1}{A} + \frac{y_p}{I}y\right)$$

Note 1. The simplified formulae given above can be used for statically determined beams.

Note. 2. In general, the initial stress is not constant along the axis of the beam, and the losses vary from section to section. In ordinary cases the stress considerations can be restricted to certain critical sections, which facilitates the calculations.

Calculation of long-term response, general aspects

The prestressing force and the self weight are permanent loads. For the varying loads (imposed loads), the design codes specify a factor which tells the long-term share of the load.

For example, in Finland the long-term share of the residential load is 30 % and that of snow load either 20% or 50 % depending on, which load combination results in the critical action.

When calculating the long-term effects, the long-term share of a load is assumed to be active from its application on without interruptions.

To calculate the effects of loads at time t, the short-term loads are applied at that time and their momentary response (deflection, stress, strain etc.) is calculated without long-term effects. (These loads may have been effective several times before, but due to their momentary nature, their effects have recovered.) The short-term response calculated in this way is added to the long-term response which is calculated separately.

As soon as $\Delta\varepsilon_0$ and $\Delta\kappa$ are known in each cross-section, the deflection increment in time interval (t_0,t) can be calculated by integration from $(\Delta w)'' = -\Delta\kappa$ numerically. It is also possible to load the beam by $\Delta\kappa = M/(EI)$, and calculate the bending moment due to this load, which gives the deflection when the boundary conditions are properly chosen (Mohr's analogy). The obtained deflection increment Δw is added to the deflection $w(t_0)$ calculated at time t_0.

Instead of numerical integration, the well-known formulae for deflection calculation are to be preferred when they are available. For this, the different load types and loads activated at different times must be considered separately.

Example. Deflection of simple pretensioned I-beam

- Tendon force P, $t_0 = x$ d, share of long-term (LT) load 100 %
- Self weight of beam, $t_0 = x$ d, LT-share 100 %
- Self-weight of structures on beam, t_0 = 1 kk, LT-share 100 %
- Imposed point load, t_0 = 2 kk, LT-share 100%
- Ímposed uniform load, t_0 = 6 kk(?), LT-share 30 %.

Deflection due to the tendon force is $$w = \frac{\kappa_P L^2}{8}$$

When aiming at an accurate results, calculate first $\varepsilon_0(t_0)$ ja $\kappa(t_0)$ due to the prestressing force and other actions at time t_0 . Thereafter, solve $\Delta\kappa$ due to long term deformations in time interval (t_0,t) as shown on pp. 95 – 96 from expressions

$$\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix} = \begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{shr} + \begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{cre} + \begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{rel}$$

$$\begin{Bmatrix}\Delta\varepsilon_0\\ \Delta\kappa\end{Bmatrix} = \frac{1}{E_c^{'}\left(A_m I_m - S_m^2\right)}\begin{bmatrix} I_m & -S_m\\ -S_m & A_m\end{bmatrix}\begin{Bmatrix}-\Delta N\\ -\Delta M\end{Bmatrix}$$

The deflection is obtained by integration of differential equation $w'' = -(\kappa(t_0)+\Delta\kappa)$ or by using well-known expressions for different types of boundary conditions and loads.

Simplified approach: Ignore the stiffening effect of steel on I and write for the curvature due to P

$$\kappa_P = -\frac{A_p y_p \sigma_p(t_0)}{E_c(t_0) I_{br}}\left[1+\varphi(t,t_0)\right] - \frac{A_p y_p \Delta\bar{\sigma}_{p,c+s+r}(t,t_0)}{E_c^{'} I_{br}}$$

This can still be simplified by replacing the age-adjusted modulus E_c' by $E_c(t_0)/(1+\varphi)$:llä, which results in

$$\kappa_P = -\frac{A_p y_p\left[\sigma_p(t_0) + \Delta\bar{\sigma}_{p,c+s+r}(t,t_0)\right]}{E_c(t_0) I_{br}/(1+\varphi)}$$

(Note. Instead of I_{br} it would be more accurate to use I_m, in such a way that in the denominator of the term representing the tendon force increment, I_m should be calculated using E_c'.

In a post-tensioned beam it would be more accurate to use I_c *instead of* I_{br} at the time of tensioning and I_m calculated using E_c' for the tendon force increment.)

The deflection due to uniformly distributed load q is calculated applying

$$w = \frac{5}{384}\frac{qL^4}{EI}$$

L is the span of the beam. Denote: αq is the long-term share of q. The deflection due to q at time t is

$$w_q = w_{\alpha q} + w_{(1-\alpha)q} = \frac{5}{384}\frac{\alpha q L^4}{E_c I_m/[1+\varphi(t_{0,q},t)]} + \frac{5}{384}\frac{(1-\alpha)qL^4}{E_c(t) I_m}$$

$t_{0,q}$ is the age of the concrete when q is applied.

The deflection due to a point load F is calculated by applying the general formulae for the point loads. E.g., if F is in the mid-span and its long-term share is αF, the deflection is given by

$$w_F = w_{\alpha F} + w_{(1-\alpha)F} = \frac{1}{48}\cdot\frac{\alpha F L^3}{E_c I_m/\left[1+\varphi(t_{0,F},t)\right]} + \frac{1}{48}\cdot\frac{(1-\alpha)FL^3}{E_c(t)I_m}$$

Example. Post-tensioned rectangular beam

Due to the friction losses, the tendon force is greater at the active end (left end in Fig. 66) and so are also the transverse forces. However, to simplify the calculations, it is possible to assume a constant tendon force in each span. In a cantilevered span, the constant value should be representative to the value at support, between the supports the value should be representative to that in the mid-span.

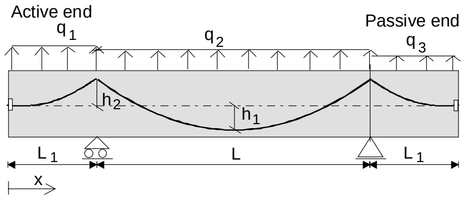


Fig. 66. The average transverse forces q_i due to the tendon forces span by span.

It is also possible to apply the maximum or minimum tendon force to find out the limits for the deflection.

Denote: $P_{0,i}$ is the tendon force representing each span at the time of prestressing and $\Delta P_{0,i}$ the corresponding loss in the time interval (t_0,t). Then the curvature due to P in each span i is

$$\kappa_{P,i} = \frac{M_P(x)}{E_c I_C/\left[1+\varphi(t_0,t)\right]} + \frac{\Delta M_P(x)}{E_c^{'} I_m} = \frac{-P_{0,i}y_P}{E_c I_C/\left[1+\varphi(t_0,t)\right]} + \frac{-\Delta P_{0,i}y_P}{E_c^{'} I_m}$$

To avoid calculation of each section separately, I_m and I_c are usually replaced by I_{br}:llä. Adopting still another approximation, i.e. replacing $E_c^{´}$ by $E_c/(1+\varphi)$:llä, we obtain an expression

$$\kappa_{P,i} = \frac{-(P_{0,i}+\Delta P_{0,i})y_P}{E_c I_{br}/\left[1+\varphi(t_0,t)\right]}$$

For parabolic tendons, y_P is a second degree polynomial of x. Other parameters are assumed to be constant within in each span.

Now the deflection w can be solved by integrating $w'' = -\kappa$ twice. There will be two integration constants per span. They can be solved from compatibility equations. E.g. for the beam in Fig. 66, six boundary conditions can be written:

- at both supports, w is continuous and = 0 at the support or
  $w- = w+ = 0$
- at both supports, w' is continuous or
  $w'- = w'+$

The deflections due to the tendon force are added to those due to external loads. The latter are calculated using elasticity modulus $E_c/(1+\varphi)$ for the long-term loads and E_c for the short-term loads.

Strains in statically undetermined beams

Not considered in this course, see e.g. the textbook of Ghali & Favre.

Eurocodes, particularly EC2

They are a series of European standards, aimed at design of load-bearing structures , e.g.

- Eurocode 0 (EC0) EN 1990-x-y … Basis of design
- Eurocode 1 (EC1) EN 1991-x-y … Loads
- Eurocode 2 (EC2) EN 1992-x-y … Concrete structures
- Eurocode 3 (EC3) EN 1993-x-y … Steel structures
- Eurocode 4 (EC4) EN 1994-x-y … Steel-concrete composite str.
- Eurocode 5 (EC5) EN 1995-x-y … Timber structures

…

- The target is uniform design philosophy and notation
- Include Nationally Determined parameters (NDP) which are given in National Annexes (NA)
- In Finland they will replace the old Finnish codes latest 1.4.2011?

In this course the methods of EC2 will be used when design codes are needed. The safety factors and other NDP:s will be given if they are needed in the exams.

<u>About notation in Eurocodes:</u>

- Subscript *c* refers to concrete, *y* or *s* to mild and *p* to prestr. steel
- Subscript *m* refers to mean, *k* to characteristic and *d* to design value
- Subscript *E* refers to action effect caused by load (action **E**ffect), *R* to resistance of structure (**R**esistance)
- Symbols for strength sand safety factor are f and γ, respectively.

<u>Example</u>. The design value for tensile strength of concrete is $f_{ctd} = \alpha_{ct} f_{ctk} / \gamma_c$

<u>Example</u>. The design value for compr. strength of concrete is $f_{cd} = \alpha_{cc} f_{ck} / \gamma_c$

Note. In Finland α_{ct} = 1,0 and α_{cc} = 0,85

<u>Example</u>. The design value for steel strength is $f_{yd} = f_{yk} / \gamma_s$

<u>Example.</u> The design value of shear force due to load must not exceed the design value of shear resistance: $V_{Ed} \le V_{Rd}$

<u>Basics of Eurocode design principles</u>

The design value of the resistance, R_d, shall be higher than the design value of action effect, E_d, or $E_d \le R_d$. The design variables X are characterised by one or more of the following values see Fig. 67,

- mean value x_m or nominal value
- lower characteristic value $x_{k,low}$, which corresponds to a certain small fractile, e.g. 5%
- upper characteristic value $x_{k,up}$, which corresponds to a certain high fractile, e.g. 95%
- instead of characteristic value, a nominal value is sometimes used.

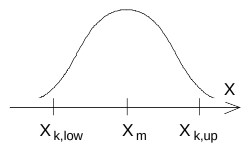


Fig. 67.

E_d is determined by actions G_i (permanent, one of them may be prestressing P), Q_i (variable action) ja A_i (accidental action).

In ordinary ultimate limit states (ULS) the following combination is considered:

$$E_d = \sum_{j\ge1} \gamma_{G,j} G_{k,j} \text{"+"} \gamma_P P \text{"+"} \gamma_{Q,1} Q_{k,1} \text{"+"} \sum_{i>1} \gamma_{Q,i} \psi_{0,i} Q_{k,i}$$

"+" means combination of actions
Σ means the combined action of variables.

γ, G_k and Q_k refer to a safety factor and characteristic value of permanent and variable action, respectively. ψ-factors are combination factors making allowance for the fact that the maxima of all loads seldom occur simultaneously.

In the serviceability limit states (SLS) the following three combinations are considered:

1. <u>Quasi-permanent</u> (e.g. when considering creep, shrinkage, aesthetics)
2. <u>Frequent</u> (in general for reversible effects like elastic deflection)
3. <u>Characteristic</u> (in general for irreversible effects)

The corresponding combinations are

$$E_d = \sum_{j\ge1} G_{k,j} \text{"+"} \text{"}P\text{"} \text{"+"} \sum_{i\ge1} \psi_{2,i} Q_{k,i}$$ quasi permanent

$$E_d = \sum_{j\ge1} G_{k,j} \text{"+"} \text{"}P\text{"} \text{"+"} \psi_{1,1} Q_{k,1} \text{"+"} \sum_{i>1} \psi_{2,i} Q_{k,i}$$ frequent

$$E_d = \sum_{j\ge1} G_{k,j} \text{"+"} \text{"}P\text{"} \text{"+"} Q_{k,1} \text{"+"} \sum_{i>1} \psi_{0,i} Q_{k,i}$$ characteristic

$Q_{k,1}$ is the leading variable action.

In Finland, two cases are considered in the ultimate limit state. First, the unfavourable permanent actions are multiplied by 1,15K_{FI}, the favourable ones by 0,9 and the variable actions by 1,5K_{FI}. Second, only the permanent actions are considered and the unfavourable ones multiplied by 1,35 K_{FI}.

$$E_d = 1{,}15 K_{FI} \sum G_{k,j,sup} \text{"+"} 0{,}9 \sum G_{k,j,inf} \text{"+"} 1{,}5 K_{FI} Q_{k,1} \text{"+"} 1{,}5 K_{FI} \sum_{i>1} \psi_{0,i} Q_{k,i}$$

$$E_d = 1{,}35 K_{FI} \sum G_{k,j,sup} \text{"+"} 0{,}9 \sum G_{k,j,inf}$$

Factor K_{FI} depends on the reliability class RC3 … RC1 as shown in the Table.

RC	K_{FI}
RC3	1,1
RC2	1,0
RC1	0,9

E.g. the design load combination for a uniformly distributed live load *q* ja permanent uniformly distributed load *p* is

$1{,}15K_{FI}p + 1{,}5K_{FI}q$ or
$1{,}35K_{FI}p$

whicchever is greater. This can be expressed as
$\gamma(p + q)$

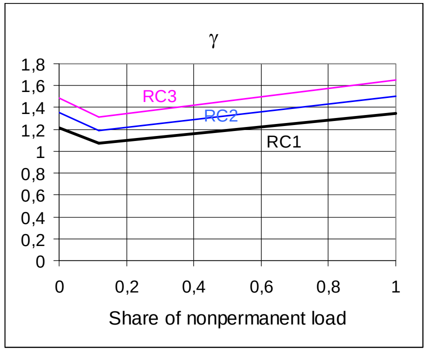


Fig. 68.

Fig. 68 illustrates γ (in Finland only) for reliability classes RC1 … RC3

The ψ-factors to be used in Finland are given by YM in YM:n asetus Eurocode-standardien soveltamisesta talonrakentamisessa 15.10.2007, which is available in www.ymparisto.fi. In the same document, all other nationally determined parameters of the Eurocodes like partial factors γ are given.
Examples of combination factors applied in Finland for imposed loads:

	ψ_0	ψ_1	ψ_2
Residential and office buildings	0,7	0,5	0,3
Storages	1,0	0,9	0,8
Snow (s_k < 2,75 kN/m²)	0,7	0,4	0,2
Snow ($s_k \geq$ 2,75 kN/m²)	0,7	0,5	0,2
Wind	0,6	0,2	0

The combination factors do not affect the ULS if there is only one imposed load. In the SLS the load combination has to be chosen by the designer because the Eurocodes do not specify it uniquely, even though some limits are given (e.g. L/250 for the deflection, L/500 for the deflection after construction unless a smaller deflection is required for other reasons).

For large loaded areas, certain reductions are allowed but they are not discussed here.

In the ULS, the design value of the material strength is typically obtained dividing the lower characteristic or nominal strength by the relevant safety factor. For specific reasons, e.g. due to the long-term effects of the loads, difference in the actual strength of the material in the structure and that measured from test specimens, differences in the shape and size of the test specimens etc., additional factors may be necessary. E.g. the concrete strength is obtained from $f_{cd} = \alpha_{cc}f_{ck}/\gamma_c$, where α_{cc} = 0,85 (in Finland).

In Finland, the safety factor for the concrete and reinforcing & prestressing steel are 1,5 and 1,15, respectively. In certain cases these may be reduced to 1,35 ja 1,10. This is posssible e.g. for CE-marked concrete elements if they meet certain strict tolerance requirements.

In the SLS the steel remains elastic, but the concrete may crack. When searching for the transition from uncracked to cracked section for evaluation of the deflection, when estimating the crack width or calculating the minimum amount of reinforcemen to prevent a brittle failure, the mean value of concrete tensile strength f_{ctm} is used.

The density and elasticity modulus of materials are considered so accurately known that the mean or nominal values are used in the design instead of characteristic ones both in the SLS and ULS.

EC2 is not so consequent as one might expect. E.g. the basic value l_{pt} for the transfer length is calculated from an expression which includes the safety factor of the concrete, even though it is between the two values $0{,}8l_{pt}$ and $1{,}2l_{pt}$ used in the design. In other words, the *mean value is calculated using the safety factor*.

Another example is the shear formula

$$V_{Rd,c} = \left[\frac{0{,}18}{\gamma_c}k(100\,\rho_1 f_{ck})^{(1/3)} + k_1\sigma_{cp}\right]b_w d$$

The safety factor is applied to constant 0,18, not to the strength of the concrete f_{ck} etc.

About the materials of prestressed concrete structures

Example. Assume concrete C20/25 , relative humidity 50 %, the notional thickness of the beam h_0 = 200 mm, cement class N, age at prestressing 28 d

Nominal drying shrinkage $\varepsilon_{cd,0}$ = -0,00054, drying shrinkage in (28 d,∞) is

$\varepsilon_{cd}(\infty)$-$\varepsilon_{cd}(28d)$ = -0,85x0,00054(1-0,20) = -0,00039

Creep coefficient φ = 2,8. Concete stress after tensioning = -0,45x20 MPa = -9 MPa $\Rightarrow$ creep is

$\varepsilon_{cc}(\infty) = \varphi\sigma_c / E_c$ = 2,8x(9/30000) = -0,00084

Loss due to creep and shrinkage is approximately

$\Delta\sigma_p$ =(200 000)x(-0,00039-0,00084) MPa = - 246 MPa

This is not far from the typical strength of the reinforcing steel in the days when the first attempts were made for prestressing. When the losses due to the friction and relaxation are added, it is no wonder that the first attempts to prestress did not succeed.

Prestressing tendons are made of wires with thickness of a few millimetres, of strands made of the wires, and of bars (seldom). The surface of the wires may be smooth or indented to improve the bond. Due to the lack of a clear yield strength, the strength is given in form $R_{p,0,2}/R_m$, where $R_{p,0,2}$ is the stress corresponding to a 0,2 % permanent strain and R_m is the failure strength. Fig. 67 shows a typical stress-strains relationship for strand 1640/1860 and the design assumptions made in B4 and EC2. For more details, see Fig. 68.

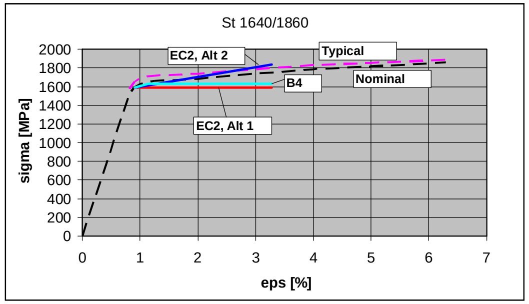


Fig. 67.

The elasticity modulus E_p of the wires and bars used in the design may be taken as 205 GPa in the EC2, but it may vary 195 – 210 GPa. Therefore is is better to assume a lower value, say 200 GPa which is the same value as for reinforcing bars. For strands, the value depends on the tightness of the strand. It is typically 185 – 200 GPa. It is given by the manufacturer. Since the manufacturer is not known in the design stage, it is most feasible to use a relatively low value, say 190 GPa in the design. (EC2: E_p = 195 GPa can be used.) A steel with low relaxation is recommended to reduce the prestresssing losses.

To reduce the losses, a relatively strong concrete is used. This keeps the elasticity modulus high, creep and shrinkage low. Rapidly hardening cement makes an early prestressing possible, which is important in factory production. Strength classes lower than C40/50 are seldom used in factories and on site only if C40 is not likely to be easily achieved.

The characteristic values of 0,1% yield limit ($f_{p0,1k}$) and strength (f_{pk}), see Fig. 68, should be known for the prestressing steel when the material model of EC2 is applied. Instead of these, $R_{p0,1}$ and R_m, with a slightly different meaning, are used. For E_p, either the certified value, or if it is not known, 195 GPa for the strands and 205 GPa for the wires and bars are used.

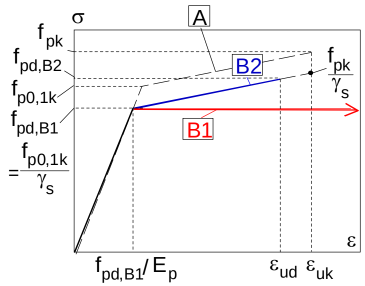


Fig. 68. EC2. A: Idealised curve. B: Design (two alternatives).

The characteristic value of the failure strain, ε_{uk} , is generally not known. Instead, the minimum failure strain is used. E.g. ε_{uk} = 3,5 % may be used, if the failure strain is guaranteed to be at least 3,5% by the manufacturer. ε_{ud} may be nationally determined. In Finland ε_{ud} = 2,0%. Either a bilinear, strain-hardening curve with limit strain ε_{ud} ,or a bilinear elastic-plastic curve without strain limit may be chosen. Later on, yielding of steel means exceedance of the 0,1% yield limit.

Failure mechanisms of a prestressed concrete beam

Fig. 71 depicts possible failure mechanisms of a prestressed concrete beam. The failure mechanisms due to excessive tendon force, inadequate liftings and transport are not considered here.

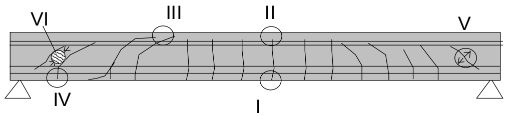


Fig. 71. Failure mechanisms

I. Bending tension failure. Yielding of steel is followed by crushing of concrete in the compressed zone.
II. Bending compression failure. Crushing of concrete in the compression zone before the steel yields.
III. Bending shear failure. A flexural crack turns into an inclined crack due to the shear force. The concrete fails above the upper end of the crack before steel yields.
IV. Anchorage failure. May take place after cracking in bending.
V. Shear tension failure in web. In absence of adequate shear reinforcement, a crack appears in the web and a failure immediately follows.
VI. Shear compression failure in web. With strong reinforcement, a concrete compression strut between diagonal cracks may fail.

General observations about bending

Fig. 72 illustrates the change of curvature of a prestressed section ($\Delta\kappa$) with imposed moment(M). Before cracking the response is (almost) linear (stage I). The steel stress increases slightly. After cracking in stage II the stress increases both in the tensioned steel and in the compressed concrete. In the plastic stage (III), the steel is yielding and the curvature increases rapidly with the moment. Even though the increase in steel stress is slow, the moment can increase because the depth of the compression zone reduces and the inner lever arm grows.The smaller the amount of the steel, the more can the inner lever arm grow. If the ultimate compressive strain of the concrete (ε_{cu}) is achieved at the same time when the steel starts to yield, we have a balanced failure. If the steel yields before (after) ε_{cu} is achieved, we have a normally reinforced (overreinforced) section, respectively. In a balanced or overreinforced section the plastic stage does not exist.

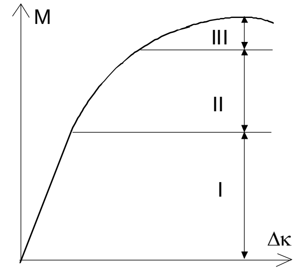


Fig. 72. I. Elastic stage. II. transition stage. III. Plastic stage.

The steel yields ≠ M_{max} has been achieved.

Relationship between strains and stresses in concrete and steel

In the ultimate limit state stress resultants of the the concrete and steel are calculated starting from the σ–ε-relationship and compatibility of strains. The different types of strain affect the stress in different ways. Calculate first the steel stress when the elastic strain $\varepsilon_{ce}$and stress σ_c of the concrete are = 0. At time t = 0 the length of the member shown in Fig. 73 is = $L(0)$. σ_c = 0 and the initial (prestress) of the steel = σ_{p0}. At time t the contraction of the member ddue to shrinkage, creep and elastic deformation (release of prestress) is = $\Delta L_{cs} + \Delta L_{cc} + \Delta L_{ce}$. After pulling the member by an amount of $-\Delta L_{ce}$, the elastic strain of the concrete is = 0, σ_c = 0 and $\sigma_p = \beta\sigma_{p0}$, where the factor β represents the losses due to the creep, shrinkage and relaxation.

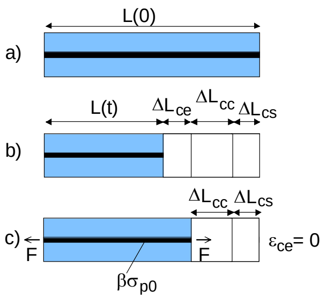


Fig 73.

Note. ΔL_{cs} , ΔL_{cc}, ja ΔL_{ce} include the effect of the external forces, too.

In a postensioned structure (Fig. 74) consider first only the case with one tendon concentrating on the tensioning only.

Right after the tensioning the steel stress is = $\beta_{KL}\sigma_{p0}$, where β_{KL} represent the effect of friction and locking losses.When the member is pulled by an amount of $-\varepsilon_{ce,P}$,i.e. the elastic deformation of the concrete is eliminated, the steel stress corresponding to the elastic zero deformation of the concrete is = $\beta_{KL}\sigma_{p0} - E_P\varepsilon_{ce,P}$. Taking into account the creep, shrinkage and relaxation, the steel stress corresponding to the elastic zero deformation of the concrete becomes $\beta\sigma_{p0} - E_P\varepsilon_{ce,P}$, where β represent the reduction factor due to all losses.

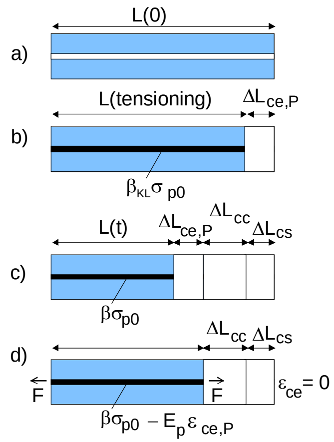


Fig. 74.

The external loads affect the losses and the losses affect the reduction factor β. The loads also affect the elastic deformation of the concrete but the elastic deformation does not affect the compatibility of the steel and the concrete. This is the same situation as in the axially prestressed and loaded member.

If the tendons are prestressed in several stages, only the elastic strain due to the own prestress and that of the other tendons prestressed simultaneously are taken into account in $\varepsilon_{ce,P}$ of each tendon. The prestress in the other tendon affect β in the same way as the external loads, i.e. only by the losses. So we can write

$\sigma_P^0 = \beta\sigma_{P0} - E_P\varepsilon_{ce,P}$ for post-tensioned tendons

$\sigma_P^0 = \beta\sigma_{P0}$ for pretensioned tendons

Here $\varepsilon_{ce,P}$ is the elastic strain of the concrete at the depth of the considered tendon. It includes only the effect of the tendons prestressed simultaneously with the tendon considered. Elastic zero-strain ε_P^0 of the concrete is obtained from

$$\varepsilon_P^0 = \frac{\sigma_P^0}{E_P}$$

In general it is safe to assume that $\varepsilon_{ce,P} = 0$.

In the actual structures there are often several tendons, passive reinforcement, bending moment etc. The following procedure can always be applied:

1. In each tendon or rebar, determine the streel strain ε_P^0 which corresponds to the elastic zero strain of the concrete $\varepsilon_{ce} = 0$.
2. Change the elastic strain of the concrete on the most compressed edge of the beam and at the depth of the outermost tendon on the opposite side. Based on these strain changes, calculate the strain changes $\Delta\varepsilon_P$ at the depth of each rebar or tendon.
3. Superimpose the srains from 1. and 2. , pick the corresponding stress from σ–ε-relationship and calculate the steel force in each tendon and rebar.
4. Based on the elastic strain diagram, calculate the compressive force in the concrete.
5. If the inner and outer forces N and moments M in the section are not in equilibrium, vary the elastic edge strains until equilibrium has been achieved . The bending resistance can be determined as explained later.

Fig 75 illustrates the means to calculate the steel stress when the assumptions of EC2 are used to the prestressing steel.

In case of design assumption B1, the total strain ε_P does not affect the steel stress when it is > yield strength ε_{yd}. This is the aim in normal design. The small role of $\varepsilon_{ce,P}$ also becomes obvious. It cannot be too significant, because it is only a small share of the elastic strain and a very small share of the total strain ε_P in the ultimate limit state.

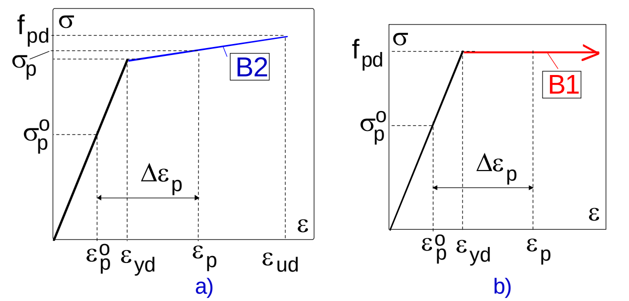


Fig. 75.

Calculation of bending resistance

Fig. 76 depicts the strain diagram of the concrete in ultimate limit state. The strain of the top fibre is set = failure strain ε_{cu} (= ε_{cu3} in EC2). The strain change $\Delta\varepsilon_p$ in the steel due to the concrete strain is assumed equal to the concrete strain ε_{cpp} at the depth of the steel. The total strain of the steel is

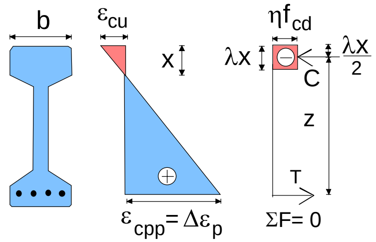


Fig. 76.

$$\varepsilon_P = \frac{\sigma_P^0}{E_P} + \varepsilon_{cpp}$$

where the meaning of σ_P^0 has been explained before.

EC2: The concrete stress distribution is replaced by a rectangular stress block with depth λx (Fig. 76). The stress is ηf_{cd} .($\eta \le 1$ ja $\lambda \le 0{,}8$ from EC2). Set first $\Delta\varepsilon_p = \varepsilon_{cpp}$ ($\Rightarrow \varepsilon_p \Rightarrow \sigma_p$) in such a way that, $\sigma_p = f_{pyd}$. Solve for $x \Rightarrow$ stress resultant of concrete $C = b\lambda x\eta f_{cd}$ and that of steel = $T = A_p f_{pyd}$.

There are two cases:

1. $C > T$ (bending tension failure): Increase ε_{cpp}, until $C = T$. The bending resistance of the section is $M_{Rd} = Tz$.

Special case 1b: When applying the strain-hardening curve (B2 in Fig. 75) and T corresponding to strain ε_{ud} is $< C$, keep the steel strain ε_p constant (= ε_{ud}) and reduce the strain in the top fibre until $C = T$. $M_{Rd} = Tz$.

2. $C < T$ (bending compression failure): keep the strain of the top fibre constant = ε_{cu} and reduce ε_{cpp} until $C = T$. $M_{Rd} = Tz$.

Note.1. If an elastic-plastic design curve is chosen and the section is not overreinforced, $C = T = A_p f_{pd} = b\eta f_{cd}\lambda x \Rightarrow \lambda x \Rightarrow$
$M_{Rd} = A_p f_{pd} z = A_p f_{pd}(d - 0{,}5\lambda x)$

Note.2. If there are several steel layers, their strains are calculated from the assumed strain diagram and the stresses and forces from the strains. All steel and concrete forces on a section must be in equilibrium. The equilibrium is obtained by varying the strains as above.

Note.3. The method is the same as for reinforced concrete section except that the steel strain corresponding to the initial prestress must be taken into account.

Shear design

A part of the shear design methods are based on experimental research. Applying such methods outside the experimentally verified range is problematic. This is particularly true for the cracked zone of a beam without shear reinforceement. On the other hand, if the beam has shear reinforcement, it is difficult to know whether the different stirrups yield simultaneously etc. The shear design method of EC2 has been criticised. Despite the criticism, the method is presented below.

Notation (EC2):

$V_{Rd,c}$ shear resistance of beam without shear reinforcement
$V_{Rd,s}$ shear resistance based on yielding of shear reinforcement
$V_{Rd,max}$ upper limit for shear resistance based on compression resistance of concrete strut

In the cracked state, $V_{Rd,c}$ is obtained from

$$V_{Rd,c} = [C_{Rd,c}k(100\rho_l f_{ck})^{1/3} + k_1\sigma_{cp}]b_w d \qquad \text{EC2: (6.2a)}$$

However, $V_{Rd,c}$ is at least

$$V_{Rd,c} = \left[v_{min} + k_1\sigma_{cp}\right]b_w d \qquad \text{EC2: (6.2b)}$$

$C_{Rd,c}$ (recommended value = 0,18 MPa/γ_c), k_1 (rec. value = 0,15) and v_{min} (rec. value = $0{,}035k^{3/2}f_{ck}^{1/2}$) are nationally determined parameters. The meaning of other symbols:

f_{ck} characteristic strength of concrete (e.g. concrete C40: f_{ck} = 40 MPa)

$$k = \min\left\{1 + \sqrt{\frac{200\,mm}{d}}; 2\right\}$$

d is the effective depth of the section

$$\rho_l = \min\left\{\frac{A_{sl}}{b_w d}; 0{,}02\right\}$$

A_{sl} cross-sectional area of reinforcement which is anchored to carry the moment calculated around the upper end of inclined crack
b_w minimum width of tensile zone of section
σ_{cp} = min{N_{Ed}/A_c ; $0{,}2f_{cd}$}, A_c is the cross-sectional area of concrete and N_{Ed} external normal force or prestressing force, compressive $N_{Ed} > 0$.

Note. Formula (6.2a) of EC2 may be valid for small bending moments but it overestimates the shear resistance when the bending moment is high (steel stress close to yielding), at least for deep hollow core slabs. This is possible because the bending moment is totally ignored.

A section is considered cracked if the tensile strength of the top or bottom fibre equals $0{,}70f_{ctm,fl}/\gamma_c$ (= $f_{ctk,fl}/\gamma_c$) where

$$f_{ctm,fl} = \max\left\{\left(1{,}6 - \frac{h}{1000\,mm}\right)f_{ctm}; f_{ctm}\right\}$$

Here f_{ctm} is the mean tensile strength of the concrete and h the depth of the section. This criterion is in accordance with the principles of EC2 even if is is not directly given there.

In uncracked state $V_{Rd,c}$ is obtained from

$$V_{Rd,c} = \frac{Ib_w}{S}\sqrt{f_{ctd}^2 + \alpha_l\sigma_{cp}f_{ctd}} \qquad \text{EC2: (6.4)}$$

I, b_w ja S are the second moment of area, the web width and the first moment of area of concrete section, respectively, calculated with respect to the centroidal axis. $f_{ctd} = f_{ctk}/\gamma_c$ is the design value of the tensile strength of the concrete. $\sigma_{cp} = +P_{Ed}/A_c$, where P_{Ed} is the design value of the prestressing force. $\alpha_l = l_x/l_{pt2}$, where l_x is the horizontal distance from the considered point to the end of the beam, and l_{pt2} is the transfer lenght of the prestressing force.

Expression 6.4 means that the maximum principal stress in the web is set equal to the design value of the tensile strength. Fig. 77.a shows the relevant stress components. σ_1 is attributable to the prestressing force, τ to the external shear force and σ_2 to the bearing pressure. The (positive) effect of σ_2 vanishes rapidly when distancing from the bearing. So the resistance is determined by $\bar{\sigma}_1$ and τ only. This results in Eq. (6.4) which is applied in the critical section of Fig. 73.b. The worst drawback of (6.4) is the fact that it ignores the effect of the varying prestress on τ, which results in overestimation of the shear resistance of some hollow core slab types by tens of percent. For this reason *(6.4) should not be used.*

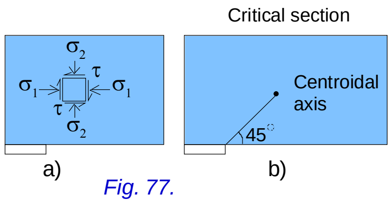


Fig. 77.

On pages16 – 17 equation $\tau = VS/(Ib)$ has been deduced. A constant normal force was assumed, i.e. $dN/dx = 0$. When the normal force is not constant as at the end of a pretensioned beam, special considerations are necessary.

Fig. 78 depicts a pretensioned simply supported beam with n tendon layers. A free body diagram, cut from the beam, is shown in Figs 79 – 80. It is loaded horizontally by axial stress σ, shear stress τ and the resultant P_{cp} of the tendon forces above the horizontal cut. Consider separately the concrete stresses due to release of the prestress (subscript $_{c,P}$) and external load (subscript $_{c,V}$ for shear force ja $_{c,M}$ for bending moment).

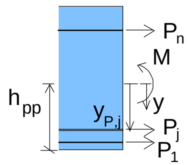


Fig. 78.

a. External load:

$$\sigma_{c,M} = \frac{M}{I_m} y \Rightarrow$$

$$\Delta\sigma_{c,M} = \frac{d\sigma_{c,M}}{dx}\Delta x = \frac{1}{I_m}\frac{dM}{dx} y \Delta x = \frac{Vy}{I_m}\Delta x$$

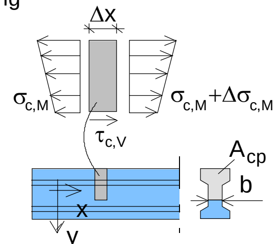


Fig. 79.

Horizontal Equilibrium of the cut in Fig. 79:

$$\tau_{c,V} b \Delta x = -\int_{A_{cp}} \Delta\sigma_{c,M} dA = -\int_{A_{cp}} \frac{d\sigma_{c,M}}{dx}\Delta x dA$$

$$= -\Delta x \frac{V}{I_m}\int_{A_{cp}} y dA \quad \Bigg| \quad : b\Delta x \text{ ja merkitään } S = -\int_{A_{cp}} y dA \ (>0) \quad \Rightarrow \tau_{c,V} = \frac{VS}{I_m b}$$

b. Tansfer of prestressing force:

Fig. 80 shows the horizontal actions due to the prestressing force acting on the free body. They are axial stress σ, horizontal shear stress τ_{cp} and the resultant P_{cp} of the tendon forces penetrating the cut.

$$\sigma_{c,P} = \frac{-\sum_j P_j}{A_m} + \frac{-\sum_j P_j y_{P,j}}{I_m} y$$

$$\frac{d\sigma_{c,P}}{dx} = -\frac{1}{A_m}\sum_j \frac{dP_j}{dx} - \frac{y}{I_m}\sum_j \frac{dP_j}{dx} y_{P,j}$$

$$\Delta\sigma_{c,P} = \frac{d\sigma_{c,P}}{dx}\Delta x \qquad \Delta P_{c,P} = \frac{dP_{c,P}}{dx}\Delta x$$

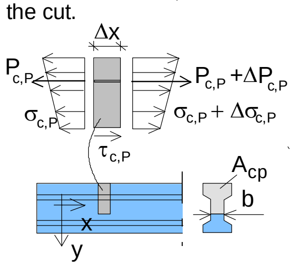


Fig. 80.

Horizontal equilibrium of the cut in Fig. 80:

$$\tau_{c,P} b \Delta x = -\int_{A_{cp}} \Delta\sigma_{c,P} dA - \Delta P_{cp} = -\int_{A_{cp}} \frac{d\sigma_{c,P}}{dx}\Delta x dA - \frac{dP_{cp}}{dx}\Delta x \quad \Bigg| \quad : b\Delta x \quad \Rightarrow$$

$$\tau_{cp} = -\frac{1}{b}\int_{A_{cp}} \frac{d\sigma_{c,P}}{dx} dA - \frac{1}{b}\frac{dP_{cp}}{dx}$$

$$= -\frac{1}{b}\int_{A_{cp}} \left(-\frac{1}{A_m}\sum_j \frac{dP_j}{dx} + -\frac{y}{I_m}\sum_j \frac{dP_j}{dx} y_{P,j} \right) dA - \frac{1}{b}\frac{dP_{cp}}{dx}$$

$$= \frac{1}{b}\frac{1}{A_m}\sum_j \frac{dP_j}{dx}\int_{A_{cp}} dA + \frac{1}{bI_m}\sum_j \frac{dP_j}{dx} y_{P,j}\int_{A_{cp}} y dA - \frac{1}{b}\frac{dP_{cp}}{dx} \quad \Bigg| \quad \text{Notation: } S = -\int_{A_{cp}} y dA \ (>0)$$

$$= \sum_j \frac{1}{b}\frac{dP_j}{dx}\left(\frac{A_{cp}}{A_m} - \frac{S}{I_m} y_{P,j}\right) - \frac{1}{b}\frac{dP_{cp}}{dx}$$

If there is only one tendon layer and it is below the cut (free body):

$$\tau_{cp} = \frac{1}{b}\frac{dP}{dx}\left(\frac{A_{cp}}{A_m} - \frac{S}{I_m} y_P\right) \qquad \text{Superposition:} \quad \tau = \tau_{cV} + \tau_{cP}$$

Only one tendon layer: $\tau = \frac{1}{b}\left[\left(\frac{A_{cp}}{A_m} - \frac{Sy_p}{I_m}\right)\frac{dP}{dx} + \frac{S}{I}V\right]$ (Yang 1994)

A_{cp} and S_{cp} are the area and first moment of the area above the considered axis. S_{cp} is calculated with respect to the centroidal axis of the section. One consequence of Yang's formula is the fact that the maximum principal stress is not necessarily close to the centroidal axis. For this reason, the bending stresses must also be taken into account when calculating the normal stress σ_1. E.g. the critical depth for a section shown in Fig. 81 is located at a level where the web with constant width and the lower chamfering meet.

So: τ is calculated from Yang's formula or from generalized Yang's formula and σ_1 from

$$\sigma_1 = \frac{N}{A_m} + \frac{M}{I_m} y$$

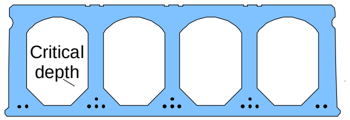


Fig. 81.

N and M include both the external actions and the actions due to prestress.

From τ and σ_1 the principal stress is calculated and compared with the tensile strength. The method is presented in detail in the first amendment (2008) to standard EN 1168 which is the European harmonized product standard for hollow core slabs.

Transfer of prestressing force and anchorage in pretensioned tendons

After the release, the stress at the end of a pretensioned tendon is = 0. At a certain distance l_{pt} from the end, there is no slip between the tendon and concrete and the prestress has reached its full value σ_{p0} . Fig. 82 illustrates the gradual increase of the tendon stress. It is generally believed that the stress gradient decreases with x as in curve A in Fig. 82. However, to simplify the calculations, the linear curve B shown in Fig. 82 is generally adopted in the design codes.

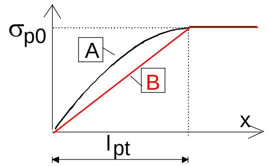


Fig. 82. Transfer of prestressing force. A. Assumed. B. Idealized.

EC2:The design value of bond strength f_{bpt} and the basic value of transfer length l_{pt} are obtained from

$$f_{bpt} = \eta_{p1}\eta_1 f_{ctd}(t_0) \qquad l_{pt} = \alpha_1\alpha_2\phi\frac{\sigma_{pm0}}{f_{bpt}}$$

$f_{ctd}(t_0)$ is the design value of concrete tensile strength at release ($t = t_0$)
ϕ diameter of tendon
σ_{pm0} steel stress immediately after release
η_i and α_i constants

EC2: Design according to Fig. 83 .

l_{pt1} = 0,8l_{pt},
l_{pt2} = 1,2l_{pt},
l_{pbd} distance needed for anchorage of stress σ_{pd},
σ_{pi} initial prestress,
$\sigma_{p\infty}$ prestress after losses,
σ_{pd} steel stress to be anchored,
x distance to the point where the transfer begins,

$$l_{pbd} = l_{pt2} + \alpha_2\phi(\sigma_{pd} - \sigma_{p\infty})/f_{bpd}$$

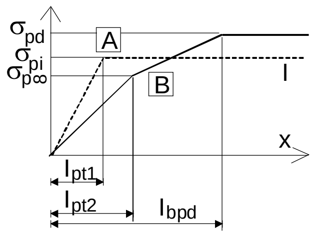


Fig. 83. A. Transfer of prestress when l_{pt1} is prevailing. B. Anchorage (l_{pt2} is prevailing).

Note. In the design, l_{pt1} or l_{pt2} is chosen, depending on which one gives the critical effect. The unfavourable effects of the tendon force, like cracking of the beam end, must be evaluated using l_{pt1}, the favourable ones using l_{pt2}.

Note. Anchorage resistance is checked only for cracked sections. A section is considered cracked if the tensile stress calculated using elementary beam theory exceeds the value $0{,}70 f_{ctm,fl}/\gamma_c$, see p. 132.

Note. The force to be anchored is calculated from the moment around the upper end of the crack, see Fig. 84. When there is no shear reinforcement and the dowel action of the main reinforcement is ignored, σ_{pd} is obtained from the equilibrium of the inner and outer bending moment around point O

$$M_{Ed}(x + z\cot\theta) = \sigma_{pd}A_p z$$

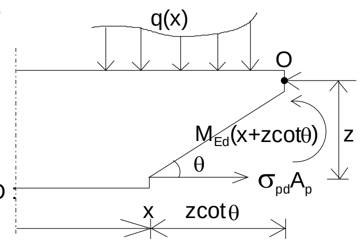


Fig. 84.

Angle θ is chosen in such a way that $1 \le \cot\theta \le 2{,}5$ or $22° \le \theta \le 45°$.

A small value of θ results in a conservative value for the anchorage resistance. This is necessary for a beam without shear reinforcement.

The shear resistance of a beam with shear reinforcement is in EC2 calculated using truss analogy. Simultaneous yielding of all stirrups crossing the inclined crack is assumed. The concrete serves as compression diagonals and chords, the shear reinforcement works as vertical or inclined ties and the main reinforcement as tensile chords.

Fig. 85 illustrates the model. θ is the angle between the shear cracks and horizontal axis (gives the direction of the compressed diagonal). α is the angle between the stirrups and the horizontal axis.

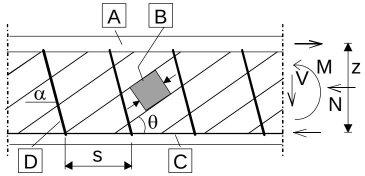


Fig. 85. A. compressed chord. B. Compr. diagonal. C. Tensile chord. D. Stirrup.

The shear resistance is now

$V_{Rd} = \min\{V_{Rd,s};V_{Rd,max}\}$ where

$V_{Rd,s} = \dfrac{A_{sw}}{s} z f_{ywd}(\cot\theta + \cot\alpha)\sin\alpha$ (yielding of stirrups)

$V_{Rd,max} = \alpha_{cw} b_w z \nu_1 f_{cd}(\cot\theta + \cot\alpha)/(1+\cot^2\theta)$ (compression failure of diagonals)

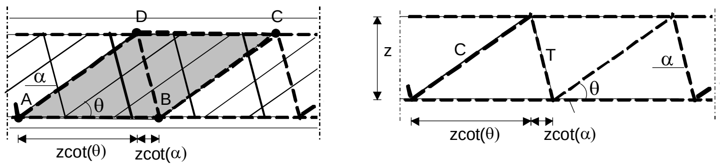
A_{sw}	is cross-sectional area of stirrup,
s	horizontal spacing of stirrups,
z	inner lever arm,
f_{ywd}	design strength of shear reinforcement,
b_w	minimum web width of beam, the zone below the stirrups is not considered
f_{cd}	design strength of concrete,
ν_1	coefficient, allows for the effect of shear on the concrete strength
α_{cw}	coefficient, allows for the effect of axial compression

	α_{cw}
No prestress	$1{,}00$
$0 \le \sigma_{cp} \le 0{,}25 f_{cd}$	$1+\sigma_{cp}/f_{cd}$
$0{,}25 f_{cd} < \sigma_{cp} \le 0{,}5 f_{cd}$	$1{,}25$
$0{,}5 f_{cd} < \sigma_{cp} \le f_{cd}$	$2{,}5(1-\sigma_{cp}/f_{cd})$

f_{ck}	ν_1
≤ 60 MPa	0,6
> 60 MPa	$\max\{0{,}9 - f_{ck}/(200\,\text{MPa}); 0{,}5\}$

σ_{cp} is the average axial stress in the concrete due to the design value of the normal force (compressive stress positive).

The shear model of EC2 is based on the truss analogy illustrated in Fig. 85. The compression diagonals are concrete struts parallel to the cracks and the tensile diagonals are stirrups, both connecting the tensile chords with the compression chords. All stirrups crossing line AB belong to tie T and all oncrete struts crossing line AB belong to the compression strut C as shown in Fig. 86.b .



a) Structure.

b) Mechanical model.

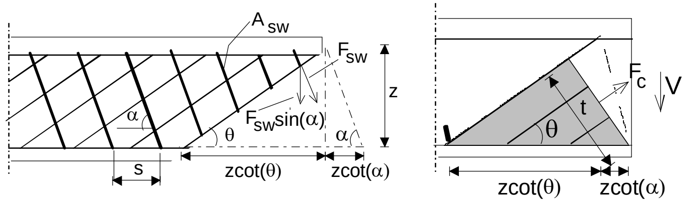
Fig. 86.

The resistance of the stirrups can be solved assuming an inclined section shown in Fig. 87.a. The cross-sectional area A ja yield force F_{sw} of the stirrups cross-sing the section are

$A = \dfrac{A_{sw}}{s} z(\cot\theta + \cot\alpha)$ $F_{sw} = A f_{ywd}$

The final expression is obtained by setting the shear force equal to $F_{sw}\sin\alpha$, the vertical component of F_{sw}.

The resistance of the compression strut can be solved from Fig. 87.b. If the concrete strength is = $\alpha_{cw}\nu_1 f_{cd}$, The resultant of the compressive forces parallel to the cracks is $F_c = \alpha_{cw}\nu_1 f_{cd}\, t b_w$. The vertical component of F_c is = $F_c\sin\theta$ and $t = z(\cot\theta+\cot\alpha)\sin\theta$. The expression of EC2 follows by setting $F_c\sin\theta$ equal to the shear force.



a)

b)

Kuva 87.

b_w is replaced by $b_{w,nom}$ if there are post-tensioned tendons in the web.

$b_{w,nom} = b_w$ for metallic, grouted ducts with $\phi \le b_w/8$,

$b_{w,nom} = b_w - 0{,}5\Sigma\phi$, for metallic, grouted ducts, $\phi > b_w/8$,

$b_{w,nom} = b_w - 1{,}2\Sigma\phi$, unbonded tendons, ungrouted ducts, non-metallic ducts even if they are grouted

$b_{w,nom}$ is calculated at the most unfavourable depth.

About unbonded tendons

To facilitate the grouting, the duct diameter must be greater than that of the tendon. To prevent corrosion, a metallic duct must be covered with concrete, the thickness of which depends on the exposure class. Post-tensioning presses a curved tendon against the duct or far from the nearest (top or bottom) fibre. Result: In a slab or shallow beam, a considerable share of the depth of the section cannot be effectively exploited.

The unbonded tendons are not grouted. Therefore, the outer diameter of the duct is only slightly greater than that of the tendon. Given a depth of the

beam or slab, the curvature can be made greater than that of the grouted tendons, which is a considerable advantage in slabs. In a flat slab (floor without beams), it is easier to fit the unbonded tendons above the columns than the bonded ones. The cost of grouting is saved. For these reasons, the unbonded tendons are widely used in flat slabs.

Fig. 88 illustrates a slab supported along four edges and prestressed by parabolic tendons of which only the midmost ones are shown. In *x*-direction, the transverse forces result in a vertical line load q_x, in *y*-direction load q_y. Uniformly distributed tendons cause a vertical uniformly distributed load *q*.

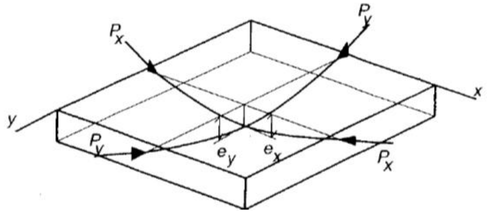


Fig. 88. Post-tensioned slab [BY210].

$$q = q_x + q_y = 8\frac{p_x e_x}{L_x^2} + 8\frac{p_y e_y}{L_y^2},$$

L_x and L_y are the span widths, $p_x = P_x/s_x$ ja $p_y = P_y/s_y$, s_x ja s_y denote the spacing of tendons in *y*- and *x*-direction.

Since an unbonded tendon is fixed to the concrete only at the ends, its strain does not follow the strain of the concrete but is almost uniformly distributed between the ends. The strain increase of the concrete is localised in the same zones as the high bending moment. The steel strain also increases but much less because it is not localised. The elongation of the tendon can be integrated from the strain increment of the concrete along the tendon. The elongation divided by the tendon length is the strain increment in the steel.

Because the integration is a tedious way to find out the steel strain and stress, approximative methods have been developed. In one of them, the deformations of the beam are assumed to take place only in plastic hinges. If no strain evaluations are made, the steel stress can be assumed to be 50 MPa higher than the prestress (EC2).

In the ultimate limit state of bending the steel stress in the tendon is higher than the prestress but remains in ordinary beams below the 0,1-limit. The concrete failure in compression is the failure mechanism. In simplified design the steel stress is first evaluated and the tendon forces calculated. Next the depth of the compression block in concrete is solved from the equilibrium of the tensile and compressive forces. Finally, the bending resistance is obtained as the product of the inner lever arm and the tendon forces.

Fig. 89. Post-tensioned flat slab [BY210].
P: Primary tendon.
S: Secondary tendon.

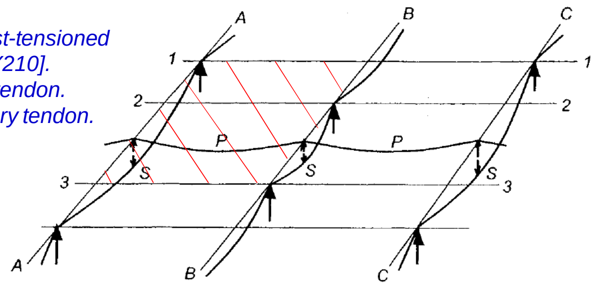


Fig. 89 depicts a prestressed flat slab with irregular column spacing. Consider the hatched zone between lines A & B and 1 & 3. The primary tendons P are distributed uniformly between lines 2 and 3. By adjusting the force and curvature of the tendons, the transverse forces are made balance a predetermined uniformly distributed load.

The secondary tendons follow the column lines A, B, … They are designed to carry the transverse downward forces due to the downward curvature of the primary tendons (support reactions) .

CONCRETE-CONCRETE COMPOSITE STRUCTURES

Typically the lower part is prestressed:

Hollow core slab + topping

Solid plank + topping

Double-tee slab + topping

I-beam + topping

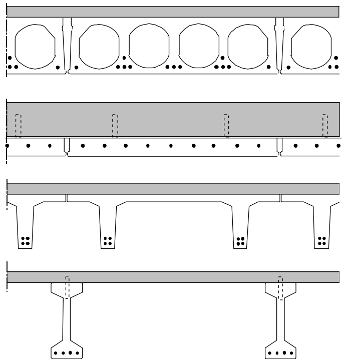


Fig. 88.

Effects of topping concrete

- Enhances the load-bearing cpaciity, particularly seismic
- Enhances the stiffness
- Facilitates HVAC installations (particularly solid plank)
- Enhances the diaphragm action (stability)
- Distributes concentrated loads
- Makes cantilevers and partial continuity possible
- Improves sound insulation and fire safety
- Offers a possibility to use reinforcement against accidental actions
- Cost-effective, if a thick screed is needed in any case
- Good workmanship needed to ensure the bond between the new and old concrete

General about the design

- The interface between the old and new concrete may often be assumed stiff unil failure
- The resistance of the interface is often strenghened by shear keys, grooves or shear reinforcement
- In the serviceability limit state the structure is uncracked in ordinary cases (prestressed lower part)
- Elementary beam theory (Hooke's law, planes perpendicular to the centroidal axis remain perpendicular) is a good approximation in the serviceability limit state.
- In the ultimate limit state, the principles of the reinforced concrete strucctures are applied.
- Propping (temporary supports during execution) is used only in special cases (solid planks+topping)

Differential shrinkage

Parts with different ages and materials have different shrinkages. The differential shrinkage $\Delta\varepsilon_{sh}$ tends to bend the structure.

I. Assume the upper part separated from the lower part and let it shrink by $\Delta\varepsilon_{sh}$ which is equal to the differential shrinkage.

II. Pull the upper part with force $P = -EA\Delta\varepsilon_{sh}$, i.e. return the part to its original length.

III. While P is acting, connect the parts.

IV. Compress the composite structure with the countervectors of P.

$$P = -E_{up}A_{up}\Delta\varepsilon_{sh} \qquad \sigma_{up,II\&III} = -E_{up}\Delta\varepsilon_{sh}$$

Note! $\Delta\varepsilon_{sh} < 0$ $\qquad \sigma_{low,III} = 0$

$$\sigma_{IV} = E_k\left(\frac{-P}{(EA)} + \frac{-Py_P}{(EI)}y\right) + \sigma_{III}$$

E_k is elasticity modulus for either part,
y_P y-coordinate of P.

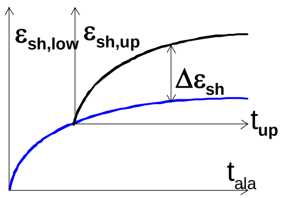


Fig. 89.

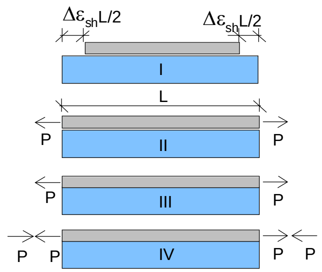


Fig. 90.

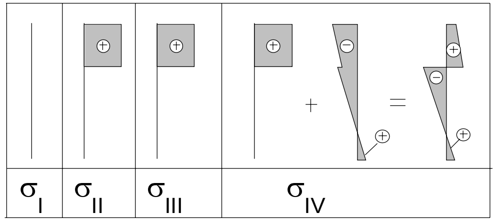


Fig. 93. Stresses of concrete in stages I – IV.

Note.

It is often accurate enough to consider uniform differential shrinkage because the upper part may dry both upwards and downwards.

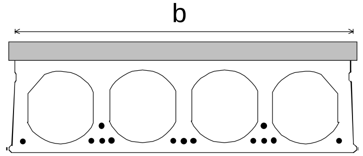
The elastic stresses and strains due to differential shrinkage $\Delta\varepsilon_{sh}$, calculated from the free shrinkage tend to overestimate the real consequences of the differential shrinkage. This is attributable to creep and relaxation in the new concrete, caused by the restraining old concrete. This effect is taken account by reducing the free differential shrinkage. The reduction factor may be of the order of 0,30. Generally valid rules are not available because

- experimental data are missing
- the calculation methods for the free shrinkage vary which also affects the reported reduction factors

Shear at the interface when the cross-section is uncracked

Close to the support the concrete may be uncracked in flexure until failure, particularly in pretensioned elements wiithout shear reinforcement. In such a case the shear stress τ in the interface can be calculated from the well-known expression

$$\tau = \frac{|(ES)_{up}|V}{(EI)b} = \frac{S_{m,up}V}{I_m b}$$



Kuva 94.

(EI) is the bending stiffness of the whole cross-section and $(ES)_{up}$ the first moment of the area of the upper part (above the interface) around the centroidal axis of the whole section. I_m and S_m refer to the corresponding transformed characteristics. *b* is the widht of the interface. Shear force $V = V_{Ed}$ includes the effect of all external forces causing shear stresses in the interface.

In EC2 symbols v_{Ed} and V_{Ed} are used for the shear stress and shear force.

Shear at the interface, when the cross-section is cracked

Consider the zone between two inclined cracks, see Fig. 95.
Equilibrium: $C=C_1+C_2= T$, $M = Cz$, $\Delta M = \Delta(Cz)=z\Delta C=z\Delta(C_1+C_2)$

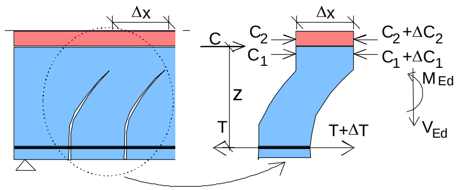


Fig. 95.

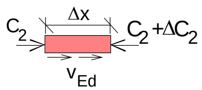


Fig. 96. v_{Ed} = shear stress

Notation: $\beta = C_2 / C$

Equilibrium of horizontal forces in Fig. 96 (*b* = width of the joint), gives

$$v_{Ed} b\Delta x = \Delta C_2 = \Delta(\beta C) = \beta\Delta C + C\Delta\beta$$

$$\leq \beta\Delta C \quad \text{(because } \Delta\beta \text{ on < 0)}$$

Generally:

$$V_{Ed} = \frac{dM_{Ed}}{dx} = \frac{d(Cz)}{dx} \approx z\frac{\Delta C}{\Delta x}$$

$$v_{Ed} \leq \beta\frac{\Delta C}{b\Delta x} \approx \beta\frac{V_{Ed}}{bz}$$

It is safe to assume that $\beta = 1$: $v_{Ed} \leq \dfrac{V_{Ed}}{bz}$

Shear resistance of interface

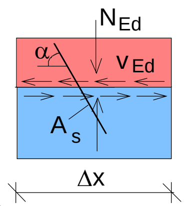


Fig. 97.

In Fig. 97, the joint is subjected to compressive stress σ_n ja and shear stress v_{Ed}. Reinforcement with cross-sectional area ρ per unit area crosses the joint. According to EC2, the shear resistance of the joint is:

$$v_{Rd} = \min\{0{,}5\nu f_{cd}; cf_{ctd} + \mu\sigma_{cn} + \rho f_{yd}(\mu\sin\alpha + \cos\alpha)\}$$

where $\nu = 0{,}6\left[1 - \dfrac{f_{ck}}{250}\right]$ (NDP, may be nationally chosen)

- Parameters *c* and μ depend on the roughness of the old concrete surface
- f_{ck} ja f_{cd} are the characteristic and design value of concrete strength, respectively
- α is the angle between the joint and the reinforcement. If α <45° or α > 90°, the reinforcement is ignored (ρ = 0)
- σ_n is compressive stress normal to the interface positive, compression positive. If is tension, it is set equal to zero.

Classification of joints:

	c	μ
Very smooth	0,025– 0,10	0,5
Smooth	0,20	0,6
Rough	0,45	0,7
Dowelled according to Fig. 98	0,50	0,9

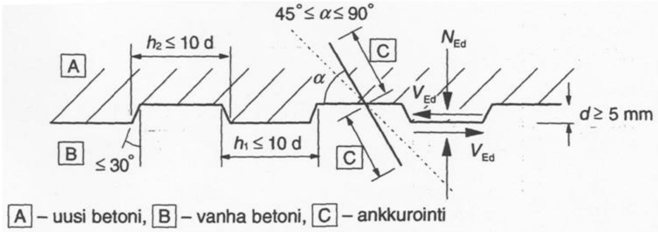


Fig. 98. Dowelled joint according to EC2. A: new concrete. B: old concrete. C: anchoring.

Bending resistance of concrete-concrete composite beam

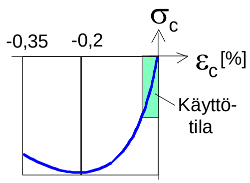


Fig. 99.

In the SLS the compressive stains are small when compared with those in the ULS. (Fig. 99). Therefore, the small differential strains due to the execution can be ignored, see Figs. 100 and 101. Conclusion: Calculate the bending resistance as for a homogenous concrete beam taking into account the different strength values.

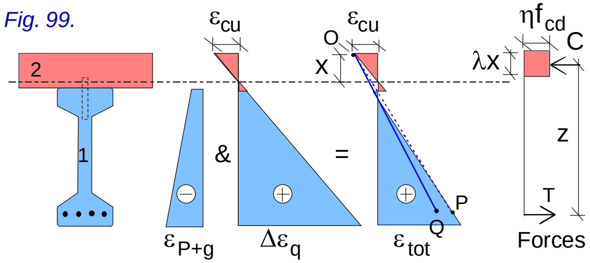


Fig 100. Compression in topping only.

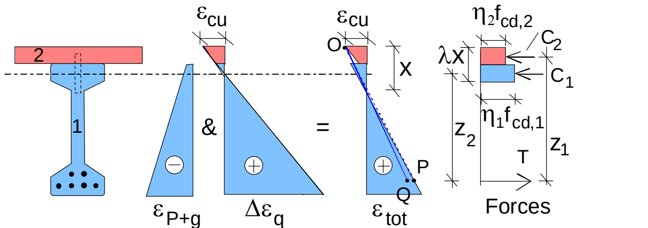


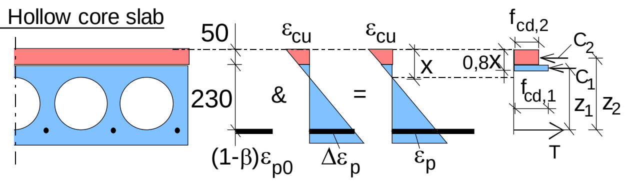
Fig. 101. Neutral axis in old concrete.

EC2: The stresses in concrete are replaced by stress blocks with depth λx. The stress equals ηf_{cd} .(η ≤ 1 ja $\lambda \leq 0{,}8$ from EC2). An approximative value for the compressed zone (x) is determined as follows:

1. Set strain on the top = failure strain ε_{cu} (point O in Figs 100 and 101).
2. Let the concrete strain ε_{cpp} at the centroid of gravity of tendons be such that the steel yields, (point Q). Calculate the tensile force T.
3. Use the line QO to calculate $x \Rightarrow \lambda x$ and compression $C = C_1 + C_2$.
4. Change $\varepsilon_{cpp} \Rightarrow$ new x, … until $C = T$, i.e. equilibrium of forces. If the strain-hardening of steel is ignored, iteration is not needed to solve x.

Example. Hollow core slab

Fig. 102.



Concrete: α_{cc}=0,85, γ_c =1,5; **Steel**: γ_c =1,15	
HCS: Width 1160 mm, strength C50/60 ⇒ λ=0,8, η =1, f_{cd1}=0,85x1x50/1,50 = 28,3 MPa, effective depth d = 230 mm	
Strands: 10 , A_p=93 mm²/ strand, prestress. σ_{p0} = 1000 MPa, losses 20%, E_p = 190 GPa, strength 1600 MPa , yield force = 1,294 MN	
Topping: width 1160 mm, strength C35/45 ⇒ λ=0,8, η =1, ε_{cu} =0,0035, f_{cd2}=0,85x1x35/1,50 = 19,8 MPa, thickness = 50 mm	

Equil. of forces ⇒ x = 68,0 mm, z_1 = 228 mm, z_2 = 255 mm, C_2 = 1,148 MN, C_1 = 0,146 MN and $M_{Rd} = C_1 z_1 + C_2 z_2$ = 326 kNm

Note 1.

Assuming the steel strength constant (=1600 MPa, no strain-hardening), gave a constant tensile force T = 1,294 MN.

The maximum value of C_2 = (19,8 MPa) x (1160x50 mm²) = 1,148 MN < T $\Rightarrow$ C_1 = (28,3 MPa)xh_1x(1,160 m) = (1,294 – 1,148) = 0,146 MN $\Rightarrow$ the depth of the stress block in the hollow core slab is h_1 = 4,44 mm $\Rightarrow$ z_1 = 230 – 4,44/2 = 227,8 mm, z_2 = 255 mm. To verify the failure mode, it must be checked that the ultimate strain ecu = 0,0035 is not exceeded. The strain of the steel at the start of yielding is 1600/190000 = 0,842 %. The strain due to the prestressing after losses is (1-0,20)x1000/190000 = 0,421 %.

So, the strain of the concrete at the centroid of the tendons is 0,842 – 0,421 = 0,421 %. Giving the top fibre of the section the strain 0,0035 gives the depth of the compression zone the value of 127 mm and the bending compression failure is out of question.

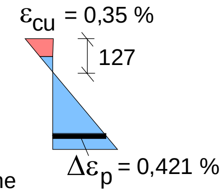


Fig. 103.

Note 2.

Assuming strain-hardening steel would mean that the steel stress varies and iteration is needed.

About harmonized product standards and CE marking

The Construction Products Directive (CPD) states that the construction products put on the market shall predominantly be CE marked if they are subject to Essential Requirements. The requirements for concrete elements deal with the mechanical resistance and stability, fire resistance and durability. The Finnish authorities have interpreted that the CE marking is not obligatory but the next update of the CPD most likely makes it clear that it is.

A rough interpretation is that the CE marking tells in a unified way the essential properties of a construction product. The properties, as well as the methods how the values or classes of the properties are determined and declared, are given in harmonized product standards (hEN), in a European guide for technical approvals (ETAG) or in a CUAP (Common Understanding of Assesment Procedure, a guide for products with few or only one manufacturer, must be unanimously approved). EN standards are made by CEN which is a European standardisation organisation, ETAGs and CUAPs by EOTA which is a European organisation for technical approvals.

A CE marked construction product can be transported from one country to another, and the authorities have no right to prevent the use of it if the properties declared in the CE marking meet the requirements of the authorities set beforehand in the country where the product will be used.

This principle is based on the fact that the authorities have had the right to require beforehand that all properties which are subject to essential requirements in the country must be given in the CE marking. In addition, the authorities are obliged to give the national requirement levels to these properties. In other words, this is a prejudgement about the essential requirements and how the properties are determined, declared and controlled. Only the requirement levels may vary from country to country. The aim is to eliminate trade barriers between countries.

The product standards and European technical approvals (ETA) are in some cases important from the structural engineer's point of view because they may specify design methods which are not given in the Eurocodes or which may be in contradiction with those given in the Eurocodes. In such a case the product standard or ETA must be followed.

Product standards have been made e.g. for linear concrete elements (beams and columns) ribbed floor elements (e.g. double-tee slabs) hollow core slabs, concrete fence elements, wall elements, retaining wall elements etc.

Examples of progress of CE marking in Finland
- cement has been CE marked for years
- first CE marked hollow core slabs in Finland in 2009
- the standard for reinforcing bars (EN 10080) may be available during the next few years (this is an optimistic guess) and before this happens, no CE marking is possible.

About cracking of concrete

One of the advantages of a prestressed concrete structure is the lack of cracks in the SLS which results in small deflections and improved durability. However, EC2 does not require that the prestressed structures be crack-free in the SLS. This is justified by several reasons, the most important being the fact that the deflections and durability are often in control even if there are small cracks.

EC2 requires control of the crack widths. The design rules for this purpose have given rise to debate and there is even more debate about the role of the crack width in durability. On the other hand, if the deflections become too large or there is a risk of steel corrosion, elimination of cracks may help, and it is always recommended to design crack-free structures if it possible at reasonable costs.

The risk of cracking is present already when the prestressing force is transferred (or even before due to the shrinkage of the concrete). Transverse tensile and compressive stresses develop within the anchorage zone, compressive and bending stresses outside it. The resulting principal tensile stresses may be high enough to make the concrete crack.

Fig. 104 illustrates the stresses due to a centric anchorage. There are transverse tensile stresses and longitudinal compressive stresses. The tensile stresses are inversely proportional to the size of the anchor plate.

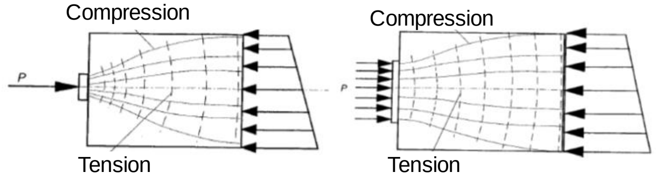


Fig. 104. a) Small loaded area. b) Large loaded area. [BY210].

The eccentricity of the force has a strong increasing effect on the cracking risk. The notation shown in Fig. 105 is used to evaluation of cracking stresses and forces in the following expressions. Fig. 106 shows the real and simplified cracking stress at the end of a beam. Fig. 107 shows the resultant of the cracking stresses.

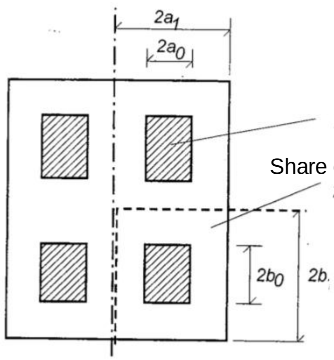


Fig. 105. Notation.[BY210].

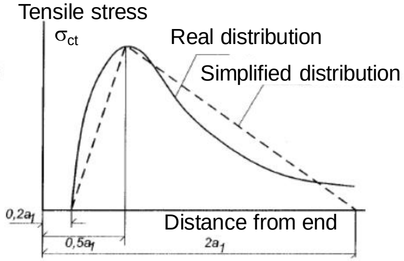
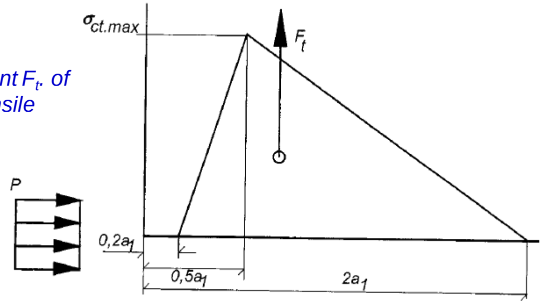


Fig. 106. Real and simplified stress distribution. [BY210].

Fig. 107. Resultant F_t. of the transverse tensile stress distribution [BY210].



In an ETA (european technical approval) for a post-tensioning system, instructions are given how to avoid cracking or excessive cracking due to the anchorage forces. These instructions are often based on strut-and-tie methods (truss analogy) and they do not assume completely crack-free concrete.

In the Finnish practice (B4) no transverse reinforcement due to the anchorage is necessary in case the condition on the right is in force. A_{co} is the area of the anchor plate and $f_{ck,K150}$ the characteristic value of the concrete cubic strength. Using the cylindrical strength f_{ck}, this is roughly equivalent to the condition

$$\frac{1{,}2P_{Ed}}{A_{c0}} \le \frac{0{,}7f_{ck,K150}}{\gamma_c}$$

$$\frac{P_{Ed}}{A_{c0}} \le 0{,}7\frac{f_{ck}}{\gamma_c}$$

Cracking force for *centric* compression is obtained from

$$F_{t1,Ed} = 0{,}25P_{Ed}\left(1-\frac{b_0}{b_1}\right)$$

(by Mörsch, b_1 and b_0 in the direction of $F_{t1,Ed}$, see Figs 105 and 108, Leonhardt: coefficient = 0,30 pro 0,25)

Leonhardt: *Eccentric* compressive force P_{Ed} causes a cracking force at the depth of the centroidal axis, which is

$$F_{t2,Ed} = \frac{0{,}015P_{Ed}}{1-\sqrt{\dfrac{2e_p}{h}}}$$

(h is the depth of the section and e_p eccentricity of P_{Ed} , see Fig. 108)

If there are two symmetrically acting forces P_{Ed}, one above and one below the centroidal axis, the cracking force is doubled.

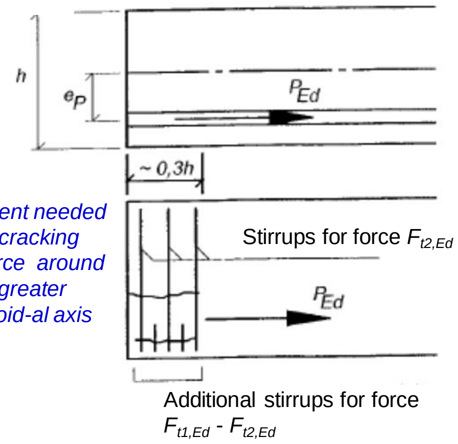
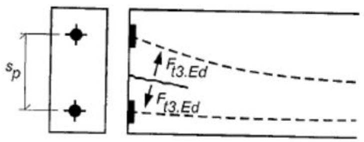


Fig. 108. Reinforcement needed to control transverse cracking when the cracking force around the tendon ($F_{t1,Ed}$) is greater than that at the centroid-al axis ($F_{t2,Ed}$).[BY210].

For the cracking force at the centroidal axis of a deep beam, BY210 gives an expression (see Fig. 87)

$$F_{t3,Ed} = 0{,}2bs_p\frac{P_{Ed}}{A_c}$$

Fig. 109. [BY210].



- b is the width of the concrete belonging to the anchor plate of the tendon group,
- s_p free distance between the tendon groups,
- P_{ED} total tendon force
- A_c area of concrete belonging to the anchor plate, see Fig. 105.

The expressions are not universal. They fit best to rectangular sections and postensioned beams. (Leonhardt's F_{2t}-formula also applies to pretensioned beams.) For beams with thin webs the cracking stresses are sensitive to the shape of the section. For this reason, the ends of the I-beams are often made rectangular (thicker). Despite this, a part of the tendons must be debonded for a certain length at the ends to avoid excessive cracking.

The formulae are not applicable to hollow core slabs because they cannot be shear reinforced. The cracking forces are carried by the tensile strength of the concrete. The design method is presented in standard EN 1168.

The release may also give rise to *bending cracks* close to the ends of pre-tensioned elements. For post-tensioned beams, the top fibre of the spans or the bottom fibre at the interior supports are sensitive to cracking when the tendon force is high when compared with the self-weight of the beam. The cracking is evaluated using expressions

$$\sigma_{c,P+G} = \frac{-\sum_k P_k(t_0)}{A} + \frac{M_P+M_G}{I}y_1 \le f_{ct}(t_0)$$

$$\sigma_{c,P+G} = \frac{-\sum_k P_k(t_0)}{A} + \frac{M_P+M_G}{I}y_2 \le f_{ct}(t_0)$$

Note. For statically determined beams: $M_P = -\sum_k P_k(t_0)y_{k,p}$

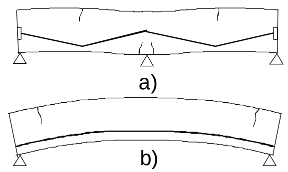


Fig. 110. a) Post-tensioning. b) Pretensioning.

$P_k(t_0)$ is the prestressing force in tendon k at the time t_0 (release or pos-tensioning), y_1, y_2 (<0) and $y_{k,p}$ refer to the y-coordinate of the bottom fibre, top fibre and tendon k, respectively, M_P and M_G are the bending moment due to the prestressing and self-weight, f_{ct} the tensile strength of the concrete and A & I are area and second moment of area of section (gross or transformed).

The inequalities should be in force at all (critical) sections. There are at least two ways of thinking:

1. *If the elements are uncracked after trasportation and installation, or if the structures post-tensioned on site are intact after some days after the tensioning, it does not matter, how much there has been safety margin against cracking at prestressing. The prestressing force decreases and the strength increases with time.*

2. *The normal safety factors shall be applied during the execution and the design value of the concrete tensile strength have to be used.*

Alternative 1 allows the designer to decide, which tensile strength and safety factors he should use. The tensile strength of the concrete may be e.g. f_{ctk} , f_{ctd}, $f_{ctk,fl}$ or 0. The nominal value may be chosen for the self-weight and prestressing force, and the safety factors for the loads may be 1,0 or those used in the ULS.

EC2 gives no clear answer, how to prevent or control the cracking due to the prestressing.

More generally, there is no European consensus concerning the cracking criteria to be applied in the design. The afore-mentioned flexural tensile strength is (according to EC2)

$$f_{ctm,fl} = \max\left\{1{,}6 - \frac{h}{1\text{m}};1\right\} f_{ctm} \quad \text{(mean value)}$$

$$f_{ctk,fl} = \max\left\{1{,}6 - \frac{h}{1\text{m}};1\right\} f_{ctk} \quad \text{(characteristic value)}$$

h is the depth of the section and f_{ctm} (f_{ctk}) the mean (characteristic) value of the tensile strength.

(According to B4, $f_{ctk,fl} = 1{,}7 f_{ctk}$. This value is independent of the section depth and supported by no research result.)

The flexural tensile strength is not used as a design criterion in the SLS, but the criteria of table 7.1N in EC2 are applied. The idea is that if no tensile stresses are accepted, the beam remains uncracked and the durability is good enough for severe exposure classes. In other cases there are tensile stresses which necessarily result in some cracks and controlling their width is the only thing that can be done.

EC2 (National annex of Finland), Table 7.1N: Maximum crack widths w_{max} [mm].

Exposure class	Reinforced and unbonded	Pretensioned and grouted post-tensioned
	Quasi permanent load combination	Frequent load combination
X0, XC1	0,4[1]	0,2
XC2, XC3, XC4 XD1, XS1	0,3	0,2[2]
XD2, XD3, XS1, XS3	0,2	No tensile stresses allowed

Note. 1. In classes X0 ja XC1 the crack width does not affect the durability, and this limit is set for aesthetical reasons. If the appearance is not relevant, the limit may be increased.

Note. 2. In these classes the quasi permanent combination must not cause tension.

Explanation: Carbonisation →C, Sea water→S, Deicing salt→D

The design rules of EC2 for the crack width are regarded as doubtful by many researchers. In many caes they result in considerably smaller crack widths as B4 and there are also some inconsistencies. This part of EC2 is likely to be modified in the near future. Therefore, the design rules for crack width are not considered in this course.

The calculation methods for the crack width are not adequate for prelimin-ary design, i.e. for finding the measures of the section and prestressing force. For this purpose, the following inequalities may be applied:

$$\sigma_{c,P+G+Q} = \frac{-\sum_k \eta_k P_k(t)}{A} + \frac{M_P + M_G + M_Q}{I} y_1 \le f_{ct}(t)$$

$$\sigma_{c,P+G+Q} = \frac{-\sum_k \eta_k P_k(t)}{A} + \frac{M_P + M_G + M_Q}{I} y_2 \le f_{ct}(t)$$

Note: For a statically determined beam:

$$M_P = -\sum_k \eta_k P_k(t) y_{k,p}$$

where the coefficients η_k take into account the losses at time t in tendon k and M_Q is the bending moment due to the external forces .

The tensile strength may be the characteristic flexural tensile strength of the concrete $f_{ctk,fl}$. If the amount of the prestressing steel is clearly higher than the minimum amount, the crack width at the cracking load will not exceed 0,2 mm. In such a case, if the stress is lower than the tensile strength, the crack width will remain below 0,2 mm. However, the crack width must be checked in the final design.

To ensure that the beam remains uncracked, it is preferred to check the inequalities on the previous page for a characteristic load combination with f_{ct} = characteristic axial strength. The risk can still be reduced by choosing the prestressing force so high that the characteristic load combination does not cause tensile stresses.

About the compression of the concrete

EC2: During prestressing, the compressive stress of the concrete must not exceed $0{,}60 f_{ck}(t_0)$ (in some circumstances in Finland $0{,}65 f_{ck}(t_0)$ is acceptable). If the permanent stress exceeds $0{,}45 f_{ck}(t)$, the nonlinearity of the creep must be taken into account.

About section design

The pretensioned concrete structures are in general precast elements. The free choice of their cross-section is restricted due to the moulds, reinforce-ments etc. The rectangular beams or other elements, for which the moulds are made case by case, give the widest freedom. On the other hand, the designer should check the possibilities of the producers of solid slabs, I-beams double-tee slabs and hollow-core slabs before proposing his own ideas. In Finland, the shape and size recommendations of the precast element industry are largely followed, which makes it easy to use elements from different producers in the same building.

The sections of hollow core slabs are fully controlled by the casting machines. In Finland the slab width is 1,2 m and the depth options 150 mm, 200 mm, 265 mm, 320 mm, 370 mm, 400 mm ja 500 mm.

The casting machines and moulds can be modified for production of innovative section shapes. Such efforts should be organised as product development projects in which all aspects comprising the mechanical, acoustical and fire performance as well as the production, transport, installation, durability, economics etc. are considered.

Due to the standardisation of the precast prestressed elements, their producers have calculated the resistance of their elements for uniformly distributed load in advance, at least for the maximum prestressing force. The results are available to the structural engineer as tables, charts or computer programs. With these aids, the sections can easily be specified in the preliminary design.

The detailed design is concentrated to the producers or to their consult-ants. Thanks to this, the reinforcement and details are those which fit best to the production. The designer of the building frame has to provide the element designer with information concerning the position of the elements, preliminary measures, openings, holes, supports, loads etc. which may affect the final design. To be succesful, he must have the basic knowledge of the behaviour of the elements.

The ultimate limit state of bending can be used for finding the dimensions of the section and the prestressing force as an alternative for the cracking considerations in the SLS. On the other hand, attempts are made to take care of the other ultimate limit states (shear, torsion, combined limit states) by additional reinforcement in such a way that they only occasionally may determine the dimensions of the section.

Post-tensioned structures are cast on site using traditional methods. Therefore, their geometry can be chosen relatively freely.

The lower limit of the section is affected by the requirement that the section must be large enough to accommodate the anchors, ducts and their cover concrete. This requirement is affected by
- The SLS: a certain combination of tendon force and arch depth is needed to keep the deflections and crack widths acceptable
- The ULS: a certain internal lever arm is needed to balance the external bending moment.

The section must also be wide enough so that
- there is enough space for the tendon
- the concrete can carry the stresses
- the stability requirements in use can be solved

Special considerations

Buckling of beams

Buckling of slender beams, particularly pitched I-beams, must be taken into account in lifting, transporting and installation. Since the beams are in the completed structure tied with concrete roofs and floors to each other, which in ordinary cases prevents the buckling, it is preferable to provide the beams for transportation with external auxiliary structures which increase their lateral bending stiffness. Advice for the design against buckling are given e.g. in BY210.

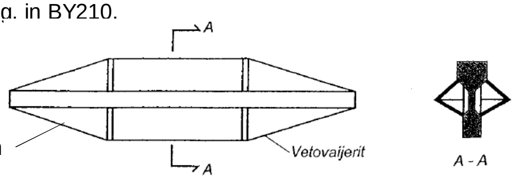


Fig. 111. [BY 210].

External tendon

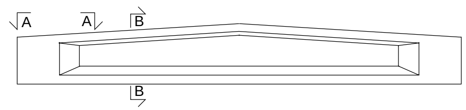


Fig. 112. Pitched I-beam.

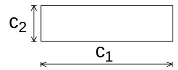


Fig. 114.

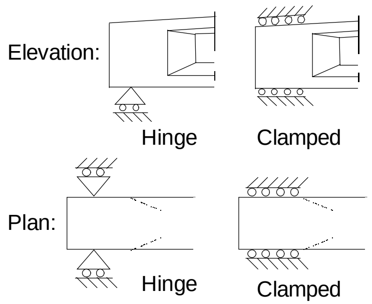


Fig. 113. Support conditions in two directions.

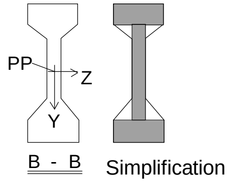


Fig. 115.

de St. Venant's torsion modulus for the rectangle in Fig. 114 is

$I_t = \frac{1}{3}c_1c_2^3$ and for the grey I-beam

it is the sum of the I_t:s of the parts. For more accurate values, see any textbook of structural mechanics.

Buckling due to uniformly distributed load

Consider a case in which a slender beam is supported at the ends. The span is = L. The temporary supports used during the lifting have been removed but the beam has not yet been tied to other stabilizing structures. The self weight of the beam shall fulfil the requirement $g \le q_{kr}/2$ where q_{kr} is the critical buckling load of the beam per unit length. BY210 gives

$$q_{kr} = \frac{\kappa_1\kappa_2\alpha_{cm}E_{cm}\sqrt{0{,}4I_tI_y}}{L^3} \qquad \kappa_1 = 1+1{,}44\frac{d_l}{L}\sqrt{\frac{I_y}{0{,}4I_t}}$$

$$\kappa_2 = \kappa_{2a} = \sqrt{1+\frac{\pi^2\beta}{4}}$$, if there is hinge support in transverse direction

$$\kappa_2 = \kappa_{2b} = \sqrt{1+\pi^2\beta}$$, if clamped at supports in transverse direction

$$\beta = \frac{2I_{fy}z^2}{0{,}4I_tL^2} \qquad I_{fy} = \frac{2}{1/I_{y,top}+1/I_{y,bot}}$$

For α_{cm} see p. 184, I_y is the second moment of area in transverse direction, E_{cm} elasticity modulus and d_l the y-coordinate of the loading point. $I_{y,top}$ ja $I_{y,bot}$ are the transverse second moment of area of the top and bottom flange and z the centroidal distance of the flange.

α_m. [BY 210] .

Rotation around longitudinal axis	Support against bending around horizontal axis	Support against bending around vertical axis	α_m
Prevented	Both ends simply supported	Hinge	28,3
Prevented	Cantilever	Hinge	12,8
Prevented	Both ends clamped	Hinge	98
Prevented	One end clamped, the other end simply supported	Hinge	54
Prevented	Both ends simply supported	Clamped	50
Prevented	Both ends clamped	Clamped	137

In BY 210 it has been shown that if the sum of the flange depths in an I beam is at least 40% of the section and the span L is not greater than $60b$ where b is the width of the flanges, the critical buckling load is at least twice the self weight. This result is obtained even though the contribution of the web to I_t and I_y is ignored.

Buckling during lifting is more complicated because the support against rotation around the longitudinal axis is flexible. In most cases the force on the lifting hooks has a horizontal component which compresses the beam and reduces the critical buckling load.

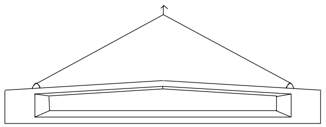


Fig. 116.

Ties on the top of pitched I-beam

On the free outer surface of a structure, the stresses are always parallel to the surface. On the top of a pitched I-beam, the surface makes a bow and so must the stresses also do. This necessitates a vertical balancing force.

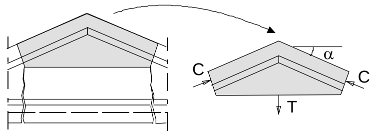


Fig. 117.

The free body shown in Fig. 117 is affected by forces C parallel to the surface. Their horizontal component is not higher than the yield force of the tendons P_{yd} or $C\cos\alpha = P_{yd}$. It follows that

$$T = 2C\sin\alpha = 2P_{yd}\tan\alpha$$

With inclination 1:16, $T = P_{yd}/8$ is obtained.

In reality, C is not exactly parallel to the surface but more horizontal because the stresses inside the beam bend before the ridge. Therefore, the actual T is slightly lower. É.g. BY 210 gives a milder expression

$$T = 1{,}8P_{yd}\sin\alpha \approx 0{,}9P_{yd}/8$$

Before the beam cracks in flexure, the vertical stresses are small. A part of the nearly horizontal stresses are transmitted through the web below the upper chord. After cracking, the interface between the web and the upper chord becomes critical, and the chord must be tied to the web with stirrups that carry the vertical tensile force, see Fig. 118.

Stirrups tying compressed rebars

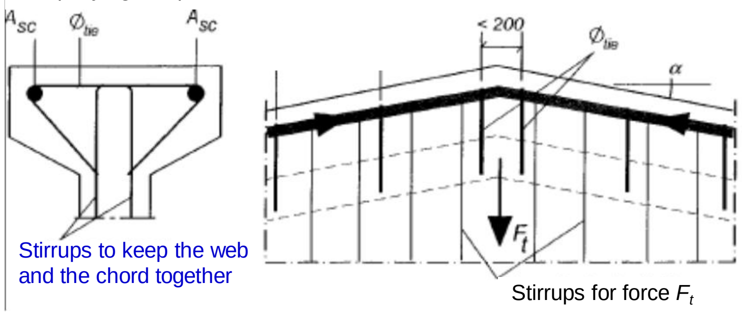


Fig. 118. Stirrups on the top of a pitched I-beam. [BY210].

Even though the rebars in the upper chord were needed only to control the cracking due to the prestressing, they, nevertheless, work as compression reinforcement in the completed structure. Due to the long-term deformations of the concrete, the compressive stresses are transmitted from the concrete to the rebars and the rebars may even yield in compression. There is a risk of buckling and transmission of the stresses from the bars to the concrete via the ends of discontinuous bars . Consequently, the concrete may spall. For this reason

- the amount of steel and bar size in the compressed chord must be kept small
- the compressed bars are tied with stirrups to prevent buckling
- the compressed bars are not spliced in the zones of highest compression
- whenever possible, the rebars on the top are replaced by upper tendons.

For more information, see www.onnettomuustutkinta.fi , B- ja C- tutkinnat,

- Kauppakeskuksen katon sortumisvaara Kuopiossa 18.3.2005
- Kauppakeskuksen sortumisvaara Savonlinnassa 31.3.2006

Shear resistance of hollow core slabs on flexible supports (= beam)

The web of a hollow core slab is sensitive to shear because it is thin, see Fig. 119, and because the production tehnology prevents the use of shear reinforcement. However, the shear resistance is seldom critical if the slabs are supported on walls or other non-flexible supports.

This is not the case when the slabs are supported on beams. When the beams deflect, the slab ends are subjected to additional transverse deformations to which the webs are sensitive. In full scale floor tests it has been observed that the shear resistance of slabs supported on beams may be less than 50% of that measured on non-flexible supports.

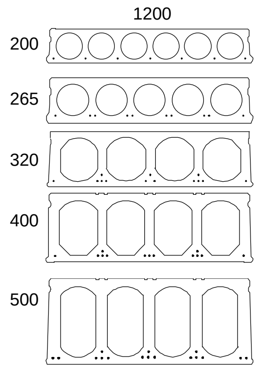


Fig. 119. Finnish hollow core slabs.

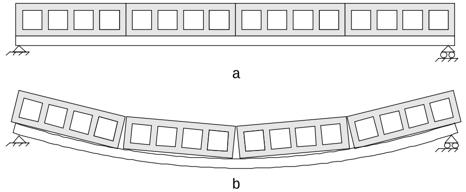


Fig. 120. Slabs on a frictionless beam. a) Initial state. b) After deflection of supporting beam.

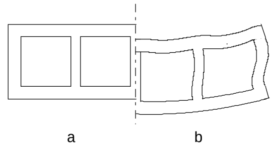


Fig. 121. Actual situation. a) initial state. b) After deflection of supporting beam.

When a frictionless beam deflects, the slabs push each other outwards as shown in Fig. 120. Since there is always friction between the soffit of the slab and the beam, and because the slabs are tied to the beam with reinforcement, the slabs will be subjected to transverse shear parallel to the beam and deformations as shown in Fig. 121. This effect is taken into account according to the Finnish Concrete Code Card 18 based on 20 full-scale floor tests. The principles of the design method of the Code Card are also explained in BY210.

Fig.122. Example of floor test. Size of test specimen 7,2 x 21 x 0,5 m^3.

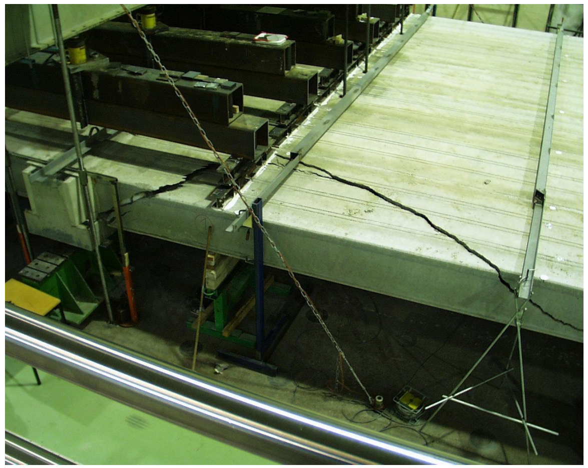


Fig. 123. Failure mode. The failure initiated next to the support of the beam.

Note.

- The tested floors have failed when the deflection of the beams has been of the order of *L*/250 or even smaller
- The reduction in shear resistance cannot be explained by the magnitude of deflection only. The interaction between the slab and the beam must also be taken into account
- For these reasons, it is not economically reasonable to give such limits for the deflection which would make it safe to escape the consideration of the reduction in shear resistance. In other words, it does not pay to control the *effects* of the deflection by controlling the deflections only.

Things affecting positively or negatively to the shear resistance

+ Stiffness of beam
+ Dowels on the top of the joint between the slab end and the beam
+ Reinforcement in the concrete topping, if it ties the topping, beam and slab together
+ Long concrete fillings in the hollow cores
- Dowels at the bottom of the joint between the slab end and the beam
- Tie reinforcement placed close to the soffit of the slab
- Temporary supports below the beams if not removed before hardening of the joint concrete.